Известия НАН Армении, Математика, том 54, н. 1, 2019, стр. 40 – 59

WEIGHTED NORM INEQUALITIES FOR AREA FUNCTIONS RELATED TO SCHRÖDINGER OPERATORS

L. TANG, J. WANG, H. ZHU

Peking University, Beijing, China
Tufts University, Medford, USA
Beijing International Studies University, Beijing, China
E-mails: *tanglin@math.pku.edu.cn; Jue.Wang@tufts.edu; zhuhua@pku.edu.cn*

Abstract. Let $L = -\Delta + V$ be a Schrödinger operator, where Δ is the Laplacian operator on $\mathbb{R}^n$, and V is a nonnegative potential belonging to certain reverse Hölder class. In this paper, we establish some weighted norm inequalities for area functions related to Schrödinger operators and their commutators.

MSC2010 numbers: 42B25, 42B20.
Keywords: Area function; Schrödinger operator; weighted norm inequality.

1. Introduction

In this paper, we consider the Schrödinger differential operator on $\mathbb{R}^n$ $(n \geq 3)$:

$$L = -\Delta + V(x),$$

where Δ is the Laplacian operator on $\mathbb{R}^n$, and V is a nonnegative potential belonging to certain reverse Hölder class.

A nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to the class B_q $(1 < q \leq \infty)$ if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \leq C\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)dy\right) \tag{1.1}$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x,r)$ denotes the ball centered at x and radius r. In particular, if V is a nonnegative polynomial, then $V \in B_\infty$. It is worth to point out that if $V \in B_q$ for some $q > 1$, then there exist $\epsilon > 0$, depending only n, and a constant C (as in (1.1)) such that $V \in B_{q+\epsilon}$. Throughout this paper, we always assume that $0 \not\equiv V \in B_{n/2}$.

[0]The research was supported by the NNSF (11771023) and (11571289) of China.

The study of the Schrödinger operator $L = -\Delta + V$ has recently attracted much attention (see [1, 2, 5, 6, 12, 15], and references therein). In particular, in Shen [12] it was proved that the Schrödinger type operators: $\nabla(-\Delta + V)^{-1}\nabla$, $\nabla(-\Delta + V)^{-1/2}$, $(-\Delta + V)^{-1/2}\nabla$ with $V \in B_n$, and $(-\Delta + V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in B_{n/2}$, are standard Calderón-Zygmund operators.

Recently, Bongioanni et al. (see [1]) proved the $L^p(\mathbb{R}^n)(1 < p < \infty)$ boundedness for commutators of Riesz transforms associated with Schrödinger operators with $BMO(\rho)$ functions (which include the class BMO functions), and then, in [2], they established the weighted boundedness of Riesz transforms, fractional integrals and Littlewood-Paley functions associated with Schrödinger operator with weights from the class A_p^ρ, which includes the class of Muckenhoupt weights. Very recently, in [13, 14], one of the authors of this paper has established weighted norm inequalities for some Schrödinger type operators, which include commutators of Riesz transforms, fractional integrals, and Littlewood-Paley functions related to Schrödinger operators (see also [3, 4]).

In this paper, we continue our research to study weighted norm inequalities for area functions related to Schrödinger operators and their commutators. To state the main result of this paper, we first introduce some definitions. The area function S_Q related to Schrödinger operators is defined by

$$S_Q(f)(x) = \left(\int_0^\infty \int_{|x-y|<t} |Q_t(f)(y)|^2\, dy \frac{dt}{t^{n+1}}\right)^{1/2},$$

where

$$(Q_t f)(x) = t^2 \left(\left.\frac{dT_s}{ds}\right|_{s=t^2} f\right)(x), \quad T_s = e^{-sL}, \quad (x,t) \in \mathbb{R}_+^{n+1} = (0,\infty) \times \mathbb{R}^n$$

The commutator of S_Q with $b \in BMO(\rho)$ is defined by

$$S_{Q,b}(f)(x) = \left(\int_0^\infty \int_{|x-y|<t} |Q_t((b(x) - b(\cdot))f)(y)|^2\, dy \frac{dt}{t^{n+1}}\right)^{1/2}.$$

The following two theorems are the main results of this paper.

Theorem 1.1. *Let $1 < p < \infty$. If $\omega \in A_p^\rho$ (to be defined in Section 2), then there exists a constant C such that*

$$\|S_Q(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

If $\omega \in A_1^\rho$, then there exists a constant $C > 0$ such that for any $\lambda > 0$

$$\omega(\{x \in \mathbb{R}^n : |S_Q(f)(x)| > \lambda\}) \le \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|\omega(x)dx.$$

The next theorem contains weighted norm inequalities for the commutator $S_{Q,b}$.

Theorem 1.2. *Let $b \in BMO(\rho)$ (to be defined in Section 2) and $1 < p < \infty$. If $\omega \in A_p^\rho$, then there exists a constant C such that*

$$\|S_{Q,b}(f)\|_{L^p(\omega)} \le C\|b\|_{BMO(\rho)}\|f\|_{L^p(\omega)}.$$

If $\omega \in A_1^\rho$, then there exists a constant $C > 0$ such that for any $\lambda > 0$

$$\omega(\{x \in \mathbb{R}^n : |S_{Q,b}(f)(x)| > \lambda\}) \le C\int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right)\omega(x)dx.$$

The rest of the paper is organized as follows. In Section 2, we introduce some notation and state some basic results. In Section 3, we establish a number of lemmas, which play a crucial role in this paper. Finally, in Section 4 , we prove our main results - Theorems 1.1 and 1.2.

Throughout the paper, we let C to denote constants that are independent of the main parameters involved, but whose value may vary from line to line. The notation $A \sim B$ means that there exists a constant $C > 1$ such that $1/C \le A/B \le C$.

2. Preliminaries

We first recall some notation. Given a ball $B = B(x,r)$ and a number $\lambda > 0$, by λB we will denote the λ-dilated ball, which is the ball with the same center x and with radius λr. Similarly, by $Q(x,r)$ we will denote the cube centered at x with the side length r, and $\lambda Q(x,r) := Q(x,\lambda r)$ (here and below only cubes with sides parallel to the coordinate axes are considered). Given a Lebesgue measurable set E and a weight ω, by $|E|$ we denote the Lebesgue measure of E and $\omega(E) := \int_E \omega dx$. For $0 < p < \infty$, by $L^p(\omega)$ we denote the L^p-weighted space with norm $\|f\|_{L^p(\omega)} := (\int_{\mathbb{R}^n} |f(y)|^p\omega(y)dy)^{1/p}$.

The function $m_V(x)$ is defined by

$$\rho(x) = \frac{1}{m_V(x)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \le 1\right\}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \ne 0$. In particular, we have $m_V(x) = 1$ for $V = 1$ and $m_V(x) \sim (1 + |x|)$ for $V = |x|^2$.

Lemma 2.1. *(see [12]). There exist constants $l_0 > 0$ and $C_0 > 1$ such that*

$$\frac{1}{C_0}\left(1+|x-y|m_V(x)\right)^{-l_0} \le \frac{m_V(x)}{m_V(y)} \le C_0\left(1+|x-y|m_V(x)\right)^{l_0/(l_0+1)}.$$

In particular, $m_V(x) \sim m_V(y)$ if $|x-y| < C/m_V(x)$.

For a ball $B = B(x_0, r)$ with center at x_0 and radius r and a number $\theta > 0$, we denote $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$.

A weight will always mean a nonnegative locally integrable function. As in [2], we say that a weight ω belongs to the class $A_p^{\rho,\theta}$ $(1 < p < \infty)$, if there is a constant C such that for all balls $B = B(x,r)$,

$$\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega(y)\,dy\right)\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1} \le C.$$

Also, we say that a nonnegative function ω satisfies the $A_1^{\rho,\theta}$ condition if there exists a constant C such that for all balls B,

$$M_V^\theta(\omega)(x) \le C\omega(x), \text{ a.e. } x \in \mathbb{R}^n,$$

where

$$M_V^\theta f(x) = \sup_{x\in B}\frac{1}{\Psi_\theta(B)|B|}\int_B |f(y)|\,dy.$$

Since $\Psi_\theta(B) \ge 1$, we obviously have $A_p \subset A_p^{\rho,\theta}$ for $1 \le p < \infty$, where A_p denotes the class of classical Muckenhoupt weights (see [7] and [9]). Note that in some cases we have the embedding $A_p \subset\subset A_p^{\rho,\theta}$ for $1 \le p < \infty$. Indeed, let $\theta > 0$ and $0 \le \gamma \le \theta$, then it is easy to check that $\omega(x) = (1+|x|)^{-(n+\gamma)} \notin A_\infty := \bigcup_{p\ge1} A_p$ and $\omega(x)dx$ is not a doubling measure, but $\omega(x) = (1+|x|)^{-(n+\gamma)} \in A_1^{\rho,\theta}$ provided that $V = 1$ and $\Psi_\theta(B(x_0,r)) = (1+r)^\theta$.

Also, we remark that in the above definitions of $A_p^{\rho,\theta}$ $(p \ge 1)$ and $M_{V,\theta}$, the balls can be replaced by cubes because $\Psi_\theta(B) \le \Psi_\theta(2B) \le 2^\theta\Psi_\theta(B)$. For $V = 0$ and $\theta = 0$, instead of $M_{0,0}f(x)$ we use the notation $Mf(x)$, which is the classical Hardy-Littlewood maximal function. It is easy to see that $|f(x)| \le M_V^\theta f(x) \le Mf(x)$ for a.e. $x \in \mathbb{R}^n$ and $\theta \ge 0$. For convenience, in the rest of this paper, for a fixed $\theta \ge 0$, instead of $\Psi_\theta(B)$ and $A_p^{\rho,\theta}$ we use the notation $\Psi(B)$ and A_p^ρ, respectively.

The next lemma follows from the definition of the class A_p^ρ $(1 \le p < \infty)$.

Lemma 2.2. *Let $1 \le p < \infty$. Then the following assertions hold.*

(i) If $1 \le p_1 < p_2 < \infty$, then $A_{p_1}^\rho \subset A_{p_2}^\rho$.

(ii) $\omega \in A_p^\rho$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}^\rho$, where $1/p + 1/p' = 1$.

In [1], Bongioanni et al. have introduced a new space $BMO(\rho)$ defined by

$$\|f\|_{BMO(\rho)} = \sup_{B\subset\mathbb{R}^n} \frac{1}{\Psi(B)|B|}\int_B |f(x)-f_B|dx < \infty,$$

where $f_B = \frac{1}{|B|}\int_B f(y)dy$, $\Psi(B) = (1+r/\rho(x_0))^\theta$, $B = B(x_0,r)$, and $\theta > 0$.

In particular, in [1] it was proved the following result for the space $BMO(\rho)$.

Lemma 2.3. *Let $\theta > 0$ and $1 \le s < \infty$. If $b \in BMO(\rho)$, then*

$$\left(\frac{1}{|B|}\int_B |b-b_B|^s\right)^{1/s} \le C_{\theta,s}\|b\|_{BMO(\rho)}\left(1+\frac{r}{\rho(x)}\right)^{\theta'},$$

for all $B = B(x,r)$ with $x\in\mathbb{R}^n$ and $r>0$, where $\theta' = (l_0+1)\theta$.

Obviously, the classical BMO is properly embedded into $BMO(\rho)$. More examples can be found in [1].

Applying Lemma 2.3, one of the the authors of this paper proved the following John-Nirenberg type inequality for space $BMO(\rho)$ (see [13]).

Proposition 2.1. *Let $f \in BMO(\rho)$. There exist positive constants γ and C such that*

$$\sup_B \frac{1}{|B|}\int_B \exp\left\{\frac{\gamma}{\|f\|_{BMO(\rho)}\Psi_{\theta'}(B)}|f(x)-f_B|\right\}dx \le C,$$

where $f_B = \frac{1}{|B|}\int_B f(y)dy$, $\Psi_{\theta'}(B) = (1+r/\rho(x_0))^{\theta'}$, $B = B(x_0,r)$, and $\theta' = (l_0+1)\theta$.

We remark that in the above definitions of A_p^ρ, $BMO(\rho)$ and M_V, the balls can be replaced by cubes.

We also will need the dyadic maximal operator $M^{\triangle}_{V,\eta}f(x)$ and the dyadic sharp maximal operator $M^{\sharp}_{V,\eta}f(x)$, which for $0<\eta<\infty$ are defined by the following formulas:

$$M^{\triangle}_{V,\eta}f(x) := \sup_{x\in Q(dyadic\ cube)} \frac{1}{\Psi(Q)^\eta|Q|}\int_Q |f(x)|\,dx$$

and

$$\begin{aligned} M^{\sharp}_{V,\eta}f(x) &= \sup_{x\in Q, r<\rho(x_0)} \frac{1}{|Q|}\int_{Q(x_0,r)} |f(y)-f_Q|\,dy + \sup_{x\in Q, r\ge\rho(x_0)} \frac{1}{\Psi(Q)^\eta|Q|}\int_{Q(x_0,r)} |f|\,dy \\ &\simeq \sup_{x\in Q, r<\rho(x_0)} \inf_C \frac{1}{|Q|}\int_{Q(x_0,r)} |f(y)-C|\,dy + \sup_{x\in Q, r\ge\rho(x_0)} \frac{1}{\Psi(Q)^\eta|Q|}\int_{Q(x_0,r)} |f|\,dx, \end{aligned}$$

where $Q'_{x_0}s$ denotes the dyadic cube $Q(x_0,r)$ and $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$.

The following versions of dyadic maximal and dyadic sharp maximal operators:

$$M^{\triangle}_{\delta,\eta}f(x) = M^{\triangle}_{V,\eta}(|f|^\delta)^{1/\delta}(x)$$

and

$$M^{\sharp}_{\delta,\eta}f(x) = M^{\sharp}_{V,\eta}(|f|^{\delta})^{1/\delta}(x)$$

will be the main tools in our scheme.

In [13], one of the the authors of this paper proved the following results.

Theorem 2.1. *Let $0 < p,\ \eta,\ \delta < \infty$ and $\omega \in A_\infty$. There exists a positive constant C such that*

$$\int_{\mathbb{R}^n} M^{\triangle}_{\delta,\eta}f(x)^p\omega(x)dx \le C\int_{\mathbb{R}^n} M^{\sharp}_{\delta,\eta}f(x)^p\omega(x)dx.$$

Further, let $\varphi : (0,\infty) \to (0,\infty)$ be a doubling function, then there exists a positive constant C such that

$$\sup_{\lambda>0}\varphi(\lambda)\omega(\{x\in\mathbb{R}^n:\ M^{\triangle}_{\delta,\eta}f(x) > \lambda\}) \le C\sup_{\lambda>0}\varphi(\lambda)\omega(\{x\in\mathbb{R}^n:\ M^{\sharp}_{\delta,\eta}f(x) > \lambda\})$$

for any smooth function f for which the left hand-hand side is finite.

Proposition 2.2. *Let $1 < p < \infty$ and $\omega \in A^{\rho}_p$. If $p < p_1 < \infty$, then*

$$\int_{\mathbb{R}^n} |M_V f(x)|^{p_1}\omega(x)dx \le C_p\int_{\mathbb{R}^n} |f(x)|^{p_1}\omega(x)dx.$$

Further, let $1 \le p < \infty$, then $\omega \in A^{\rho}_p$ if and only if

$$\omega(\{x\in\mathbb{R}^n:\ M_V f(x) > \lambda\}) \le \frac{C_p}{\lambda^p}\int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx.$$

From Proposition 2.2 it follows that M_V may be unbounded on $L^p(\omega)$ for all $\omega \in A^{\rho}_p$ and $1 < p < \infty$. We will need a variant of maximal operator $M_{V,\eta}$ $(0 < \eta < \infty)$ defined as follows:

$$M_{V,\eta}f(x) = \sup_{x\in B}\frac{1}{(\Psi(B))^{\eta}|B|}\int_B |f(y)|\,dy.$$

Theorem 2.2. *Let $1 < p < \infty$ and $p' = p/(p-1)$, and let $\omega \in A^{\rho}_p$. Then there exists a constant $C > 0$ such that*

$$\|M_{V,p'}f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Finally, we recall some basic definitions and facts about Orlicz spaces, referring to [11] for a complete account.

A function $B(t) : [0,\infty) \to [0,\infty)$ is called a Young's function if it is continuous, convex, increasing and satisfies $\Phi(0) = 0$ and $B \to \infty$ as $t \to \infty$. For a Young's function B, we define the B-average of a function f over a cube Q by means of the following Luxemberg norm:

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0:\ \frac{1}{|Q|}\int_Q B\left(\frac{|f(y)|}{\lambda}\right)dy \le 1\right\}.$$

If A, B and C are Young's functions such that

$$A^{-1}(t)B^{-1}(t) \le C^{-1}(t),$$

where A^{-1} is the Young's complementary function associated with A, then we have

$$\|fg\|_{C,R} \le 2\|f\|_{A,R}\|g\|_{B,R}.$$

The examples to be considered in our study will be $A^{-1}(t) = \log(1+t)$, $B^{-1}(t) = t/\log(e+t)$ and $C^{-1}(t) = t$. Then $A(t) \sim e^t$ and $B(t) \sim t\log(e+t)$, which give the generalized Hölder's inequality:

$$\frac{1}{|Q|}\int_Q |fg|\,dy \le \|f\|_{A,Q}\|g\|_{B,Q}.$$

For these examples, if $b \in BMO(\rho)$ and b_Q denotes its B-average over the cube Q, in view of Proposition 2.1, we get

$$\|(b - b_Q)/\Psi_{\theta'}(Q)\|_{expL,Q} \le C\|b\|_{BMO(\rho)},$$

where $\theta' = (1 + l_0)\theta$.

Also, we define the corresponding maximal functions:

$$M_B f(x) = \sup_{Q:x\in Q} \|f\|_{B,Q}$$

and

$$M_{V,B} f(x) = \sup_{Q:x\in Q} \Psi(Q)^{-1}\|f\|_{B,Q}.$$

3. Some lemmas

In this section, we establish some estimates, which will play a crucial role in the proofs of the main results of this paper. We first introduce some notation and definitions. We define the space $\mathrm{B} = L^2(\mathbb{R}^{n+1}_+, dydt/t^n)$ to be the set of measurable functions $a : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ endowed the norm $|a|_{\mathrm{B}} = (\int_{\mathbb{R}^{n+1}_+} |a(y,t)|^2\,dydt/t^n)^{1/2} < \infty$. By $\mathcal{M}(\mathbb{R}^n)$ we denote the set of measurable functions $a : \mathbb{R}^n \to \mathbb{C}$, and by $\mathcal{M}(\mathbb{R}^n, \mathrm{B})$ we denote the set of Bochner-measurable functions $h : \mathbb{R}^n \to \mathrm{B}$. The space $L^p(\mathbb{R}^n, \mathrm{B})$ is defined to be the set of functions $h \in \mathcal{M}(\mathbb{R}^n, \mathrm{B})$ endowed the finite norm:

$$\|h\|_{L^p(\mathbb{R}^n,\mathrm{B})} = \left(\int_{\mathbb{R}^n} |h(x)|^p_{\mathrm{B}}\,dx\right)^{1/p}.$$

We define $s_Q(f)(x) = \left(\int_0^\infty |Q_t f(x)|^2 \frac{dt}{t}\right)^{1/2}$. It is known that $\|s_Q(f)\|_2 = \frac{1}{\sqrt{8}}\|f\|_2$ (see Lemma 4.1 of [6]).

Let $\varphi \le 1$ be a nonnegative infinitely differentiable function on $\mathbb{R}_+$ such that $\varphi(s) = 1$ for $0 < s < 1$ and $\varphi(s) = 0$ for $s \ge 2$. Then the function $\varphi_t(x, y) := \frac{1}{t}\varphi(\frac{|x-y|}{t})$ satisfies

$$|\varphi_t(x,y) - \varphi_t(x',y)| \le C\frac{|x-x'|}{t^2}\chi_{[0,2]}\left(\frac{\min\{|x-y|, |x'-y|\}}{t}\right), \tag{3.1}$$

for $|x - y| > 2|x - x'|$.

Now we consider an operator $\tilde{S} : \mathcal{M}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{R}^n, \mathbb{B})$ defined as follows:

$$\tilde{S}f(x) = \left\{\tilde{S}_{(y,t)}f(x) := t^{1/2}\varphi_t(x,y)Q_tf(y)\right\}_{(y,t)\in\mathbb{R}^{n+1}_+}, \tag{3.2}$$

which has an associated kernel given by

$$\tilde{K}(x,z) = \left\{t^{1/2}\varphi_t(x,y)Q_t(y,z)\right\}_{(y,t)\in\mathbb{R}^{n+1}_+}. \tag{3.3}$$

We first recall some properties of the function Q_t.

Lemma 3.1. *(see [6]) There exist positive constants c and $\delta_0 \le 1$ such that for every $l \ge 0$ there is a constant C_l so that the following inequalities hold:*

(a) $|Q_t(x,y)| \le C_l t^{-n}\left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-l}\exp\left(-\frac{c|x-y|^2}{t^2}\right)$,

(b) $|Q_t(x+h,y) - Q_t(x,y)| \le C_l\left(\frac{|h|}{t}\right)^{\delta_0} t^{-n}\left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-l}\exp\left(-\frac{c|x-y|^2}{t^2}\right)$
for all $|h| \le t$;

Lemma 3.2. *Let $\tilde{K}(x,z)$ and δ_0 be as above, then for any $l \ge 0$ we have*

$$|\tilde{K}(x,z)|_{\mathbb{B}} \le \frac{C_l}{(1 + |x-z|(\rho(x)^{-1} + \rho(z)^{-1}))^l}\frac{1}{|x-y|^n}, \tag{3.4}$$

(3.5)
$$|\tilde{K}(x,z) - \tilde{K}(x',z)|_{\mathbb{B}} \le \frac{C_l}{(1 + |x-z|(\rho(x)^{-1} + \rho(z)^{-1}))^l}\frac{|x-x'|^{\delta_0}}{|x-z|^{n+\delta_0}}, \text{ if } |x-z| > 2|x-x'|,$$

(3.6)
$$|\tilde{K}(z,x) - \tilde{K}(z,x')|_{\mathbb{B}} \le \frac{C_l}{(1 + |x-z|(\rho(x)^{-1} + \rho(z)^{-1}))^l}\frac{|x-x'|^{\delta_0}}{|x-z|^{n+\delta_0}}, \text{ if } |x-z| > 2|x-x'|.$$

Proof. We adapt the arguments applied in the proof of Theorem 4.1 of [8]. Without loss of generality, we can assume that $\rho(z) < |x - z|$. We first prove the inequality

(3.4). By Lemma 3.1(a), for any $N, l > 0$ we can write

$$\begin{aligned}|\tilde{K}(x,z)|_{\mathbb{B}}^2 &= \int_0^\infty\int_{\mathbb{R}^n} \frac{|\varphi_t(x,y)|^2|Q_t(y,z)|^2}{t^{n-1}}\,dydt\\ &\le C\int_{\mathbb{R}^n}\int_{|x-y|/2}^\infty \frac{1}{t^{n+1}}\left(\frac{(1+|y-z|/t)^{-N}}{t^n}\right)^2\left(\frac{t}{\rho(z)}\right)^{-2l} dtdy\\ &\le C\int_{A_1\cup A_2\cup A_3}\int_{|x-y|/2}^\infty \frac{1}{t^{3d+1}}(1+|y-z|/t)^{-2N}\left(\frac{t}{\rho(z)}\right)^{-2l} dtdy\\ &:= J_1+J_2+J_3,\end{aligned}$$

where the sets A_1, A_2 and A_3, constituting a partition of $\mathbb{R}^n$, are given by

$A_1 = \{y\in\mathbb{R}^n : |y-z| > 2|x-z|\}$,
$A_2 = \{y\in\mathbb{R}^n : \frac{1}{2}|x-z| < |y-z| \le 2|x-z|\}$,
$A_3 = \{y\in\mathbb{R}^n : |y-z| \le \frac{1}{2}|x-z|\}$.

For $y\in A_1$, we have $|x-z|\le\frac{1}{2}|y-z|\le|x-y|\le 2|y-z|$, and hence

$$\begin{aligned}J_1 &\le C\int_{|x-y|\ge|x-z|}\int_{|x-y|/2}^\infty \frac{1}{t^{3n+1}}\left(\frac{t}{\rho(z)}\right)^{-2l} dtdy\\ &\le C\rho(z)^{2l}\int_{|x-y|\ge|x-z|}\frac{1}{|x-y|^{3n+2l}}\,dy\\ &\le \frac{C}{|x-z|^{2n}}\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}.\end{aligned}$$

For $y\in A_2$, we have $|x-y|\le 3|x-z|$ and $|x-z|\sim|y-z|$, and hence

$$\begin{aligned}J_2 &\le C\int_{A_2}\left(\int_{|x-y|/2}^{3|x-z|}\frac{1}{t^{3n+1}}\frac{t^{2N}}{|y-z|^{2N}}\left(\frac{t}{\rho(z)}\right)^{-2l}dt+\int_{3|x-z|}^\infty\frac{1}{t^{3n+1}}\left(\frac{t}{\rho(z)}\right)^{-2l}dt\right)dy\\ &:= J_{2a}+J_{2b}.\end{aligned}$$

For J_{2a}, we have

$$\begin{aligned}J_{2a} &\le \frac{C\rho(z)^{2l}}{|x-z|^{2N}}\int_{|x-y|<3|x-z|}\int_{|x-y|/2}^\infty\frac{1}{t^{3n-2N+1+2l}}\,dtdy\\ &\le \frac{C}{|x-z|^{2N}}\int_{|x-y|<3|x-z|}\frac{1}{|x-y|^{3n-2N+2l}}\,dy\\ &\le \frac{C}{|x-z|^{2n}}\left(\frac{|x-z|}{\rho(z)}\right)^{-2l},\end{aligned}$$

in which we take $n < N-l < \frac{3}{2}n$, and for J_{2b}, we get

$$J_{2b}\le\frac{C\rho(z)^{2l}}{|x-z|^{3n+2l}}\int_{|x-y|<3|x-z|}dy\le\frac{C}{|x-z|^{2n}}\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}.$$

Thus, from the above inequalities it follows that

$$J_2\le\frac{C}{|x-z|^{2n}}\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}.$$

Finally, for $y \in A_3$, we have $|y-z| < \frac{1}{2}|x-z|$ and $|x-z| \sim |x-y|$, and hence

$$\begin{aligned} J_3 &\leq C \int_{|y-z|<\frac{1}{2}|x-z|} \int_{|x-y|/2}^{\infty} \frac{\rho(z)^{2l} dt}{t^{3n+1+2l}} dy \\ &\leq \frac{C\rho(z)^{2l}}{|x-z|^{3n+2l}} \int_{|x-y|<3|x-z|} dy \\ &\leq \frac{C}{|x-z|^{2n}} \left(\frac{|x-z|}{\rho(z)} \right)^{-2l}. \end{aligned}$$

From the above inequalities and Lemma 2.3, we obtain the inequality (3.4).

To prove the inequality (3.5), let us consider $|x-z| > 2|x-x'|$, denote $a = \min\{|x-y|, |x'-y|\}$, and define $B = \{y : |x-y| > 2|x-x'|\}$. Then, by Lemma 3.1(b) and the inequality (3.1), we can write

$$\begin{aligned} &|\bar{K}(x,z) - \bar{K}(x',z)|_{\mathbb{B}}^2 = \int_{\mathbb{R}^n} \int_0^{\infty} |\varphi_t(x,y) - \varphi_t(x',y)|^2 |Q_t(y,z)|^2 \frac{dtdy}{t^{n-1}} \\ &\leq C \int_B \int_{a/2}^{\infty} \frac{|x-x'|^2}{t^{3n+1+2}} \frac{t^{2N}}{(t+|y-z|)^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy \\ &+ C \int_{B^c} \int_{a/2}^{\infty} \frac{1}{t^{3n+1}} \frac{t^{2N}}{(t+|y-z|)^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy := I + II. \end{aligned}$$

For $y \in B$, we have $|x'-y| > |x-y|/2$, and hence $a > |x-y|/2$. Denoting $B_1 = B \cap \{y : |x-y| \geq |x-z|/2\}$, we get

$$\begin{aligned} I &\leq C|x-x'|^2 \int_{B_1} \int_{|x-y|/4}^{\infty} \frac{1}{t^{3n+1+2}} \frac{t^{2N}}{(t+|y-z|)^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy \\ &+ C|x-x'|^2 \int_{B \setminus B_1} \int_{|x-y|/4}^{\infty} \frac{1}{t^{3n+1+2}} \frac{t^{2N}}{(t+|y-z|)^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy \\ &:= I_1 + I_2. \end{aligned}$$

For $y \in B_1$, we have

$$\begin{aligned} I_1 &\leq C|x-x'|^2 \int_{|x-y|\geq|x-z|/2} \int_{|x-y|/4}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2l+2}} dtdy \\ &\leq C|x-x'|^2 \int_{|x-y|\geq|x-z|/2} \frac{\rho(z)^{2l}}{|x-y|^{3n+2l+2}} dy \\ &\leq C \left(\frac{|x-z|}{\rho(z)} \right)^{-2l} \frac{|x-x'|^2}{|x-z|^{2n+2}}. \end{aligned}$$

For $y \in B \setminus B_1$, we have $|y-z| \sim |x-z|$, and hence

$$\begin{aligned} I_2 &\leq C|x-x'|^2 \int_{B \setminus B_1} \int_{|x-y|/4}^{|x-z|} \frac{1}{t^{3n+1+2}} \frac{t^{2N}}{|y-z|^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy \\ &+ C|x-x'|^2 \int_{B \setminus B_1} \int_{|x-z|}^{\infty} \frac{1}{t^{3n+1+2}} \frac{t^{2N}}{|y-z|^{2N}} \left(\frac{t}{\rho(z)} \right)^{-2l} dtdy := I_{2a} + I_{2b}. \end{aligned}$$

Then

$$\begin{aligned} I_{2a} &\leq C\frac{|x-x'|^2}{|x-z|^{2N}}\int_{B\setminus B_1}\left(\int_{|x-y|/4}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2-2N+2l}}\,dt\right)dy \\ &\leq C\frac{|x-x'|^2}{|x-z|^{2N}}\int_{|x-y|\leq|x-z|/2}\frac{\rho(z)^{2l}}{|x-y|^{3n+2-2N+2l}}\,dy \\ &\leq C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^2}{|x-z|^{2n+2}}, \end{aligned}$$

where $n+1<N-l<\frac{3n+2}{2}$, and

$$\begin{aligned} I_{2b} &\leq C|x-x'|^2\int_{B\setminus B_1}\int_{|x-z|}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2+2l}}\,dtdy \\ &\leq C\frac{\rho(z)^{2l}|x-x'|^2}{|x-z|^{3n+2+2l}}\int_{|x-y|\leq|x-z|/2}dy \\ &\leq C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^2}{|x-z|^{2n+2}}. \end{aligned}$$

In this way, we get

$$I\leq C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^2}{|x-z|^{2n+2}}.$$

To estimate II, we notice that if $y\in B^c$, then $|x'-y|\leq 3|x-x'|$, and hence

$$II\leq\left(\int_{B_c}\int_{|x-y|/2}^{\infty}+\int_{|x'-y|\leq 3|x-x'|}\int_{|x'-y|/2}^{\infty}\right)\frac{\rho(z)^{2l}}{t^{3n+2l+1}}\frac{t^{2N}}{(t+|y-z|)^{2N}}\,dtdy:=II_1+II_2.$$

Since the above two integrals are similar, we estimate only II_1, the estimate for II_2 can be obtained similarly.

We consider the set $(B^c)_1=B^c\cap\{y:|x-y|\geq|x-z|/2\}$, and notice that for $y\in(B^c)_1$, we have $|x-y|\sim|x-z|$ and $|y-z|\leq 2|x-z|$, and for $y\in B^c\setminus(B^c)_1$, we have $|x-z|\sim|y-z|$ and $|x-y|<|x-z|/2$. Thus, we can write

$$\begin{aligned} II_1 &\leq C\int_{(B^c)_1}\int_{|x-y|/2}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2l}}\,dtdyC\int_{B^c\setminus(B^c)_1}\int_{|x-y|/2}^{|x-z|}\frac{\rho(z)^{2l}}{t^{3n+1+2l}}\frac{t^{2N}}{|y-z|^{2N}}\,dtdy \\ &\quad+\int_{B^c\setminus(B^c)_1}\int_{|x-z|}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2l}}\,dtdy:=II_{1a}+II_{1b}+II_{1c}. \end{aligned}$$

Then, for II_{1a} we have

$$\begin{aligned} II_{1a} &\leq C\int_{(B^c)_1}\frac{\rho(z)^{2l}}{|x-y|^{3n+2l}}dy \\ &\leq\frac{C\rho(z)^{2l}}{|x-z|^{2n+2+2l}}\int_{|x-y|\leq 2|x-x'|}\frac{dy}{|x-y|^{n-2}} \\ &\leq C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^2}{|x-z|^{2n+2}}. \end{aligned}$$

For II_{1b}, we take $N = n+1$, to obtain

$$\begin{aligned} II_{1b} &\le C \int_{B^c \setminus (B^c)_1} \int_{|x-y|/2}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2l}} \frac{t^{2n+2}}{|y-z|^{2n+2}}\, dt dy \\ &\le \frac{C\rho(z)^{2l}}{|x-z|^{2n+2+2l}} \int_{|x-y|\le 2|x-x'|} \frac{1}{|x-y|^{n-2}}\, dy \\ &\le C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l} \frac{|x-x'|^2}{|x-z|^{2n+2}}. \end{aligned}$$

For II_{1c}, since $|x-z| > 2|x-y|$, we have

$$\begin{aligned} II_{1c} &\le C \int_{B^c \setminus (B^c)_1} \frac{\rho(z)^{2l}}{|x-z|^{3n+2l}}\, dy \\ &\le \frac{C\rho(z)^{2l}}{|x-z|^{2n+2+2l}} \int_{|x-y|\le 2|x-x'|} \frac{dy}{|x-y|^{n-2}} \\ &\le C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l} \frac{|x-x'|^2}{|x-z|^{2n+2}}. \end{aligned}$$

Combining the above inequalities and using Lemma 2.1, we get the inequality (3.5).

Now we proceed to prove the inequality (3.6). To this end, we consider $|x-z| > 2|x-x'|$ and define $E = \{y : |y-z| \ge |x-z|/2\}$. Note that $Q_t(x,y) = Q_t(y,x)$, and hence we can apply Lemma 3.1(b) to obtain

$$\begin{aligned} |\tilde{K}(z,x) - \tilde{K}(z,x')|_{\mathbb{B}}^2 &= \int_{\mathbb{R}^n} \int_0^{\infty} |\varphi_t(z,y)|^2 |Q_t(y,x) - Q_t(y,x')|^2 \frac{dt}{t^{n-1}} dy \\ &\le \int_{\mathbb{R}^n} \int_{|y-z|/2}^{\infty} \frac{1}{t^{d+1}} |Q_t(x,y) - Q_t(x',y)|^2\, dt dy \\ &\le C|x-x'|^{2\delta_0} \int_{\mathbb{R}^n} \int_{|y-z|/2}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}}\, dt dy \\ &\le C|x-x'|^{2\delta_0} \int_E \int_{|y-z|/2}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}}\, dt dy \\ &\quad + C|x-x'|^{2\delta_0} \int_{E^c} \int_{|y-z|/2}^{\infty} \frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}}\, dt dy \\ &:= III_1 + III_2. \end{aligned}$$

For III_1, we then have

$$III_1 \le C|x-x'|^{2\delta_0} \int_E \frac{\rho(z)^{2l}}{|y-z|^{3n+2\delta_0+2l}}\, dy \le C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l} \frac{|x-x'|^{2\delta_0}}{|x-z|^{2n+2\delta_0}}.$$

If $y \in E^c$, then $|y-z| < |x-z|/2 < |x-y| < 2|x-z|$, and hence

$$I_2 \le C|x-x'|^{2\delta_0} \int_{E^c} \left(\int_{|y-z|/2}^{|x-z|} + \int_{|x-z|}^{\infty} \right) \frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}} \frac{t^{2N}}{(t+|x-y|)^{2N}}\, dt dy := III_{2a} + III_{2b}.$$

For III_{2a} and III_{2b}, we have the following estimates:

$$\begin{aligned} III_{2a} &\leq C\frac{|x-x'|^{2\delta_0}}{|x-z|^{2N}}\int_{E^c}\int_{|y-z|/2}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0-2N+2l}}\,dtdy \\ &\leq C\frac{|x-x'|^{2\delta_0}}{|x-z|^{2N}}\int_{|y-z|\leq|x-z|/2}\frac{\rho(z)^{2l}}{|y-z|^{3n+2\delta_0-2N+2l}}\,dy \\ &\leq C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^{2\delta_0}}{|x-z|^{2n+2\delta_0}}, \end{aligned}$$

(we take $d+\delta_0 < N-l < (3n+2\delta_0)/2$), and

$$\begin{aligned} III_{2b} &\leq C|x-x'|^{2\delta_0}\int_{E^c}\int_{|x-z|}^{\infty}\frac{\rho(z)^{2l}}{t^{3n+1+2\delta_0+2l}}\,dtdy \\ &\leq C\frac{\rho(z)^{2l}|x-x'|^{2\delta_0}}{|x-z|^{3n+2\delta_0+2l}}\int_{|y-z|\leq|x-z|/2}dy \leq C\left(\frac{|x-z|}{\rho(z)}\right)^{-2l}\frac{|x-x'|^{2\delta_0}}{|x-z|^{2n+2\delta_0}}. \end{aligned}$$

Combining the above inequalities, we get the inequality (3.6). □

Lemma 3.3. *Let $0 < p, \eta < \infty$ and let $\omega \in A^{\rho}_{\infty}$, then the following inequalities hold:*

$$\int_{\mathbf{R}^n}|\widetilde{S}(f)(x)|^p_{\mathbf{B}}\omega(x)dx \leq C\int_{\mathbf{R}^n}|M_{V,\eta}f(x)|^p\omega(x)dx$$

$$\sup_{\lambda>0}\lambda^p\omega(\{x\in\mathbf{R}^n:\ |\widetilde{S}(f)(x)|_{\mathbf{B}}>\lambda\}) \leq C\sup_{\lambda>0}\lambda^p\omega(\{x\in\mathbf{R}^n:\ M_{V,\eta}f(x)>\lambda\}),$$

Proof. By Fubini's theorem and the property of $s_Q f$, we have

$$\||\widetilde{S}(f)|_{\mathbf{B}}\|_2 \leq C\|s_Q(f)\|_2 \leq C\|f\|_2.$$

By Lemma 3.2 and the theory of vector valued singular integrals the result will be proved by showing that the kernel $\widetilde{K}$ of $\widetilde{S}$ is a standard vector valued Calderón-Zygmund kernel, and so $|\widetilde{S}|_{\mathbf{B}}$ is bounded on $L^p(\mathbf{R}^n)(1<p<\infty)$ and of weak type $(1,1)$. In view of the inequality $|f(x)| \leq M^{\triangle}_{\delta,\eta}f(x)$ a.e $x\in\mathbf{R}^n$, and Theorem 2.1, to prove the lemma, we need only to show that for any $0<\eta<\infty$ and $0<\delta<\eta/(\eta+1)$ the following inequality holds:

$$M^{\sharp}_{\delta,\eta}(|\widetilde{S}(f)|_{\mathbf{B}})(x) \leq CM_{V,\eta}(f)(x),\ \text{a.e}\ x\in\mathbf{R}^n, \tag{3.7}$$

We fix $x\in\mathbf{R}^n$ and assume that $x\in Q=Q(x_0,r)$ (dyadic cube). Decompose $f=f_1+f_2$, where $f_1=f\chi_{\bar{Q}}$ with $\bar{Q}=Q(x,8\sqrt{n}r)$. To prove the inequality (3.7), we consider the following two possible cases: $r<\rho(x_0)$ and $r\geq\rho(x_0)$.

Case 1: $r<\rho(x_0)$. Let $C_Q=|\widetilde{S}(f)(x_0)|_{\mathbf{B}}$. Since $0<\delta<1$, we can write

$$\begin{aligned} \left(\frac{1}{|Q|}\int_Q\left||\widetilde{S}(f)(y)|^{\delta}_{\mathbf{B}}-C^{\delta}_Q\right|dy\right)^{1/\delta} &\leq \left(\frac{1}{|Q|}\int_Q\left||\widetilde{S}(f)(y)|_{\mathbf{B}}-|\widetilde{S}(f_2)(x_0)|_{\mathbf{B}}\right|^{\delta}dy\right)^{1/\delta} \\ \leq C\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_1)(y)|^{\delta}_{\mathbf{B}}\,dy\right)^{1/\delta} &+C\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_2)(y)-\widetilde{S}(f_2)(x_0)|^{\delta}_{\mathbf{B}}\,dy\right)^{1/\delta} := I+II. \end{aligned}$$

To estimate the term I, we recall that $|\widetilde{S}(f)|_{\mathbf{B}}$ is of weak type $(1,1)$, and note that $\rho(x) \sim \rho(x_0)$ for any $x \in \bar{Q}$ and $\Psi(\bar{Q}) \sim 1$. Hence we can apply Kolmogorov's inequality (see [10]), to obtain

$$I \le \frac{C}{|Q|}\||\widetilde{S}(f)|_{\mathbf{B}}\|_{L^{1,\infty}} \le \frac{C}{|\bar{Q}|}\int_{\bar{Q}} |f(y)|\,dy \le CM_{V,\eta}f(x). \tag{3.8}$$

To estimate the term II, we let $Q_k = Q(x_0, 2^{k+1}r)$ and $\alpha = \eta+1$. Then, taking $l \ge \theta\alpha$ and using (3.5), we obtain

$$\begin{aligned} II &\le \frac{C}{|Q|}\int_Q |\widetilde{S}(f_2)(y) - \widetilde{S}(f_2)(x_0)|_{\mathbf{B}}\,dy \\ &\le \frac{C}{|Q|}\int_Q\int_{\mathbf{R}^n\setminus\bar{Q}} |\widetilde{K}(y,\omega) - \widetilde{K}(x_0,\omega)|_{\mathbf{B}}|f(\omega)|d\omega dy \\ &\le \frac{C}{|Q|}\int_Q\int_{|x_0-\omega|>2r} |\widetilde{K}(y,\omega) - \widetilde{K}(x_0,\omega)|_{\mathbf{B}}|f(\omega)|d\omega dy \\ &\le \frac{C}{|Q|}\int_Q\sum_{k=4}^{\infty}\int_{2^k r\le|x_0-\omega|<2^{k+1}r} |\widetilde{K}(y,\omega) - \widetilde{K}(x_0,\omega)|_{\mathbf{B}}|f(\omega)|d\omega dz dy \end{aligned}$$

$$\begin{aligned} &\le C_l\sum_{k=2}^{\infty}\frac{2^{-k\delta_0}}{(1+2^k r m_V(x_0))^l(2^{k+1}r)^n}\int_{Q_k}|f(\omega)|d\omega \\ &\le C_l\sum_{k=1}^{\infty}\frac{2^{-k\delta_0}}{(1+2^k r/\rho(x_0))^{l-\alpha\theta}}\frac{1}{(1+2^k r/\rho(x_0))^{\alpha\theta}|Q_k|}\int_{Q_k}|f(\omega)|d\omega \\ &\le C_l\sum_{k=1}^{\infty}2^{-k\delta_0}M_{V,\eta}(f)(x) \le C_l M_{V,\eta}(f)(x). \end{aligned} \tag{3.9}$$

Case 2: $r \ge \rho(x_0)$. In this case, noting that $\alpha_1 := \eta/\delta \ge \eta+1$, we get

$$\begin{aligned} \frac{C}{\Psi(Q)^{\alpha_1}}\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_1)(y)|_{\mathbf{B}}^{\delta}\,dy\right)^{1/\delta} &\le \frac{C}{\Psi(Q)^{\alpha_1}}\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_1)(y)|_{\mathbf{B}}^{\delta}\,dy\right)^{1/\delta} \\ &\quad+\frac{C}{\Psi(Q)^{\alpha_1}}\left(\frac{1}{|Q|}\int_Q|\widetilde{S}(f_2)(y)|_{\mathbf{B}}^{\delta}\,dy\right)^{1/\delta} \\ &=: I_1 + II_1. \end{aligned}$$

For I_1, similar to I, we have the following estimate

$$\begin{aligned} I_1 &\le \frac{C}{\Psi(Q)^{\alpha_1}}\frac{1}{|Q|}\||\widetilde{S}(f_1)|_{\mathbf{B}}\|_{L^{1,\infty}} \\ &\le \frac{C}{\Psi(Q)^{\eta}(\Psi(Q)|\bar{Q}|)}\int_{\bar{Q}}|f(y)|\,dy \le CM_{V,\eta}f(x). \end{aligned} \tag{3.10}$$

As for II_1, taking $l = \alpha_1\theta + 1$, and using (3.5) and Lemma 2.1, we get

$$\begin{aligned}
II_1 &\leq \frac{C}{|Q|}\int_Q |\widetilde{S}(f_2)(y)|_{\mathbb{B}}\, dy \leq \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus\bar{Q}} |\widetilde{K}(y,\omega)|_{\mathbb{B}}|f(\omega)|d\omega dy \\
&\leq \frac{C}{|Q|}\int_Q\sum_{k=2}^{\infty}\int_{2^k r\leq|x_0-\omega|<2^{k+1}r} |\widetilde{K}(y,\omega)|_{\mathbb{B}}|f(\omega)|d\omega dy \\
&\leq C_l\sum_{k=2}^{\infty}\frac{1}{(1+2^k r/\rho(x_0))^l|Q_k|}\int_{Q_k}|f(\omega)|d\omega \qquad (3.11)\\
&\leq C_l\sum_{k=1}^{\infty}\frac{1}{(1+2^k r/\rho(x_0))^{l-\theta\alpha_1}}\frac{1}{(1+2^k r/\rho(x_0))^{\theta\alpha_1}|Q_k|}\int_{Q_k}|f(\omega)|d\omega \\
&\leq C_l\sum_{k=1}^{\infty}2^{-k}M_{V,\eta}(f)(x) \leq C_l M_{V,\eta}(f)(x).
\end{aligned}$$

From (3.8)–(3.11), we get (3.7). Lemma 3.3 is proved. □

Lemma 3.4. *Let $b \in BMO(\rho)$ and $(l_0+1) \leq \eta < \infty$, and let $0 < 2\delta < \epsilon < 1$. Then for any $f \in C_0^\infty(\mathbb{R}^n)$ and the following inequality holds:*

(3.12)
$$M^{\sharp}_{\delta,\eta}(|[b,\widetilde{S}]f|_{\mathbb{B}})(x) \leq C\|b\|_{BMO(\rho)}(M^{\triangle}_{\epsilon,\eta}(|\widetilde{S}(f)|_{\mathbb{B}})(x) + M_{L\log L,V,\eta}(f)(x)), \text{ a.e. } x \in \mathbb{R}^n,$$

Proof. Observe first that for any constant λ we have

$$[b,\widetilde{S}]f(x) = (b(x)-\lambda)\widetilde{S}(f)(x) - \widetilde{S}((b-\lambda)f)(x).$$

As above, we fix $x \in \mathbb{R}^n$ and assume that $x \in Q = Q(x_0, r)$ (dyadic cube). Decompose $f = f_1 + f_2$, where $f_1 = f\chi_{\bar{Q}}$ with $\bar{Q} = Q(x, 8\sqrt{n}r)$.

To prove the inequality (3.12), again we consider the following two possible cases: $r < \rho(x_0)$ and $r \geq \rho(x_0)$.

Case 1: $r < \rho(x_0)$. We first fix $\lambda = b_{\bar{Q}}$, the average of b on $\bar{Q}$. Since $0 < \delta < 1$, we then can write

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \big|\, |[b,\widetilde{S}]f(y)|^{\delta}_{\mathbb{B}} - |\widetilde{S}((b-b_{\bar{Q}})f)(x_0)|^{\delta}_{\mathbb{B}}\big|\, dy\right)^{1/\delta} \\
&\leq \left(\frac{1}{|Q|}\int_Q \big||[b,\widetilde{S}]f(y)|_{\mathbb{B}} - \widetilde{S}((b-b_{\bar{Q}})f_2)(x_0)\big|^{\delta}_{\mathbb{B}}\, dy\right)^{1/\delta} \\
&\leq C\left(\frac{1}{|Q|}\int_Q |(b(y)-b_{\bar{Q}})|\widetilde{S}f(y)|_{\mathbb{B}}|^{\delta}\, dy\right)^{1/\delta} + C\left(\frac{1}{|Q|}\int_Q |\widetilde{S}((b-b_{\bar{Q}})f_1)(y)|^{\delta}_{\mathbb{B}}\, dy\right)^{1/\delta} \\
&\quad + C\left(\frac{1}{|Q|}\int_Q |\widetilde{S}((b-b_{\bar{Q}})f_2)(y) - \widetilde{S}((b-b_{\bar{Q}})f_2)(x_0)|^{\delta}_{\mathbb{B}}\, dy\right)^{1/\delta} =: I + II + III.
\end{aligned}$$

Then for any $1 < \gamma < \epsilon/\delta$, note that $\rho(x) \sim \rho(x_0)$ for any $x \in \bar{Q}$ and $\Psi(\bar{Q}) \sim 1$. Hence, by Lemma 2.3, for the term I we obtain the estimate

$$\tag{3.13} \begin{aligned} I &\leq C\left(\frac{1}{|\bar{Q}|}\int_{\bar{Q}}|b(y)-b_{\bar{Q}}|^{\delta\gamma'}\,dy\right)^{\gamma'/\delta}\left(\frac{1}{|\bar{Q}|}\int_{Q}|\widetilde{S}f(y)|_{\mathrm{B}}^{\delta\gamma}\,dy\right)^{\delta\gamma} \\ &\leq C\|b\|_{BMO(\rho)}M^{\Delta}_{\epsilon,\eta}(|\widetilde{S}f(y)|_{\mathrm{B}})(x), \end{aligned}$$

where $1/\gamma' + 1/\gamma = 1$.

To estimate the term II, we recall that $|\widetilde{S}|_B$ is of weak type $(1,1)$, and note that $\rho(x) \sim \rho(x_0)$ for any $x \in \bar{Q}$ and $\Psi(\bar{Q}) \sim 1$. Hence, by Kolmogorov's inequality and Proposition 2.1, we get

$$\tag{3.14} \begin{aligned} II &\leq \frac{C}{|Q|}\||\widetilde{S}((b-b_{\bar{Q}})f_1)|_{\mathrm{B}}\|_{L^{1,\infty}} \\ &\leq \frac{C}{|\bar{Q}|}\int_{\bar{Q}}|(b-b_{\bar{Q}})f(y)|\,dy \leq CM_{L\log L,V,\eta}f(x). \end{aligned}$$

For the term III, we let $b_{Q_k} = b_{Q(x_0,2^{k+1}r)}$ and $\theta' = (l_0+1)\theta$. Then, in view of Lemmas 2.1 and 3.2, we can write

(3.15)

$$\begin{aligned} III &\leq \frac{C}{|Q|}\int_{Q}|\widetilde{S}((b-b_{\bar{Q}})f_2)(y)-\widetilde{S}((b-b_{\bar{Q}})f_2)(x_0)|_{\mathrm{B}}\,dy \\ &\leq \frac{C}{|Q|}\int_{Q}\int_{|x_0-\omega|>2r}|\widetilde{K}(y,\omega)-\widetilde{K}(z,\omega)|_{\mathrm{B}}|(b(\omega)-b_{\bar{Q}})f(\omega)|d\omega dy \\ &\leq \frac{C}{|Q|}\int_{Q}\sum_{k=2}^{\infty}\int_{2^k r\leq|x_0-\omega|<2^{k+1}r}|\widetilde{K}(y,\omega)-\widetilde{K}(z,\omega)|_{\mathrm{B}}|(b(\omega)-b_{\bar{Q}})f(\omega)|d\omega dy \\ &\leq C_l\sum_{k=1}^{\infty}\frac{2^{-k\delta_0}}{(1+2^k r/\rho(x_0))^{l}|Q_k|}\int_{Q_k}|b(\omega)-b_{\bar{Q}}|f(\omega)|d\omega \\ &\leq C_l\sum_{k=1}^{\infty}\frac{2^{-k\delta_0}}{(1+2^k r/\rho(x_0))^{l-(\eta+1)\theta'}} \\ &\qquad\times\frac{1}{(1+2^k r/\rho(x_0))^{(\eta+1)\theta'}|Q_k|}\int_{Q_k}|b(\omega)-b_{Q_k}||f(\omega)|d\omega \\ &\quad+C_l\sum_{k=1}^{\infty}\frac{2^{-k\delta_0}}{(1+2^k r/\rho(x_0))^{l-\theta'(\eta+2)}} \\ &\qquad\times\frac{1}{(1+2^k r/\rho(x_0))^{\theta'(\eta+2)}|Q_k|}|b_{\bar{Q}}-b_{Q_k}|\int_{Q_k}|f(\omega)|d\omega \\ &\leq C_l\sum_{k=1}^{\infty}2^{-k\delta_0}\|b\|_{BMO(\rho)}M_{L\log L,V,\eta}(f)(x)+C_l\|b\|_{BMO(\rho)}M_{V,\eta}(f)(x)\sum_{k=1}^{\infty}k2^{-k\delta_0} \\ &\leq C_l\|b\|_{BMO(\rho)}M_{L\log L,V,\eta}(f)(x), \end{aligned}$$

where $l = (\eta+2)\theta'$, and in the last inequality we have used the following inequalities:

$$M_{V,\eta}(f)(x) \leq M_{L\log L,V,\eta}(f)(x) \quad\text{and}\quad |b_{\bar{Q}}-b_{Q_k}| \leq C(1+2^k r/\rho(x_0))^{\theta}\|b\|_{BMO(\rho)}.$$

Case 2: $r \geq \rho(x_0)$. Since $0 < 2\delta < \epsilon < 1$, we have $a = \eta/\delta$ and $\epsilon/\delta > 2$. Hence, we can write

$$\begin{aligned}
&\frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q |[b,\widetilde{S}]f(y)|_{\mathbf{B}}^{\delta}\,dy\right)^{1/\delta}\\
&= \frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q |(b(y)-b_{\tilde{Q}})\widetilde{S}(f)(y)-\widetilde{S}((b-b_{\tilde{Q}})f)(y)|_{\mathbf{B}}^{\delta}\,dy\right)^{1/\delta}\\
&\leq C\frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q |(b(y)-b_{\tilde{Q}})\widetilde{S}(f)(y)|_{\mathbf{B}}^{\delta}\,dy\right)^{1/\delta}\\
&\quad + C\frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q |\widetilde{S}((b-b_{\tilde{Q}})f_1)(y)|_{\mathbf{B}}^{\delta}\,dy\right)^{1/\delta}\\
&\quad + C\frac{1}{\Psi(Q)^a}\left(\frac{1}{|Q|}\int_Q |\widetilde{S}((b-b_{\tilde{Q}})f_2)(y)|_{\mathbf{B}}^{\delta}\,dy\right)^{1/\delta} := I_1 + II_1 + III_1.
\end{aligned}$$

Then noting that $l_0 + 1 \leq \eta$ for any $2 \leq \gamma < \epsilon/\delta$, by Lemma 2.3 we obtain the following estimate for I_1:

$$\begin{aligned}
I_1 &\leq C\frac{1}{\Psi_{\theta'}(Q)}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |b(y)-b_{\tilde{Q}}|^{\delta\gamma'}\,dy\right)^{1/(\gamma'\delta)}\\
&\quad\times\frac{\Psi_{\theta'}(Q)}{\Psi(Q)^{a-\eta/(2\delta)}}\left(\frac{1}{\Psi(Q)^{\eta}|Q|}\int_Q |\widetilde{S}(f)(y)|_{\mathbf{B}}^{\delta\gamma}\,dy\right)^{1/(\delta\gamma)}\\
&\leq C\|b\|_{BMO(\rho)}M^{\Delta}_{\epsilon,\eta}(|\widetilde{S}(f)|_{\mathbf{B}})(x),
\end{aligned}\tag{3.16}$$

where $1/\gamma' + 1/\gamma = 1$.

To estimate II_1, we recall that $|\widetilde{S}|_{\mathbf{B}}$ is of weak type $(1,1)$, and use Kolmogorov's inequality and Proposition 2.1, to obtain

$$\begin{aligned}
II_1 &\leq \frac{C}{\Psi(Q)^a}\frac{1}{|Q|}\||\widetilde{S}((b-b_{\tilde{Q}})f_1)|_{\mathbf{B}}\|_{L^{1,\infty}}\\
&\leq \frac{C}{\Psi(Q)^a}\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |(b-b_{\tilde{Q}})f(y)|\,dy\\
&\leq CM_{L\log L,V,\eta}f(x).
\end{aligned}\tag{3.17}$$

Finally, to estimate III_1, we let $b_{Q_k} = b_{Q(x_0, 2^{k+i}r)}$ and $\theta' = (l_0+1)\theta$. Then, we use Lemmas 2.1 and 3.2 with $l = (\eta+2)\theta' + 1$, to obtain

(3.18)

$$
\begin{aligned}
III &\le \frac{C}{|Q|}\int_Q |\widetilde{S}((b-b_{\tilde{Q}})f_2)(y)|_{\mathbf{B}}\,dy \\
&\le \frac{C}{|Q|}\int_Q\int_{\mathbf{R}^n\setminus\tilde{Q}} |\widetilde{K}(y,\omega)|_{\mathbf{B}}|(b(\omega)-b_{\tilde{Q}})f(\omega)|d\omega dy \\
&\le \frac{C}{|Q|}\int_Q\sum_{k=2}^{\infty}\int_{2^k r\le|x_0-\omega|<2^{k+1}r} |\widetilde{K}(y,\omega)||(b(\omega)-b_{\tilde{Q}})f(\omega)|d\omega dy \\
&\le C_l\sum_{k=1}^{\infty}\frac{1}{(1+2^k r/\rho(x_0))^l|Q_k|}\int_{Q_k}|b(\omega)-b_{\tilde{Q}}|f(\omega)|d\omega \\
&\le C_l\sum_{k=1}^{\infty}\frac{1}{(1+2^k/\rho(x_0))^{l-(\eta+2)\theta'}} \\
&\qquad\times\frac{1}{(1+2^k r/\rho(x_0))^{(\eta+2)\theta'}|Q_k|}\int_{Q_k}|b(\omega)-b_{Q_k}||f(\omega)|d\omega \\
&\quad + C_l\sum_{k=1}^{\infty}\frac{1}{(1+2^k r/\rho(x_0))^{l-(\eta+2)\theta'}} \\
&\qquad\times\frac{1}{(1+2^k r/\rho(x_0))^{(\eta+2)\theta'}|Q_k|}|b_{\tilde{Q}}-b_{Q_k}|\int_{Q_k}|f(\omega)|d\omega \\
&\le C_l\sum_{k=1}^{\infty}2^{-k}\|b\|_{BMO(\rho)}M_{L\log L,V,\eta}(f)(x) + C_l\|b\|_{BMO(\rho)}M_{V,\eta}(f)(x)\sum_{k=1}^{\infty}k2^{-k} \\
&\le C_l\|b\|_{BMO(\rho)}M_{L\log L,V,\eta}(f)(x).
\end{aligned}
$$

From (3.13)–(3.18), we get (3.12). Lemma 3.4 is proved. □

Finally, we recall the following results proved in [13, 14].

Lemma 3.5. *Let $0 < \eta < \infty$ and $M_{V,\eta/2}f$ be locally integrable. Then there exist positive constants C_1 and C_2 independent of f and x such that*

$$C_1 M_{V,\eta}M_{V,\eta+1}f(x) \le M_{L\log L,V,\eta+1}f(x) \le C_2 M_{V,\eta/2}M_{V,\eta/2}f(x).$$

Lemma 3.6. *Let $2 \le \eta < \infty$, $\omega \in A_1^{\rho}$ and $B(t) = t\log(e+t)$. Then there exists a constant $C > 0$ such that for all $t > 0$*

$$\omega(\{x\in\mathbf{R}^n:\ M_{B,V,\eta}f(x) > t\}) \le C\int_{\mathbf{R}^n} B\left(\frac{|f(x)|}{t}\right)\omega(x)dx. \tag{3.19}$$

Proof. Let K be any compact subset in $\{x\in\mathbf{R}^n : M_{L\log L,\varphi,\eta}(f)(x) > \lambda\}$). For any $x \in K$, by a standard covering lemma, it is possible to choose cubes $Q_1,\cdots,Q_m$ with pairwise disjoint interiors such that $K \subset \bigcup_{j=1}^m 3Q_j$ and $\|f\|_{L\log L,\varphi,Q_j} > \lambda$, $j = 1,\cdots,m$. This implies

$$\Psi(Q_j)^2|Q_j| \le \int_{Q_j}\frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right)dy.$$

From the last inequality we obtain

$$\begin{aligned}\omega(3Q_j) &\le C\Psi(Q_j)\omega(Q_j) = C\Psi(Q_j)^2|Q_j|\frac{\omega(Q_j)}{\Psi(Q_j))|Q_j|}\\ &\le C\frac{\omega(Q_j)}{\Psi(Q_j))|Q_j|}\int_{Q_j}\frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right)dy\\ &\le C\inf_{Q_j}\omega(x)\int_{Q_j}\frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right)dy\\ &\le C\int_{Q_j}\frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right)\omega(y)\,dy,\end{aligned}$$

implying (3.19). □

4. Proof of main results

Proof of Theorem 1.1. We first notice that

$$S_Q(f)(x) \le C|\widetilde{S}(f)(x)|_{\mathbf{B}} \text{ for every } x \in \mathbf{R}^n. \tag{4.1}$$

Thus, the desired results follow from (4.1), Lemmas 3.2, Theorem 2.2 and Proposition 2.2. □

Proof of Theorem 1.2. We first notice that

$$S_{Q,b}(f)(x) \le C|\widetilde{[b,\widetilde{S}]}f(x)|_{\mathbf{B}} \text{ for every } x \in \mathbf{R}^n. \tag{4.2}$$

Using arguments similar to those applied in [10], the inequality (4.2), Lemmas 3.3-3.6, Proposition 2.1, and Theorems 1.1, 2.1 and 2.2, we can obtain the desired results. □

Remark. It can be shown that the analogs of Theorems 1.1 and 1.2 hold for spaces $BMO_{\theta_1}(\rho)$ and A_p^{ρ,θ_2} if $\theta_1 \ne \theta_2$.

Список литературы

[1] B. Bongioanni, E. Harboure and O. Salinas, "Commutators of Riesz transforms related to Schrödinger operators", J. Fourier Ana Appl. **17**, 115 – 134 (2011).

[2] B. Bongioanni, E. Harboure and O. Salinas, "Class of weights related to Schrödinger operators", J. Math. Anal. Appl. **373**, 563 – 579 (2011).

[3] B. Bongioanni, E. Harboure and O. Salinas, "Weighted inequalities for commutators of Schrödinger Riesz transforms", J. Math. Anal. Appl. **392**, 6 – 22 (2012).

[4] B. Bongioanni, A. Cabral and E. Harboure and O. Salinas, "Lerner's inequality associated to a critical radius function and applications", J. Math. Anal. Appl. **407**, 35 – 55 (2013).

[5] J. Dziubański and J. Zienkiewicz, "Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality", Rev. Math. Iber. **15**, 279 – 296 (1999).

[6] J. Dziubański, G. Garrigós, J. Torrea and J. Zienkiewicz, "*BMO* spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality", Math. Z. **249**, 249 – 356 (2005).

[7] J. García-Cuerva and J. Rubio de Francia, Weighted Norm inequalities and Related Topics, Amsterdam- New York, North-Holland (1985).

[8] S. Hartzstein, O. Salinas, "Weighted *BMO* and Carleson measures on spaces of homogeneous", J. Math. Anal. Appl. **342**, 950 – 969 (2008).

[9] B. Muckenhoupt, "Weighted norm inequalities for the Hardy maximal functions", Trans. Amer. Math. Soc. **165**, 207 – 226 (1972).

[10] C. Pérez, "Endpoint estimates for commutators of singular integral operators", J. Funct. Anal. **128**, 163 – 185 (1995).

[11] M. M. Rao and Z. D. Ren, Theory of Orlicz Spaces, Monogr., Textbooks Pure Appl. Math. **146**, Marcel Dekker, Inc., New York (1991).

[12] Z. Shen, "L^p estimates for Schrödinger operators with certain potentials", Ann. Inst. Fourier. Grenoble, **45**, 513 – 546 (1995).

[13] L. Tang, "Weighted norm inequalities for Schrödinger type operators", Forum Math. **27**, 2491 – 2532 (2015). .

[14] L. Tang, "Weighted norm inequalities for commutators of Littlewood-Paley functions related to Schrödinger operators", arXiv:1109.0100.

[15] J. Zhong, Harmonic Analysis for Some Schrödinger Type Operators, Ph. D. Thesis, Princeton University (1993).

Поступила 21 июля 2016

После доработки 21 июля 2016

Принята к публикации 24 января 2017