A MOMENT CONDITION AND NON-SYNTHETIC DIAGONALIZABLE OPERATORS ON THE SPACE OF FUNCTIONS ANALYTIC ON THE UNIT DISK

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Abstract. Examples are given of (continuous, linear) operators on the space of functions analytic on the open unit disk in the complex plane having the monomials as eigenvectors, but which fail spectral synthesis (that is, which have closed invariant subspaces which are not the closed linear span of any collection of eigenvectors).

1. INTRODUCTION

The main topic of this paper concerns invariant subspaces of a particular class of complete operators $T : \mathfrak{X} \to \mathfrak{X}$ acting on a complete metrizable vector space $\mathfrak{X}$ (recall that a subspace $\mathcal{M}$ of a complete metrizable vector space $\mathfrak{X}$ is *invariant* for an operator $T : \mathfrak{X} \to \mathfrak{X}$ if $Tx \in \mathcal{M}$ whenever $x \in \mathcal{M}$). Any complete operator has an abundance of invariant subspaces, namely the closed linear span of arbitrary collections of its eigenvectors. In fact, it may be tempting to believe that these are all of the invariant subspaces of a complete operator. However, this is not always the case, even when $\mathcal{H}$ is a Hilbert space having an orthonormal basis of eigenvectors for the operator (see Wolff's Example below). Any complete operator, all of whose invariant subspaces are the closed linear span of some collection of its eigenvectors, is said to **admit spectral synthesis**. The operators of study in this paper are the so-called *diagonal operators*, which by definition act on the space $\mathcal{H}(\mathbf{D})$ of functions analytic on the open unit disk in the complex plane and have as eigenvectors the monomials. The purpose of this paper is to produce a rich class of examples of diagonal operators on $\mathcal{H}(\mathbf{D})$ which fail spectral synthesis.

28

The problem of determining which complete operators admit spectral synthesis remains open, even when $\mathfrak{X}$ is a Hilbert space having an orthonormal basis of eigenvectors for T. In fact, it wasn't until 1921, with the advent of an example due to Wolff, that it was known that there existed examples of non-synthetic operators of this type. In particular, if $T : \mathcal{H} \rightarrow \mathcal{H}$ is an operator acting on a Hilbert space $\mathcal{H}$ having an orthonormal basis of eigenvectors for T with associated eigenvalues $\{\lambda_n\}$, then

$$(1.1) \qquad T\left(\sum_{n=0}^{\infty} a_n e_n\right) \equiv \sum_{n=0}^{\infty} a_n \lambda_n e_n$$

for all $\sum_{n=0}^{\infty} a_n e_n$ in $\mathcal{H}$. Moreover, it is not difficult to see (p. 270 of [1]) that T fails spectral synthesis if and only if there exists a non-trivial sequence $\{w_n\} \in \ell^1$ for which the Moment Condition

$$0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$$

holds for all $k \geq 0$.

Wolff's elegant construction [2] of such an example (which uses only Laurent series) may also be found in [3].

There are numerous conditions known to be equivalent to the Moment Condition (1.1) holding for all $k \geq 0$ whenever $\{\lambda_n\}$ is a bounded sequence of distinct complex numbers. For instance, it follows from the Fubini-Tonelli Theorem that

$$0 \equiv \sum_{n=0}^{\infty} \frac{w_n}{z - \lambda_n} = \sum_{k=0}^{\infty} \left[\frac{1}{z^{k+1}} \sum_{n=0}^{\infty} w_n \lambda_n^k\right]$$

whenever $|z| > \sup |\lambda_n|$. Moreover, condition (1.1) holds for all $k \geq 0$ if and only if the Dirichlet series $g(z) \equiv \sum_{n=0}^{\infty} w_n e^{\lambda_n z}$ vanishes identically on the complex plane since $g \equiv 0$ if and only if

$$g^{(k)}(0) = \sum_{n=0}^{\infty} w_n \lambda_n^k$$

for all $k \geq 0$. This, in turn, is equivalent to the measure $\mu \equiv \sum_{n=0}^{\infty} w_n \delta_{\{\lambda_n\}}$ (the sum of weighted point masses) annihilating the monomials since

$$\int z^k d\mu = \sum_{n=0}^{\infty} w_n \lambda_n^k.$$

If the points $\{\lambda_n\}$ lie in a Jordan region Ω and accumulate only on its boundary, then $\sum_{n=0}^{\infty} w_n/(z - \lambda_n) \equiv 0$ whenever $|z| > \sup |\lambda_n|$ where $\{w_n\}$ is a non-trivial sequence in ℓ^1 if and only if $\{\lambda_n\}$ is a dominating sequence for Ω; that is, if and only if

$$\sup \{|f(z)| : z \in \Omega\} = \sup \{|f(\lambda_n)| : n \geq 0\}$$

for all functions f bounded and analytic on Ω (see Theorem 3 on p. 167 of Brown, Shields, and Zeller [4]). If Ω is the open unit disk, then this condition is equivalent to almost every point of the unit circle (with respect to Lebesgue arc length measure) being the non-tangential limit point of $\{\lambda_n\}$. Deep connections to operator theory are provided by work of Sarason [5] and [6] who shows that that the Borel series $\sum_{n=0}^{\infty} w_n/(z - \lambda_n) \equiv 0$ whenever $|z| > \sup |\lambda_n|$ for some non-trivial sequence $\{w_n\}$ in ℓ^1 if and only if there exists a closed invariant subspace for the diagonal operator D having eigenvalues $\{\lambda_n\}$ which is not invariant for the adjoint D^* of D. This condition, in turn, is equivalent to the weakly closed algebra generated by D and the identity operator not containing D^*. For more on the connections between Borel series and complete normal operators, please see Wermer [1], Scroggs [7] and Nikolskii [3], [8].

The study of Borel series has a rich and fabled history. Of particular interest has been conditions for a function analytic on a region to be representable as a Borel series, and conditions for such a representation, if one exists, to be unique. In particular, the seminal work of Leontev [9], Korobeinik [10], Leont' eva [11], and Brown, Shields, and Zeller [4], amongst others, has examined the extent to which the existence of non-trivial expansions of zero by Dirichlet series $\sum_{n=0}^{\infty} w_n e^{\lambda_n z} \equiv 0$ on regions Ω in the complex plane imply (and, under additional conditions, is equivalent to) the ability to represent an arbitary function $f(z)$ analytic on Ω as a Dirichlet series $f(z) = \sum_{n=0}^{\infty} a_n e^{\lambda_n z}$ on Ω. It follows from the preceding comments that the non-uniqueness of any such representation is equivalent to the existence of Borel series which vanish identically on Ω. In 1959, Makarov [12] showed that for every sequence of complex numbers $\{\lambda_n\}$ for which $|\lambda_n| \to \infty$, there exists a sequence of complex numbers $\{w_n\}$ for which the moment condition (1.1) holds for all $k \geq 0$ where the coefficients $\{w_n\}$ satisfy the decay rate $0 < \sum_{n=0}^{\infty} |w_n| \cdot |\lambda_n^k| < \infty$. In addition to Wolff's example [2], in which the coefficients $\{w_n\}$ are in ℓ^1, Denjoy [13] in 1924 and Leont' eva [11] in the late 1960's gave examples of Borel series which vanish identically where the coefficients satisfy various decay rates just shy of exponential decay (see p. 26 of [14]).

There has also been particular interest regarding the converse, namely the so-called unicity problem, which is to determine the rate at which $\{|w_n|\}$ must decrease so that $\sum_{n=0}^{\infty} w_n/(z - \lambda_n)$ does not extend analytically to a region containing $\{\lambda_n\}$. Borel [15], Carleman [16], Gonchar [17], and Poincare all determined decay rates in

the unicity problem in their investigations on Borel series, which were focused mainly on issues regarding quasianalyticity and analytic continuation. In 1968 Makarov gave such a decay rate depending on a given arbitrary sequence $\{\lambda_n\}$ (see 5.7.8(c)(vii) on p. 128 of [3]). A rather definitive result was obtained by Sibilev in 1995 when the eigenvalues $\{\lambda_n\}$ are bounded (see the theorem on p. 146 of [18]). For more on the history of Borel series and a discussion of generalized analytic continuation, please see the recent monograph of Ross and Shapiro [14].

The purpose of this paper is to provide a rich class of examples of diagonal operators acting on $\mathcal{H}(D)$ which fail spectral synthesis. The main result of this paper, Theorem 1, appears in Section 2 below and improves upon previous results in the literature. When endowed with the topology of uniform convergence on compacta, $\mathcal{H}(D)$ is an example of a complete locally convex topological vector space. Using the Radius of Convergence Formula, it follows that a function $\sum_{n=0}^{\infty} a_n z^n$ is in $\mathcal{H}(D)$ if and only if $\limsup |a_n|^{1/n} < 1$. Moreover, if $\{\lambda_n\}$ is any sequence of distinct complex numbers, then the map for which $D(z^n) \equiv \lambda_n z^n$ extends by linearity to an operator on all of $\mathcal{H}(D)$ if and only if $\limsup |\lambda_n|^{1/n} \leq 1$ (see [19]). In particular, the set of eigenvalues of a diagonal operator on $\mathcal{H}(D)$ need not be bounded. It's known that the diagonal operator $D(\sum_{n=0}^{\infty} a_n z^n) \equiv \sum_{n=0}^{\infty} a_n \lambda_n z^n$ fails spectral synthesis if and only if the moment condition (1.1) holds for all $k \geq 0$ for some non-trivial sequence $\{w_n\}$ of complex numbers for which $\limsup |w_n|^{1/n} < 1$ (please see Theorem 3 on p. 1214 of [19] for this and other conditions equivalent to non-synthesis).

In [20], Anderson, Khavinson, and Shapiro, give a detailed analysis of the moment condition (1.1) for all $k \geq 0$ where the eigenvalues $\lambda_n \equiv n^p$ are powers of n with $p > 0$. Their study focuses on questions concerning the analytic continuation of Dirichlet series and Fredholm's method for examining gap series and its connections to partial differential equations. They show, amongst other results, that the moment condition $0 \equiv \sum_{n=0}^{\infty} w_n (n^p)^k$ holds for all $k \geq 0$ where $0 < \limsup |w_n|^{1/n} < 1$ if and only if $p > 2$, and moreover, that no solution exists for integral $p > 2$ for which

$$0 < \limsup |w_n|^{1/n} < e^{-\pi \tan^{-1}(\pi/p)}$$

(see Theorem 3.1 on p. 464 of [20]). In view of which, the moment condition holding and hence a diagonal operator admitting spectral synthesis is intimately related to the growth rate of the eigenvalues $\{\lambda_n\}$ of the diagonal operator. In some cases, the growth rate of the eigenvalues alone determines the spectral synthesis; for instance,

Leontev [9] has shown that the moment condition (1.1) does not hold for all $k \geq 0$ whenever $\limsup |w_n|^{1/n} < 1$ if $\{\lambda_n\}|$ exhibits linear growth (that is, whenever $0 < \liminf |\lambda_n|/n \leq \limsup |\lambda_n|/n < \infty$) whether or not the λ_n are positive). However, it is known that the distribution of the points λ_n throughout the complex plane, as well as their growth, typically plays a role in determining spectral synthesis. For example, the diagonal operator on $\mathcal{H}(\mathbf{D})$ having eigenvalues $\{\sqrt{n}\}$ admits spectral synthesis by Theorem 3.1 of [20], while the diagonal operator on $\mathcal{H}(\mathbf{D})$ with eigenvalues $\{\lambda_n\}$ comprising the integer lattice $\mathbf{Z} \times i\mathbf{Z} \equiv \{m + in : m, n \in \mathbf{Z}\}$ fails spectral synthesis (see [21]), although $|\lambda_n| \approx n^{1/2}$.

The result of Anderson, Khavinson, and Shapiro mentioned above suggests that the slower the growth rate of $\{|\lambda_n|\}$, the harder it is for the moment condition to hold, and hence for the associated diagonal operator on $\mathcal{H}(\mathbf{D})$ with eigenvalues $\{\lambda_n\}$ to fail spectral synthesis. Nonetheless, in this paper, we demonstrate that there exist diagonal operators acting on $\mathcal{H}(\mathbf{D})$ whose eigenvalues have growth rate $|\lambda_n| \approx n^\beta$ for any $\beta < 1$ which fail spectral synthesis. The examples produced do not require that the eigenvalues $\{\lambda_n\}$ assume any particular form, only that they satisfy a particular growth rate and are regularly distributed (in a sense made precise in the next section).

2. EXAMPLES OF NON-SYNTHETIC DIAGONAL OPERATORS ON $\mathcal{H}(\mathbf{D})$

In this section, we show that a diagonal operator on $\mathcal{H}(\mathbf{D})$ fails spectral synthesis whenever its eigenvalues have order of growth less than one, are regularly distributed with respect to a proximate order $\rho(r)$, and satisfy a separation criterion, definitions of which we now provide for the convenience of the reader.

The relationship between the growth of an entire function and the distribution of its zeros is well-known. It is often convenient to measure the growth of an entire function using a so-called *proximate order*, or function $\rho(r)$ for which $\lim_{r \to \infty} \rho(r) \equiv \rho \geq 0$ and $\lim_{r \to \infty} r\rho'(r) \ln r = 0$ (see p. 32 of [22]). A set of points in the complex plane is said to have an *angular density* $\Delta(\psi)$ *of index* $\rho(r)$ if for all but a countable set of values η and θ for which $0 < \eta < \theta \leq 2\pi$, the limit

$$\Delta(r, \theta) \equiv \lim_{r \to \infty} \frac{n(r, \eta, \theta)}{r^{\rho(r)}}$$

exists where here $n(r, \eta, \theta)$ denotes the number of points of the set lying within the sector $\{z : |z| \leq r; \eta < \arg z < \theta\}$ (see p. 89 of [22]).

A sequence $\{a_n\}$ of distinct complex numbers satisfies **Condition (C)** if there exists a positive number $d > 0$ such that the set of closed balls

$$\overline{\{B(a_n; d|a_n|^{1-\rho(|a_n|)/2})\}}$$

are pairwise disjoint, while the sequence satisfies **Condition (C')** if the points all lie inside sectors with a common vertex at the origin but with no other points in common, and which are such that if one arranges the points of the set $\{a_n\}$ within any one of these sectors in order of increasing moduli, then for all points which lie inside the same sector it is true that $|a_{k+1}| - |a_k| > d|a_k|^{1-\rho(|a_k|)}$ (see p. 95 of [22]). In the following theorem, the conditions that the points are regularly distributed with respect to $\rho(r)$ and $\rho < 1/2$ ensure that there exist coefficients $\{w_n\}$ for which $0 = \sum_{n=0}^{\infty} w_n e^{\lambda_n x}$ while the separation condition ensures that $\limsup |w_n|^{1/n} < 1$.

Theorem 2.1. *Let $\rho(r)$ be any proximate order for which $\rho \equiv \lim_{r\to\infty} \rho(r) \in (0,1/2)$ and let $\{a_n\}$ be any sequence of distinct complex numbers whose angular density $\Delta(\psi)$ has index $\rho(r)$, satisfies either Condition (C) or Condition (C'), and is such that $\liminf |a_n|^{\rho(|a_n|)}/n > 0$. Then the diagonal operator having eigenvalues $\{|a_n|^{1/q} e^{i(\arg a_n + 2\pi j)/q} : 0 \leq j < q; 0 \leq n\}$ fails to admit spectral synthesis on $\mathcal{H}(D)$ whenever q is any integer for which $q > 1/\rho$.*

An outline of the proof is as follows: Let $\{a_n\}$ be any sequence of complex numbers satisfying the hypotheses of Theorem 2.1. Then $S(z) \equiv f(z^q)$ is an entire function having only simple zeros at the points

$$\lambda_n = |a_n|^{1/q} e^{i(\arg a_n + 2\pi j)/q}$$

for $0 \leq j < q$ and $0 \leq n$ and $f(z) = \Pi_{n=0}^{\infty}(1 - z/\lambda_n)$ is a canonical product having only simple zeros at the points λ_n. Since the points $\{a_n\}$ are separated, it follows that $|S(\lambda)| \geq e^{c|\lambda|^\beta}$ for all λ on a sequence of circles C_r whose radii increase to infinity, where here $\beta > 1$. Using this estimate and the Residue Theorem, we see that

$$0 \leftarrow \int_{C_r} \frac{e^{\lambda z}}{S(\lambda)} d\lambda \leftarrow \sum_{n=0}^{\infty} \frac{e^{\lambda_n}}{S'(\lambda_n)}.$$

It follows from estimates for S near the points λ_n obtained using the Inverse Function Theorem and Schwarz's Lemma, that $\limsup (1/|S'(\lambda_n)|^{1/n}) < 1$. Hence, the moment condition holds and the result follows (see Theorem 3 on p. 1214 of [19]).

Proof. Let $\{a_n\}$ be any enumeration of the set of points

$$\{|a_n|^{1/q} e^{i(\arg a_n + 2\pi j)/q} : 0 \leq j < q; 0 \leq n\}$$

where here $n(r)$ denotes the number of points a_n for which $|a_n| \leq r$. Hence $n(r) \geq .5\Delta r^{\rho(r)}$ for all r sufficiently large. Since $\{|a_t|\}$ is increasing, it follows that $|a_t| \leq r$ where $t \equiv .5\Delta r^{\rho(r)}$ for all r sufficiently large. Since $\rho(r) \to \rho$, we have that $t = .5\Delta r^{\rho(r)} \geq .5\Delta r^{\rho/2}$ for all r sufficiently large. Thus $|a_{.5\Delta r^{\rho/2}}| \leq |a_t| \leq r$ or $|a_n| \leq r = (2n/\Delta)^{2/\rho}$ for all n sufficiently large. Hence

$$\hat{r} \leq \{(2+d)|a_{n+1}|\}^{1/q} \leq (2+d)^{1/q}(2/\Delta)^{2/(qp)}(n+1)^{2/(qp)}$$

for all n sufficiently large. Moreover,

$$|S(\hat{r}_n e^{i\theta})| \geq e^{(\epsilon/2)}\hat{r}^{qp(\hat{r}_n^q)}$$

for all n sufficiently large, and so it follows that

$$\sum_{k=0}^{\infty} \frac{e^{\lambda_k z}}{S'(\lambda_k)} = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{C_n} \frac{e^{\lambda z}}{S(\lambda)} d\lambda = 0$$

for all $z \in \mathbb{C}$ since $qp(\hat{r}_n^q) \to qp > 1$. In order to deduce that the diagonal operator having eigenvalues $\{\lambda_k\}$ fails to admit spectral synthesis, by Theorem 3 on p. 1214 of [19], it suffices to show that $\limsup (1/|S'(\lambda_k)|)^{1/k} < 1$. To this end, let k be any positive integer. Then

$$\lambda_n = |a_n|^{1/q} e^{i(\arg a_n + 2\pi j)/q}$$

for some integer n and $j \in \{0, 1, ..., q-1\}$. Since $S(z) = f(z^q)$, we have that $S'(z) = qz^{q-1}f'(z^q)$, and so

$$|S'(\lambda_k)| = |q\lambda_k^{q-1}f'(a_n)| = q|a_n|^{q-1}|f'(a_n)|.$$

We now estimate $|f'(a_n)|$ using the Inverse Function Theorem and Schwarz's Lemma. Since the closed balls $\overline{\{B(a_n, r_n)\}}$ are pairwise disjoint, there exist radii $\tilde{r}_n \in (r_n, 1+r_n)$ for which the open balls $\{B(a_n, \tilde{r}_n)\}$ are pairwise disjoint and stay inside E^c. Hence

$$|f(re^{i\theta})| \geq e^{(\epsilon/2)r^{\rho(r)}} \geq e^{(\epsilon/2)\{|a_n|-\tilde{r}_n\}\rho(\{|a_n|-\tilde{r}_n\})} \equiv \alpha_n$$

whenever $re^{i\theta} \in \partial B(a_n, \tilde{r}_n)$ with r sufficiently large. It follows from the Inverse Function Theorem (p. 234 of [23]) that the restriction

$$f|_{f^{-1}(B(0, \alpha_n))} : f^{-1}(B(0, \alpha_n)) \to B(0, \alpha_n)$$

of f to $f^{-1}(B(0, \alpha_n))$ has analytic inverse f^{-1}. Hence

$$g(z) \equiv (1/\tilde{r}_n)(f^{-1}(\alpha_n z) - a_n) : B(0, 1) \to B(0, 1)$$

35

is analytic with $g(0) = 0$. By Schwarz's Lemma, we have that

$$1 \geq |g'(0)| = |\frac{\alpha_n}{\tilde{r}_n f'(f^{-1}(0))}| = \frac{\alpha_n}{\tilde{r}_n |f'(a_n)|}$$

whence

$$\frac{1}{|f'(a_n)|} \leq \frac{\tilde{r}_n}{e^{(\epsilon/2)\{|a_n| - \tilde{r}_n\}\rho(\{|a_n| - \tilde{r}_n\})}}$$

for all r sufficiently large. Since $(n-1)q \leq k < nq$, we have that

$$\limsup (1/|S'(\lambda_k)|)^{1/k} = \limsup \frac{1}{\{q|a_n|^{(q-1)/q} \cdot |f'(a_n)|\}^{1/k}}$$

$$\leq \limsup \frac{1}{\{q|a_n|^{(q-1)/q} \cdot |f'(a_n)|\}^{1/(nq)}}.$$

Since $\Delta \equiv \lim_{r \to \infty} n(r)/r^{\rho(r)}$, we have that $n(r) \leq 1.5\Delta r^{\rho(r)}$ for all r sufficiently large. Since $\{|a_n|\}$ is increasing, it follows that $|a_n| \geq r$ where $t \equiv 1.5\Delta r^{\rho(r)}$ for all r sufficiently large. Since $\rho(r) \to \rho$, we have that $t = 1.5\Delta r^{\rho(r)} \leq 1.5\Delta r^{2\rho}$ for all r sufficiently large, and so $|a_n| = |a_{1.5\Delta r^{2\rho}}| \geq r$ where $s \equiv 1.5\Delta r^{2\rho}$. Hence $|a_n| \geq (2n/(3\Delta))^{1/(2\rho)}$ for all n sufficiently large. Hence

$$\limsup (1/|S'(\lambda_k)|) \leq \limsup \frac{1}{|f'(a_n)|^{1/(nq)}}$$

$$\leq (\limsup \frac{\tilde{r}_n^{1/n}}{e^{(1/n)(\epsilon/2)\{|a_n| - \tilde{r}_n\}\rho(\{|a_n| - \tilde{r}_n\})}})^{1/q}.$$

However, for all n sufficiently large,

$$\tilde{r}_n \leq 1 + r_n \leq 1 + d|a_n|^{1-\rho} \leq 2d|a_n| \leq 2d(\Delta n/2)^{2/\rho}$$

and so

$$\limsup (1/|S'(\lambda_k)|)^{1/k} \leq (\limsup \frac{1}{e^{(1/n)(\epsilon/2)\{|a_n| - \tilde{r}_n\}\rho(\{|a_n| - \tilde{r}_n\})}})^{1/q}.$$

Since $\rho(r) \to \rho$, we have that $\tilde{r}_n \leq d|a_n|^{1-\rho/4}$ for all n sufficiently large and so $|a_n| - \tilde{r}_n \geq .5|a_n|$ for all n sufficiently large. Since $r^{\rho(r)}$ is increasing for all r sufficiently large (see p. 33 of [22]) and $L(r) \equiv r^{\rho(r) - \rho}$ is slowly increasing (see p. 33 of [22], it follows that

$$\{|a_n| - \tilde{r}_n\}^{\rho(|a_n| - \tilde{r}_n)} \geq (.5|a_n|)^{\rho(.5|a_n|)} \geq (.5|a_n|)^{\rho(.5|a_n| - \rho)}\{.5|a_n|\}^{\rho}$$

$$= L(.5|a_n|)\{.5|a_n|\}^{\rho} \geq \frac{9}{2^{\rho}} L(|a_n|) \cdot |a_n|^{\rho} \geq \frac{1}{2^{1+\rho}} |a_n|^{\rho(|a_n|)}$$

for all n sufficiently large. Since

$$\liminf |a_n|^{\rho(|a_n|)}/n \equiv \delta > 0$$

by hypothesis, it follows that

$$\limsup (1/|S'(\lambda_k)|)^{1/k} \leq (\limsup \frac{1}{e^{(1/n)(\epsilon/2)\{|a_n| - \hat{r}_n\}\rho(|z_n| - \hat{r}_n)\}}})^{1/q} \leq e^{-c\delta/(2q)} < 1.$$

The result follows.

Examples. If $\rho > 0$, then $a_n \equiv n^{1/\rho}$ is a sequence of complex numbers having proximate order $\rho(r) \equiv \rho$ with $\liminf |a_n|^{\rho(|a_n|)}/n > 0$. If $\rho > 0$, then $a_n \equiv n^{1/\rho} \ln n$ is a sequence of complex numbers having proximate order $\rho(r) \equiv \rho + (\ln \ln r)/\ln r$ with $\liminf |a_n|^{\rho(|a_n|)}/n > 0$.

It follows from Theorem 3.1 of [20] that the diagonal operator D on $\mathcal{H}(D)$ having eigenvalues $\{n^{1/3}\}$ admits spectral synthesis. However, it follows from the preceding theorem that the diagonal operator on $\mathcal{H}(D)$ having eigenvalues $\{n^{1/3}e^{2\pi i j/6} : 0 \leq j < 6\}$ consisting of six copies of $\{n^{1/3}\}$ placed on the six rays $\{re^{2\pi i j/6} : r > 0\}$ where $0 \leq j < 6$ fails spectral synthesis. In fact, a similar conclusion holds for any sequence of eigenvalues $\{n^\beta\}$ whenever $\beta < 1$. In particular, if $\beta < 1$, then for any integer $q > 2/\beta$, we have that $\rho = 1/(q\beta) < 1/2$. Hence the diagonal operator on $\mathcal{H}(D)$ having eigenvalues $\{|a_n|^{1/q}e^{2\pi i j/q} : 0 \leq j < q\}$ fails spectral synthesis by the preceding theorem, where here $a_n \equiv n^{1/\rho}$. In this case, $a_n^{1/q} = n^\beta$. In fact, we need only choose points $\{a_n\}$ having proximate order $\rho(r) \equiv r$, which places only mild conditions on how they are distributed throughout the complex plane. This example is in contrast to examples mentioned earlier where the points $\{n^{1/3}e^{2\pi i j/6} : 0 \leq j < 6\}$ lie on six rays, or the eigenvalues $\mathbb{Z} \times i\mathbb{Z} \equiv \{m + in : m, n \in \mathbb{Z}\}$ form a lattice.

It is possible to obtain examples of diagonal operators on $\mathcal{H}(D)$ which fail spectral synthesis by perturbing the eigenvalues of a diagonal operator on $\mathcal{H}(D)$ which is known to fail spectral synthesis; however, some care must be taken. Recall that a linear map D for which $D(z^n) = \lambda_n z^n$ extends to an operator on all of $\mathcal{H}(D)$ if and only if $\limsup |\lambda_n|^{1/n} \leq 1$. In this case, D fails spectral synthesis if and only if there exists a non-trivial sequence of complex numbers $\{w_n\}$ for which the moment condition $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ holds for all $k \geq 0$ where here $\limsup |w_n|^{1/n} < 1$.

It may be tempting to believe in this case that adding points to this list of eigenvalues produces another diagonal operator which fails spectral synthesis (simply by making their coefficients zero in the moment condition). However, this requires moving the position of the existing eigenvalues $\{\lambda_n\}$ and the coefficients $\{w_n\}$ which in turn typically changes the values of both $\limsup |\lambda_n|^{1/n}$ and $\limsup |w_n|^{1/n}$. This poses difficulties even when simply rearranging the eigenvalues. For instance, suppose

that D is a diagonal operator on $\mathcal{H}(D)$ which fail spectral synthesis. It follows from
the result due to Sibilev mentioned above that the eigenvalues $\{\lambda_n\}$ are unbounded
(see p. 146 of [18] or [19]). In view of which, there is some rearrangement $\{\lambda_{i(n)}\}$
of $\{\lambda_n\}$ for which $\lim\sup|\lambda_{i(n)}|^{1/n} = \infty$. That is, there does exist a continuous
linear map $\tilde{D}$ for which $\tilde{D}(z^n) = \lambda_{i(n)}z^n$ for all $i \geq 0$. Even if such a rearrangement
yields a new diagonal operator $\tilde{D}$ on $\mathcal{H}(D)$ having eigenvalues $\{\lambda_{i(n)}\}$, it need not
be the case that $\tilde{D}$ fails spectral synthesis (see, for example, Example 4.3 on p. 58
of [21]). It is known, however, that adding or deleting any finite list of eigenvalues
of a non-synthetic diagonal operator on $\mathcal{H}(D)$ produces a new diagonal operator
on $\mathcal{H}(D)$ failing synthesis (see, for example, [19]), but that adding a countable list
of eigenvalues to an operator admitting synthesis may produce an operator failing
synthesis. For example, the diagonal operator on $\mathcal{H}(D)$ having eigenvalues $\{n^{1/3}\}$
fails spectral synthesis while the diagonal operator on $\mathcal{H}(D)$ having eigenvalues $\{n\}$
admits spectral synthesis (see Theorem 3.1 of [19]). The extent to which rearranging,
adding, or deleting eigenvalues effects the synthesis or non-synthesis of a diagonal
operator is explored in [21].

Список литературы

[1] J. Wermer, "On invariant subspaces of normal operators," Proc. Amer. Math. Soc., 3, 270 – 277, (1952).

[2] J. Wolff, "Sur les series $\sum A_k/(z-x_k)$", Comptes Rendus, 173, 1057 – 1058, 1327 – 1328, (1921).

[3] N. K. Nikol'skiĭ, Operators, Functions, and Systems: An Easy Reading, Vol. I, Mathematical surveys and monographs, 92, American Mathematical Society, Providence, RI (2002).

[4] L. Brown, A. Shields, and K. Zeller, "On absolutely convergent exponential sums", Trans. Amer. Math. Soc., 96, 162 – 183 (1960).

[5] D. Sarason, "Invariant subspaces and unstarred operators algebras", Pacific J. Math., 17, 511 – 517 (1966).

[6] D. Sarason, "Weak-star density of polynomials", J. Reine Angew. Math., 252, 1 – 15 (1972).

[7] J. E. Scroggs, "Invariant subspaces of normal operators", Duke Math. J., 26, 95 – 112 (1959).

[8] N. K. Nikol'skiĭ, "The present state of the spectral analysis-synthesis problem I", in "Fifteen Papers on Functional Analysis" (Amer. Math. Soc. Transl., 124, 97 – 129, Providence, RI (1984).

[9] A. F. Leontev, Exponential Series, Nauka, Moscow (1976).

[10] Yu. F. Korobeinik, "Representing systems", Russian Math. Surveys, 36, 75 – 137 (1981).

[11] T. A. Leont'eva, "Representations of analytic functions by series of rational functions", Mat. Zametki, 4, 191 – 200 (1968), English translation in Math. Notices, 2, 695 – 702, (1968).

[12] B. M. Makarov, "On the moment problem in certain function spaces", Doklady Akad. Nauk SSSR, 127, 957 – 960, (1959).

[13] A. Denjoy, "Sur les series de fractions rationelle", Bull. Soc. Math. France, 62, 418 – 434 (1924).

[14] W. T. Ross and H. S. Shapiro, Generalized Analytic Continuation, University Lecture Series, 25, American Mathematical Society, Providence, RI (2002).

[15] E. Borel, "Remarques sur la note de M. Wolff", C. R. Acad. Sci. Paris, 173, 1056 – 1057 (1921).

[16] T. Carleman, "Sur les series $\sum \frac{A_n}{x-a_n}$", C. R. Acad. Sci. Paris, 174, 588 – 591 (1922).

[17] A. A. Gonchar, "On quasianalytic continuation of analytic functions through a Jordan arc", Dokl. Akad. Nauk SSSR, **166**, 1028 – 1031 (1966), English translation Soviet Math. Dokl., **7**, 213 – 216 (1966).

[18] R. V. Sibilev, "Uniqueness theorem for Wolff-Denjoy series", Algebra i Analiz, **7**, 170 – 199 (1995), English Translation in St. Petersburg Math. J., **7**, 145 – 168 (1996).

[19] I. N. Deters and S. M. Seubert, "Spectral synthesis of diagonal operators on the space of functions analytic on a disk", J. Math. Anal. Appl., **334**, 1209 – 1219 (2007).

[20] J. M. Anderson, D. Khavinson, H. S. Shapiro, "Analytic continuation of Dirichlet series", Rev. Mat. Iberoamerica, **11**, 453 – 476 (1995).

[21] K. Overmoyer, "Applications of entire function theory to the spectral synthesis of diagonal operators", a dissertation, Bowling Green State University (2011).

[22] B. Ya. Levin, Lectures on Entire Functions, Transl. Math. Monogr., **150**, Amer. Math. Soc., Providence, RI (1996).

[23] T. W. Gamelin, Complex Analysis, Undergraduate Texts in Mathematics, Springer-Verlag, New York (2001).