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SWW SEQUENCES AND THE INFINITE ERGODIC RANDOM WALK

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Abstract. This article is concerned with demonstrating the power and simplicity of *sww* (special weakly wandering) sequences. We calculate an *sww* growth sequence for the infinite measure preserving random walk transformation. From this we obtain the first explicit *eww* (exhaustive weakly wandering) sequence for the transformation. The exhaustive property of the *eww* sequence is a "gift" from the *sww* sequence and requires no additional work. Indeed we know of no other method for finding explicit *eww* sequences for the random walk map or any other infinite ergodic transformation. The result follows from a detailed analysis of the proof of Theorem 3.3.12 in [1] as applied to the random walk transformation from which an *sww* growth sequence is obtained. We explain the significance of *sww* sequences in the construction of *eww* sequences.

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1. Introduction

Every nonsingular invertible transformation T of a Lebesgue space $(X, \mathcal{B}, m)$ with no finite invariant measure equivalent with m has exhaustive weakly wandering sequences (defined below). However, for most transformations no explicit exhaustive weakly wandering sequences are known, and in particular given a specific T, it is not clear how to find an explicit exhaustive weakly wandering sequence for it.

In this article, we examine the infinite measure preserving random walk transformation (see next section) and derive an *sww* growth sequence for it (defined below). Using this we prove that the integer sequence $\{16^{i+4} : i = 1, 2, 3, \ldots\}$, and every infinite subsequence of it, is an exhaustive weakly wandering sequence for the random walk transformation. The method employed here is general enough that it applies to a wide range of other maps.

For the sake of completeness and clarity of exposition, we repeat some results that were presented in [1].

1.1. History. The definition of mixing in ergodic theory, for a transformation T preserving a probability measure μ is

$$\lim_{n\to\infty} \mu(T^n A \cap B) = \mu(A)\mu(B) \tag{1.1}$$

for all measurable sets A and B. There have been several attempts to extend this definition of mixing to ergodic transformations that preserve a σ-finite infinite measure. The first attempt in this direction was by Eberhard Hopf when in 1937 in his famous book Ergodentheorie [2] he devoted to it, section 17 titled "Ein Beispiel für Mischung bei unendlichem $m(\Omega)$". His goal was to extend the notion of mixing for finite measure preserving transformations to infinite measure preserving transformations. He presented a slight variation of the random walk transformation on the integers. His example started with the classic random walk on the nonnegative integers: for $n > 0$, $n \to \{n-1, n+1\}$ with probability $\{\frac{1}{2}, \frac{1}{2}\}$ and $0 \to \{0, 1\}$ with probability $\{\frac{1}{2}, \frac{1}{2}\}$. This he considered as a map of the infinite strip $[0,\infty) \times [0,1)$ which preserved the infinite Lebesgue measure. Being in an infinite measure space he replaced equation (1.1) with a ratio version. However he was only able to prove (equation 17.1 in [2])

$$\frac{m(A \cap T^n B)}{m(C \cap T^n D)} \to \frac{m(A)m(B)}{m(C)m(D)}, \quad n \to \infty \tag{1.2}$$

for Jordan measurable sets of finite measure with $m(C)m(D) \neq 0$. Then he concluded that if the above were shown to be true for all measurable sets of finite measure then "metric transitivity" (that is, ergodicity) of T would follow. He then ended the section with "Dieser Beweis verlagt jedoch tiefere Hilfsmittel."

Now we know this cannot be done. In 1964, Hajian and Kakutani [3] defined weakly wandering sets and showed that all infinite measure preserving ergodic transformations possess weakly wandering sets. These are sets with an infinite number of mutually disjoint images under the transformation T. Replacing the sets C and D with the same weakly wandering set in equation (1.2) shows the convergence fails. Further historical details and attempts at defining mixing in infinite measure spaces can be found in Lenci [4].

In what follows we do more. We discuss the random walk transformation T on the integers and exhibit some properties of it that show how far T is from possessing any type of "mixing"feature. We do this by showing the existence and construction of specific *eww* sequences (defined below) that T possesses. We also exhibit some number theoretic properties of these sequences.

1.2. Definitions and Preliminaries. We consider transformations T that are invertible onto maps defined on a σ-finite Lebesgue measure space $(X, \mathcal{B}, m)$. As usual, all statements are to be understood as "modulo sets of measure zero and all sets will be measurable by assumption or construction. We assume all the transformations T we consider are measurable $[A \in \mathcal{B} \iff TA \in \mathcal{B}]$, and non-singular $[m(A) = 0 \iff m(TA) = 0]$. We say T is a measure preserving transformation if $m(TA) = m(A)$ for all $A \in \mathcal{B}$. Two measures m and μ defined on the same measurable space $(X, \mathcal{B})$ are equivalent ($m \sim \mu$) if m and μ have the same sets of measure zero. There are many equivalent definitions of ergodicity. We use the following.

- T is *ergodic* if $TA = A$ implies $m(A) = 0$ or $m(X \setminus A) = 0$.

An ergodic transformation T is an *infinite ergodic* transformation if it is a measure preserving transformation defined on the infinite measure space $(X, \mathcal{B}, m)$.

Following [1], we consider the following infinite sequences of integers $\{n_i\}$ associated to an infinite ergodic transformation T.

Definition 1.1.

- $\{n_i\}$ is a *weakly wandering* (*ww*) sequence for T if for some set A of positive measure $T^{n_i}A \cap T^{n_j}A = \emptyset$ for $i \neq j$.
- $\{n_i\}$ is an *exhaustive weakly wandering (eww)* sequence for T if for some set A of positive measure $X = \bigcup_{i=0}^{\infty} T^{n_i}A(disj)$.
- $\{n_i\}$ is a *special* (or at times called *strongly) weakly wandering (sww)* sequence for T if there exists a set A of positive measure such that for $i, j, k, l \geq 0$ and $i > j$ we have $T^{n_i - n_k + k'}A \cap T^{n_j - n_l + l'}A = \emptyset$ whenever one of the indices $\{i, j, k, l\}$ is larger than all the others or $i = l > k$.
- We call the set A above, a *ww, eww,* or *sww set* respectively (for T, with the sequence $\{n_i\}$), and at times we say $\{n_i\}$ is a *ww, eww,* or *sww sequence* (for T with the set A).

The definition of *ww* sequences first appeared in [3] where it was shown that they exist for every infinite ergodic transformation. There are many examples of infinite ergodic transformations in the literature; however, for almost any example, it has not been possible to exhibit a specific *ww* sequence for the transformation. There is one notable exception: the infinite ergodic example T in [5] which was constructed for the purpose of exhibiting an explicit *ww* sequence for it. In that example it was noticed that the constructed *ww* sequence happened to be an *eww* sequence. Except for the transformation T in [5] and some similar ones, it is not that easy to construct specific *ww* sequences for any of the known infinite ergodic transformations — though we know they must exist. However, to our knowledge, it is practically impossible to construct *eww* sequences for any of those transformations. The construction of *ww* sequences entails showing for some sequence and some set W the mutually disjoint images of the set W. For *eww* sequences on the other hand one needs to show further that the mutually disjoint images of the set W fill up the whole space X.

In [6], Jones and Krengel present a proof that *eww* sequences exist for all infinite ergodic transformations. In outline, their proof is a complicated back-and-forth induction existence proof. They build their sequence one integer at a time while simultaneously adjusting their set. The set is built up in a two step process. At each step they must ***take a bit away*** from the set so that it will be disjoint for the next integer and then they have to ***add a bit back*** in order to build up the set to be exhaustive. As a practical matter, no one to date has been able to use this method to construct an actual *eww* sequence for any transformation.

To overcome this difficulty *sww* sequences were introduced in [1]. The definition of an *sww* sequence appears to be more complicated than that of an *eww* sequence. However it is designed in such a way that it can be easily applied. The construction of *sww* sequences is similar to the construction of *ww* sequences. By this we mean that both sequences are concerned only with the construction of a set A whose images under the sequence are mutually disjoint, and this is relatively easy. Once the set A is constructed in the case of an *sww* sequence, a second easily performed automatic construction produces the ***derived*** set W. For ergodic transformations the fact that the mutually disjoint images of the derived set W are exhaustive follows from the definitions.

In addition, *sww* sequences give a lot more. When the transformation is ergodic, not only is the *sww* sequence an *eww* sequence for the associated derived set, but every infinite subsequence of it is again an *eww* sequence with a similarly defined derived set W. This hereditary property follows from the definitions of both *ww* and *sww* sequences but not for *eww* sequences. That is, if the images of a set A are mutually disjoint under a sequence, they are still mutually disjoint for any infinite subsequence of it; but the set may not be exhaustive for the same sequence. For example, the *eww* sequence given in [5] has many infinite subsequences which are not *eww* (just remove any single non-zero integer from the sequence).

1.3. The Derived Set. To clarify the comments made above, and to make this article self contained, we make some general observations and discuss a few results that are covered in [1] and will be used in the sequel.

For a sequence of integers $\{n_i : i > 0\}$ and any set A with $m(A) > 0$,

let $n_0 = 0$, $A_0 = A$, and $W_0 = T^{-n_0} A_0$,

$A_1 = TA \setminus \bigcup_{r=0}^{\infty} T^{n_r} W_0$, and $W_1 = \bigcup_{i=0}^{1} T^{-n_i} A_i$,

and in general for $p \geq 2$

$$A_p = T^p A \setminus \bigcup_{r=0}^{\infty} T^{n_r} W_{p-1}, \text{ and } W_p = \bigcup_{i=0}^{p} T^{-n_i} A_i. \tag{1.3}$$

Let us call the set $W = \bigcup_{p=0}^{\infty} W_p$ the *derived* set from the set A and the sequence $\{n_i\}$. Then for any $p > 0$ we have

$\bigcup_{r=0}^{\infty} T^{n_r} W \supset T^p A \cup \bigcup_{r=0}^{\infty} T^{n_r} W_{p-1}$ which implies $\bigcup_{r=0}^{\infty} T^{n_r} W \supset \bigcup_{p=0}^{\infty} T^p A.$

From the above we conclude with the following remark:

Remark 1.1. Let W be the derived set from the set A and the sequence $\{n_i\}$. If $\{n_i\}$ is an *sww* sequence then

$$\bigcup_{i=0}^{\infty} T^{n_i} W (\text{disj}) \supset \bigcup_{p=0}^{\infty} T^p A. \tag{1.4}$$

To show (1.4) it is enough to show $T^{n_i} W \cap T^{n_j} W = \emptyset$ for $i, j \geq 0$ and $i > j$. For this it is sufficient to show that

$$T^{n_i - n_k} A_k \cap T^{n_j - n_l} A_l = \emptyset \quad \text{for} \quad i, j, k, l \geq 0, \quad \textit{and} \quad i > j. \tag{1.5}$$

It is clear from (1.3) that for any integer $r > 0$

$$A_p \cap T^{n_r - n_s} A_s = \emptyset \quad \text{if} \quad p > s. \tag{1.6}$$

If $i = k > \max\{j, l\}$ then (1.5) follows from (1.6). In all the other cases we note that $A_p \subset T^p A$ for all $p \geq 0$, and (1.5) then follows from the properties defining the *sww* sequence $\{n_i\}$.

For the next theorem we define the following:

Definition 1.2. Let $\{0 < N_1 < N_2 < \cdots\}$ be an increasing sequence of positive integers. Then, for any increasing sequence of positive integers $\{0 = n_0 < n_1 < n_2 < \cdots\}$

(I) If $n_i - n_{i-1} \geq N_i$ for all $i \geq 1 \implies \{n_i\}$ is a *ww* sequence for T then $\{N_i\}$ is a *ww growth sequence* for T,
and

(II) If $n_i - 2n_{i-1} \geq N_i$ for all $i \geq 1 \implies \{n_i\}$ is an *sww* sequence for T then $\{N_i\}$ is an *sww growth sequence* for T.

Theorem 1.1. *Let T be a measurable and nonsingular transformation defined on $(X, \mathcal{B}, m)$, and suppose there is a set A of positive measure satisfying $\lim_{n\to\infty} m(T^n A \cap A) = 0$. Then there exists an increasing sequence of positive integers $\{N_i\}$ which is both a ww and an sww growth sequence for T.*

Proof. The proof of the Theorem is contained in detail in [1]. Here we sketch a proof and show the similarity of the role of *ww* and *sww* sequences in constructing the *ww* and *sww* set A_0 for each. Later, we apply this construction to find an explicit *sww* growth sequence for the random walk transformation on the integers.

Let A be a set of positive measure with $m(A) < \infty$, and suppose

(1.7) $$\lim_{n\to\infty} m(T^n A \cap A) = 0.$$

For positive $\epsilon < m(A)$, and for $i \geq 1$ let $\epsilon_i = \dfrac{\epsilon}{2(2i+1)i^3 2^i}$.

Using (1.7) we choose an increasing sequence of positive integers $\{0 < N_1' < N_2' < \cdots\}$ such that for each $i \geq 1$, $m(T^n A \cap A) \leq \epsilon_i$ for all $n \geq N_i'$. We let $N_i = N_i' + i$ for $i \geq 1$.

To show $\{N_i\}$ is a *ww* growth sequence we let $\{0 = n_0 < n_1 < n_2 < \cdots\}$ be any increasing sequence of integers satisfying $n_i - n_{i-1} \geq N_i$for $i \geq 1$.

(1.8) For $i > 0$ and $0 \leq j < i$ we have $n_i - n_j \geq n_i - n_{i-1} \geq N_i$.

Next we let,

$$A' = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i-1} T^{n_i - n_j} A \cap A.$$

It is not too difficult to show that $m(A') \leq \epsilon$ and the set $A_0 = A \setminus A'$ is a *ww* set with the sequence $\{n_i\}$.

To show $\{N_i\}$ is an *sww* growth sequence we let $\{0 = n_0 < n_1 < n_2 < \cdots\}$ be any increasing sequence of integers satisfying $n_i - 2n_{i-1} \geq N_i$ for $i \geq 1$. For each

$i \geq 1$ we consider the set of integers $S_i = \{s : s = an_i + bn_j + cn_k + dn_l + e\}$ where $a \in \{1, 2\}$, $b, c, d \in \{0, \pm 1\}$, $e \in \{0, \pm 1, \pm 2, \ldots, \pm i\}$, $0 \leq j, k, l < i$.

In the sets S_i we also require that at most two of the numbers b, c, d be negative. Then the cardinality of S_i, $|S_i| \leq 2i^3 3^3 (2i + 1)$. Since $\{n_i : i \geq 0\}$ is an increasing sequence of positive integers we have for $s \in S_i$

$s = an_i + bn_j + cn_k + dn_l + e \geq n_i - 2n_{i-1} - i \geq N_i - i = N_i'$.

Similarly as before we let

$$A' = \bigcup_{i=1}^{\infty} \bigcup_{s \in S_i} T^s A \cap A.$$

Again, it is not difficult to show that $m(A') \leq \epsilon$ and the set $A_0 = A \setminus A'$ is an *sww* set with the sequence $\{n_i\}$. □

Remark 1.2. For an ergodic transformation T the following is an immediate consequence of the definition:

$$m(A) > 0 \Rightarrow \bigcup_{p=0}^{\infty} T^p A = X. \tag{1.9}$$

Then (1.4) in Remark 1.1 together with (1.9) above imply that all *sww* sequences for an ergodic transformation are *eww* sequences for T.

Next for an ergodic transformation T we extend the definition of *ww* and *sww* growth sequences to ***eww growth sequences*** for T.

(III) An increasing sequence $\{0 < N_1 < N_2 < \cdots\}$ of positive integers is an *eww growth sequence* for an ergodic transformation T if any increasing sequence $\{0 = n_0 < n_1 < n_2 < \cdots\}$ of positive integers that satisfies $n_i - 2n_{i-1} \geq N_i$ for $i \geq 1 \implies \{n_i\}$ is an *eww* sequence for T.

Then for ergodic transformations every *sww* growth sequence is an *eww* growth sequence. Finally we conclude with the following Corollary to Theorem 1.1.

Corollary 1.1. *Every infinite ergodic transformation T that posesses a set A of positive measure with $\lim_{n\to\infty} m(T^n A \cap A) = 0$ has eww growth sequences.*

2. Infinite Measure Preserving Random Walk on the Integers

2.1. Random Walk on the Integers. We begin, as did Hopf (page 61 of [2]), with the Baker's transformation S defined on the unit square $Z = \{(x, y) : 0 \leq x < 1,\ 0 \leq y < 1\}$:

$$S(x,y) = \begin{cases} (2x, y/2) & \text{if } 0 \leq x < \frac{1}{2}, \\ (2x - 1, (y+1)/2) & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

The map S is obviously a finite (probability) measure preserving invertible transformation. Hopf [2] proved it was mixing in the sense that (1.1) holds for all measurable sets. The proof begins by analyzing how S operates on dyadic rectangles. It is then a standard approximation argument to extend the mixing result from dyadic rectangles to all measurable sets. From this mixing property it follows that the map is ergodic (i.e. metrically transitive). There are now multiple proofs of the ergodicity of the Baker's map and in fact a lot more is known. For example it is well-known to be Bernoulli. We extend the transformation S to the two-sided infinite strip $\{(x,y) : -\infty < x < \infty,\ 0 \le y < 1\}$ by a skew product construction as follows. Identify each square $\{(x,y) : n \le x < n+1,\ 0 \le y < 1\}$ as (Z,n). The infinite strip $(-\infty,\infty)\times[0,1)$ with area measure is the space $Z\times\mathbb{Z} = \bigcup_{n\in\mathbb{Z}}(Z,n)$ with the measure which is the product of the Lebesgue area measure on Z and the counting measure on the integers. Consider the skewing function $\phi : Z \to \{-1,1\}$ defined by $\phi(x,y) = -1$ if $0 \le x < 1/2$ and $\phi(x,y) = 1$ if $1/2 \le x < 1$. The random walk transformation on $Z \times \mathbb{Z}$ is (see p. 62–63, [1])

$$T((x,y),n) = (S(x,y), n + \phi(x,y))$$

We refer to this map T as infinite measure preserving random walk on $\mathbb{Z}$. This example is a variation of Hopf's example on the one sided infinite strip $[0,\infty) \times [0,1)$.

Theorem 2.1. *The infinite measure preserving random walk transformation is ergodic.*

Although Hopf never completed the proof of the ergodicity of the random walk transformation on the non-negative integers its ergodicity and that of the random walk on the integers T are now well known (see [7] and [4]). An elementary proof of the ergodicity of T can be given by examining the induced transformation on $(Z,0)$ and recognizing it as a finite measure preserving Bernoulli map (similar to recognizing the Baker's map as Bernoulli). More precisely, the induced map on every square $Z \times \{n\}$ is a Bernoulli map, and each square $Z \times \{n\}$ can be mapped to any portion of every other square.

2.2. An sww sequence for Random Walk. We are now in a position to derive an explicit *sww* growth sequence for T and we emphasize how simple and short it is once one has the *sww* definition. Specifically we duplicate the steps of the proof given in Theorem 1.1 to the random walk transformation T.

The necessary inequalities used in calculating the *sww growth* sequence come from the next lemma.

Lemma 2.1. *For the infinite measure preserving random walk transformation T described above, the set $(Z,0)$ satisfies the inequality*

$$\frac{1}{\sqrt{5k}} < m\left(T^{2k}(Z,0)\cap(Z,0)\right) < \frac{1}{\sqrt{2k}} \quad \text{for all } k>0.$$

Proof. For an odd integer $n>0$ we have $m\left(T^{n}(Z,0)\cap(Z,0)\right)=0$, and for an even integer $n=2k$, $k\geq 1$ we have:

$$m\left(T^{2k}(Z,0)\cap(Z,0)\right)=\binom{2k}{k}\frac{1}{2^{2k}}=\frac{(2k)!}{k!k!2^{2k}}=\frac{k!2^k(2k-1)!!}{k!2^k(2k)!!}=\frac{(2k-1)!!}{(2k)!!}.$$

Using induction it is easy to show:

$$\frac{(2k-1)!!}{(2k)!!}<\frac{1}{\sqrt{2k+1}} \quad \text{for } k\geq 1.$$

It is also easy to show:

$$\frac{1}{\sqrt{4k+1}}<\frac{(2k-1)!!}{(2k)!!} \quad \text{for } k\geq 1.$$

Combining the above we get:

$$\frac{1}{\sqrt{5k}}<\frac{1}{\sqrt{4k+1}}<m\left(T^{2k}(Z,0)\cap(Z,0)\right)<\frac{1}{\sqrt{2k+1}}<\frac{1}{\sqrt{2k}} \quad \text{for } k\geq 1.$$

□

We use the lemma above to get an *sww* growth sequence for the random walk transformation. This will also be an *sww* and an *eww* sequence.

Theorem 2.2. *The sequence $\{N_i=16^{i+4}: i\geq 1\}$ is both an sww growth sequence and an eww sequence for T the infinite measure preserving random walk transformation.*

Proof. For the random walk transformation T we showed in Lemma 2.1

$$m(T^{2k}(Z,0)\cap(Z,0))<\frac{1}{\sqrt{2k}}.$$

Therefore specializing the part of the proof given after the statement of Theorem 1.1 to the random walk T, the set $A=(Z,0)$ and $\epsilon=1/2$, we can choose N_i' so that for all $n\geq N_i'$

$$m(T^{2n}(Z,0)\cap(Z,0))<\frac{1}{\sqrt{2N_i'}}\leq\frac{1}{4(2i+1)i^3 2^i}.$$

From this we conclude that $N_i'\geq 8(2i+1)i^6 4^i$ and we have the growth sequence $8(2i+1)i^6 4^i+i$. This can be "neatened"to the growth sequence $16(2i+1)i^6 4^i$ which can be bounded by

$$N_i=4^{2i+6}=16^{i+3},\ i\geq 1.$$

This is also an *sww* growth sequence.

Clearly this implies that the sequence 16^{i+4} is also a growth sequence.

Since T is ergodic, we use Condition II of Definition 1.2 comparing $\{16^{i+4}\}$ to the previous growth sequence $\{16^{i+3}\}$ obtaining

$$16^{i+4} - 2 \cdot 16^{(i-1)+4} = (16 - 2) \cdot 16^{i+3}$$

which shows that $\{16^{i+4}\}$ is also an *sww* sequence for the random walk transformation. □

3. Application to Tilings of the Integers

As a special case consider the integers $\mathbf{Z}$ with the counting measure μ and denote the translation transformation $T : (\mathbf{Z}, \mu) \to (\mathbf{Z}, \mu)$, $T(n) = n + 1$. This is an ergodic, infinite measure preserving, invertible transformation, albeit with an atomic measure, and we can consider the analog of Theorem 1.1 for this map.

First we note that an infinite subset of integers $\{n_i : i \geq 1\}$ (denoted simply by $\{n_i\}$) is weakly wandering for T in this context means there exists another subset $\{m_j\}$ of the integers such that

$$\{n_i\} + \{m_j\} = \{n_i\} \oplus \{m_j\}$$

By this it is meant that the sum is *direct*, $n_i + m_i = n_j + m_j$ if and only if $n_i = n_j$ and $m_i = m_j$.

Further, to say that $\{n_i\}$ is *eww* means there exists $\{m_j\}$ which is direct with $\{n_i\}$ and the sum contains all integers, i.e., $\{n_i\} \oplus \{m_j\} = \mathbf{Z}$. This says that $\{n_i\}$ *tiles* the integers $\mathbf{Z}$ and we call $\{n_i\}$ a *tile*.

The case when $\{n_i\}$ (or $\{m_j\}$) is finite is a very active area of research with many open questions. This finite case has been studied using a wide range of techniques including cyclotomic polynomials, fourier analysis and the theory of finite cyclic groups. None of these methods however apply in the case when both $\{n_i\}$ and $\{m_j\}$ are infinite. This is the situation in which we are interested in obtaining an analog of Theorem 1.1.

In [1] it is shown that the following provides an analog of part II of Theorem 1.1 and replaces the eww growth condition by a limit.

Theorem 3.1. *Any infinite sequence $\{n_i\} = \{n_0 = 0 < n_1 < n_2 \cdots\}$ of nonnegative integers satisfying $\lim_{i\to\infty} n_i - 2n_{i-1} = \infty$ tiles the integers.*

A surprising consequence of this, which emphasizes the difference between finite and infinite tiles, is that such an infinite tile has the hereditary property that any finite set of non-zero integers can be removed and the resulting sequence still tiles the integers. This is not true for finite tiles and is not true for all infinite tiles.

This theorem was first proved using ergodic theory techniques for the translation transformation T [8], but J. Schmerl (private communication) gave a strictly combinatorial proof which appears in [1].

Note that, the analogous *ww* growth condition for part I of Theorem 1.1 is not true: There exist sequences of integers $\{n_i\}$ which satisfy $\lim(n_i - n_{i-1}) = \infty$, yet there is no infinite subset $\{m_i\}$ with which $\{n_i\}$ is direct let alone tiles the integers.

4. Questions

In this section we gather a few questions about the random walk transformation.

Question 1. The *eww* sequence obtained in Theorem 2.2 has the derived set W as an *eww* set associated with it. Is the measure of W infinite or finite?

Question 2. Transformations can have many different *eww* sequences and sets. Does the random walk transformation T have another *eww* sequence whose *eww* set has finite measure?

Question 3. If S is a nonsingular transformation which commutes with the random walk T is it measure preserving?

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