Известия НАН Армении, Математика, том 53, н. 6, 2018, стр. 83 – 93

CERTAIN CLASSES OF ANALYTIC FUNCTIONS RELATED TO THE CRESCENT-SHAPED REGIONS

R. K. RAINA, P. SHARMA AND J. SOKÓL

University of Agriculture and Technology, Udaipur, India
University of Lucknow, India
University of Rzeszów, Rzeszów, Poland
E-mails: *rkraina_7@hotmail.com; sharma_poonam@lkouniv.ac.in; jsokol@prz.edu.pl*

Abstract. In this paper, we study certain classes of analytic functions which satisfy a subordination condition and are associated with the crescent-shaped regions. We first give certain integral representations for the functions belonging to these classes and also present a relevant example. Making use of some known lemmas, we derive sufficient conditions for the functions to be in these classes. Some results on coefficient estimates are also obtained.

MSC2010 numbers: 30C45.

Keywords: analytic function; convex function; starlike function; subordination; k-uniformly convex; k-uniformly starlike.

1. Introduction and preliminaries

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$, and for $m \in \mathbb{N} = \{1, 2, \ldots\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, m] = \left\{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \ldots\right\}.$$

Denote $\mathcal{H}[1, 1]$ by $\mathcal{H}$, and let a class of normalized analytic functions $\mathcal{A}_m$ be defined by

$$\mathcal{A}_m = \left\{f \in \mathcal{H}(\mathbb{U}) : f(z) = z + a_{m+1} z^{m+1} + \ldots\right\}$$

with $\mathcal{A}_1 = \mathcal{A}$. A subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Also, let $\mathcal{S}^*$ and $\mathcal{C}$ be the subclasses of $\mathcal{S}$ comprising of functions f which are, respectively, starlike and convex and which map the unit disk $\mathbb{U}$ onto starlike and convex regions, respectively. Analytically, a function f belonging to the classes $\mathcal{S}^*$ and $\mathcal{C}$ satisfies the inequalities

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \ \text{ and } \ \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{U}), \tag{1.1}$$

respectively.

We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω analytic in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \mathbb{U}$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.2}$$

Denote by $\mathcal{P}$ the class of functions p analytic in $\mathbb{U}$, such that $p(0) = 1$ and $\Re\{p(z)\} > 0$ $(z \in \mathbb{U})$.

In [12], Ma and Minda unified the approach to several classes by introducing a generalized subclass $\mathcal{P}(\phi)$ of $\mathcal{P}$. In terms of the subordination, a function $p \in \mathcal{P}$ is said to be in the class $\mathcal{P}(\phi)$ if it satisfies the condition $p \prec \phi$, where ϕ is analytic univalent with positive real part in $\mathbb{U}, \phi(\mathbb{U})$ is symmetric with respect to the real axis and is starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. Furthermore, the classes defined by

$$\mathcal{S}^*(\phi) = \{f \in \mathcal{A} : zf'(z)/f(z) \in \mathcal{P}(\phi)\}$$

and

$$\mathcal{C}(\phi) = \{f \in \mathcal{A} : 1 + zf''(z)/f'(z) \in \mathcal{P}(\phi)\}$$

become, respectively, the classes $\mathcal{S}^*$ and $\mathcal{C}$ when we choose $\phi(z) = (1+z)/(1-z)$. Recently, several results were obtained when $\phi(z) = p_k(z)$, where $p_k(z)$ is convex univalent in $\mathbb{U}$ and has the form:

$$p_k(z) = \begin{cases} 1 + \frac{2}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\operatorname{arc\,tanh}\sqrt{z}\right\}, & 0 \le k < 1, \\ 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int\limits_0^{u(z)}\frac{dx}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\right), & k > 1, \end{cases}$$

where $u(z) = \frac{z-\sqrt{t}}{\sqrt{t}(1-\sqrt{t}z)}$, $z \in \mathbb{U}$ and $t \in (0,1)$ is chosen such that

$$k = \cosh\left(\pi R(\sqrt{1-t^2})/R(t)\right),$$

$R(t)$ is the Legendre's complete elliptic integral of the first kind (see [7] – [10]). Also, we obtain the class $\mathcal{S}^*(p_k)$ of k-uniformly starlike and the class $\mathcal{C}(p_k)$ of k-uniformly convex functions defined, respectively, by

$$\mathcal{S}^*(p_k) = \{f \in \mathcal{A} : zf'(z)/f(z) \in \mathcal{P}(p_k)\}$$

and

$$\mathcal{C}(p_k) = \{f \in \mathcal{A} : 1 + zf''(z)/f'(z) \in \mathcal{P}(p_k)\}.$$

Motivated by the above defined class $\mathcal{P}(\phi)$, we consider the function $\phi(z) = z + \sqrt{1+z^2}$ $(z \in \mathbb{U})$, and define the following classes:

$$\Delta = \left\{p \in \mathcal{H} : p(z) \prec z + \sqrt{1+z^2} \quad (z \in \mathbb{U})\right\}, \tag{1.3}$$

$$\Delta^* = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2} \quad (z \in \mathbb{U})\right\} \tag{1.4}$$

and

$$\Delta^{\mathcal{C}} = \left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec z + \sqrt{1+z^2} \quad (z \in \mathbb{U})\right\}. \tag{1.5}$$

Observe that the function $\phi(z) = z + \sqrt{1+z^2}$ $(z \in \mathbb{U})$ is analytic univalent and

$$\Re\mathrm{e}\,\left\{\phi(e^{it})\right\} = \begin{cases} \cos t + \sqrt{2\cos t}\,|\cos t/2| & \text{for} \quad t \in [0, \pi/2) \cup (3\pi/2, 2\pi), \\ \cos t + \sqrt{2\,|\cos t|}\,\sin t/2 & \text{for} \quad t \in (\pi/2, 3\pi/2), \\ 0 & \text{for} \quad t = \pi/2, 3\pi/2, \end{cases} \tag{1.6}$$

which shows that $\Re\mathrm{e}\,\{\phi(z)\} > 0$ in $\mathbb{U}$. We note that $\phi(e^{it})$ for $t \in [0, 2\pi)$ traverses a *crescent* which is symmetric and convex along the real axis (see Fig.1 below, and [16]). Also, in view of (1.2), we infer that

$$\Delta \subset \mathcal{P}, \tag{1.7}$$

implying that $\Delta^* \subset \mathcal{S}^*$ and $\Delta^{\mathcal{C}} \subset \mathcal{C}$. The subordination condition in (1.3) may equivalently be given by

$$\left|(p(z))^2 - 1\right| < 2|p(z)| \quad (z \in \mathbb{U}), \tag{1.8}$$

and the class Δ^* as a subclass of $\mathcal{S}^*$ with the equivalent condition (1.8) is the class, which was recently studied in [16]. Also, in view of Theorem 2.4 from [16], we observe that if $p \in \Delta$, then

$$|p(z) + 1| > \sqrt{2} \quad \text{and} \quad |p(z) - 1| < \sqrt{2} \quad (z \in \mathbb{U}). \tag{1.9}$$

The condition (1.9) implies that if $p \in \Delta$, then $p(\mathbb{U})$ is the interior of the right-crescent formed by the circles (see Fig. 1):

$$|w + 1| = \sqrt{2} \quad \text{and} \quad |w - 1| = \sqrt{2}, \tag{1.10}$$

and is starlike with respect to 1. Thus, the above defined classes Δ, Δ^* and $\Delta^{\mathcal{C}}$ are associated with the right-crescent, formed by the circles given by (1.10), and our attempt to define these classes are in the line of investigations of various other classes defined earlier and having geometric interpretations. For example, the class of functions $f \in \mathcal{A}$ satisfying $zf'(z)/f(z) \prec \sqrt{1+z}$ $(z \in \mathbb{U})$ was considered in [17] and is associated with the Bernoulli lemniscate, while the class of functions $f \in \mathcal{A}$ satisfying $zf'(z)/f(z) \prec \sqrt{1+cz}$ was considered in [1]. It should be noted that in some recent papers (see, e.g., [2, 3, 4, 5]), certain function classes were considered, which were defined by the condition $zf'(z)/f(z) \prec \widehat{q}(z)$, where $\widehat{q}(z)$ was not univalent, and these classes created difficult problems when considering their geometric properties.

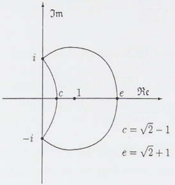

Fig. 1. $\phi(e^{it})$.

The aim of this paper is to study the class Δ and, in particular, the classes Δ^* and $\Delta^{\mathfrak{C}}$, and to obtain integral representations for functions from these classes and to derive sufficient conditions for functions to belong to these classes by making use of certain well-known results. Some results on the coefficients estimates for these functions are also derived.

The following two lemmas will be used in the proof of our main results.

Lemma 1.1 (see [14], Theorem 3.4h, p. 132, and [13]). *Let q be univalent in $\mathbb{U}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\varphi(w) \neq 0$, when $w \in q(\mathbb{U})$. Set $Q(z) = zq'(z) \cdot \varphi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$, and suppose that either*

(i) h is convex, or (ii) Q is starlike.

In addition, assume that

(iii) $\mathfrak{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$.

If p is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z) \cdot \varphi[p(z)] \prec \theta[q(z)] + zq'(z) \cdot \varphi[q(z)] = h(z),$$

then $p \prec q$, *and* q *is the best dominant in the sense that* $p \prec s \Rightarrow q \prec s$, $\forall s$.

If we set $\theta[w] \equiv 0$ in the above Lemma 1.1, then the conditions (ii) and (iii) become identical and we have the following simplified version of Lemma 1.1.

Lemma 1.2 (see [14], Corollary 3.4h.1, p. 135). *Let* q *be univalent in* $\mathbb{U}$, *and* φ *be analytic in a domain* $\mathbb{D}$ *containing* $q(\mathbb{U})$. *If* $zq'(z)\cdot\varphi[q(z)]$ *is starlike, then*

$$zp'(z)\cdot\varphi[p(z)] \prec zq'(z)\cdot\varphi[q(z)] \quad \Rightarrow \quad p \prec q,$$

and q *is the best dominant.*

2. Main results

We first give certain integral representations for functions belonging to the classes Δ^* and Δ^e, which we use to construct a specific example demonstrating the application of these classes. These integral representations follow directly from the definitions of Δ^* and Δ^e. Thus, we have the following assertions.

(i) A function $f \in \Delta^*$ if and only if there exists an analytic function $p \in \Delta$ such that

$$f(z) = z\exp\left(\int_0^z \frac{p(t)-1}{t}\mathrm{d}t\right).$$

(ii) A function $f \in \Delta^e$ if and only if there exists an analytic function $p \in \Delta$ such that

$$f(z) = \int_0^z \left\{\exp\left(\int_0^u \frac{p(t)-1}{t}\mathrm{d}t\right)\right\}\mathrm{d}u.$$

Choosing $p(z) = z + \sqrt{1+z^2}$ in the above representations (i) and (ii), we find that

$$f_0(z) = z\exp\left(\int_0^z \frac{t+\sqrt{1+t^2}-1}{t}\mathrm{d}t\right) \in \Delta^*$$

and

$$g_0(z) = \int_0^z \left\{\exp\left(\int_0^u \frac{t+\sqrt{1+t^2}-1}{t}\mathrm{d}t\right)\right\}\mathrm{d}u \in \Delta^e,$$

and obviously, we have $zg_0'(z) = f_0(z)$. Further, observe that for some A $(-1 < A < 1)$, the function $p(z) = \frac{1+Az}{1-Az}$ maps the unit disc $\mathbb{U}$ onto the disc

$$\left|p(z) - \frac{1+|A|^2}{1-|A|^2}\right| < \frac{2|A|}{1-|A|^2}, \tag{2.1}$$

and hence, if

$$p(z) = \frac{1+Az}{1-Az} \in \Delta,$$

then the endpoints $\left(\frac{1-|A|}{1+|A|}, 0\right)$ and $\left(\frac{1+|A|}{1-|A|}, 0\right)$ of the diameter of the disc in (2.1), satisfy the inequality:

$$\sqrt{2} - 1 \leq \frac{1-|A|}{1+|A|} \leq \frac{1+|A|}{1-|A|} \leq \sqrt{2} + 1.$$

Therefore, we can assert that for some A $(-1 < A < 1)$,

$$p(z) = \frac{1+Az}{1-Az} \in \Delta \Leftrightarrow A \in \left[1 - \sqrt{2}, \sqrt{2} - 1\right],$$

and hence, by using the representations given in (i) and (ii) for $p(z) = \frac{1+Az}{1-Az}$, we have the following example.

Example 2.1. *Let* $1 - \sqrt{2} \leq A \leq \sqrt{2} - 1$, *then*

$$f_1(z) = \frac{z}{(1-Az)^2} \in \Delta^* \quad \text{and} \quad f_2(z) = \frac{z}{1-Az} \in \Delta^e.$$

Now, by using Lemmas 1.1 and 1.2, we establish some sufficient conditions involving subordination for the class Δ, where Δ is defined by (1.3):

Theorem 2.1. *Let* $q(z) = z + \sqrt{1+z^2}$. *If* $p \in \mathcal{H}$ *satisfies*

$$p(z) + \frac{zp'(z)}{p(z)} \prec q(z) + \frac{zq'(z)}{q(z)}, \quad z \in \mathbb{U}, \tag{2.2}$$

then

$$p \in \Delta. \tag{2.3}$$

Proof. To prove the result, we need to show that

$$p(z) \prec q(z), \quad z \in \mathbb{U}. \tag{2.4}$$

Let

$$\theta[w] = w, \quad \varphi[w] = \frac{1}{w} \tag{2.5}$$

be analytic in the domain $\mathbb{D}$ such that $q(\mathbb{U}) \subset \mathbb{D}$. Then, it follows that

$$zq'(z) \cdot \varphi[q(z)] = \frac{zq'(z)}{q(z)} = \frac{z}{\sqrt{1+z^2}} =: Q(z). \tag{2.6}$$

Now, for $z_1, z_2 \in \mathbb{U}$ such that $Q(z_1) = Q(z_2)$, we have

$$\frac{z_1^2}{1+z_1^2} = \frac{z_2^2}{1+z_2^2} \Rightarrow z_1^2 = z_2^2 \Rightarrow z_1 = \pm z_2.$$

The case $z_1 = -z_2$ is impossible because $Q(z)$ is an odd function. So, we have only the case $z_1 = z_2$, and hence $Q(z)$ is univalent in $\mathbb{U}$. Also, $Q(z)$ is starlike in $\mathbb{U}$ because

$$\frac{zQ'(z)}{Q(z)} = \frac{1}{1+z^2}$$

and

$$\Re\left\{\frac{zQ'(z)}{Q(z)}\right\} = \Re\left\{\frac{1}{1+z^2}\right\} > \frac{1}{2}, \quad z \in \mathbb{U}. \tag{2.7}$$

Further, in view of (2.6), for function $h(z) = \theta\,[q(z)] + Q(z) = q(z) + Q(z)$ we have

$$\frac{zh'(z)}{Q(z)} = q(z) + \frac{zQ'(z)}{Q(z)},$$

and hence, from (1.6) and (2.7), we infer that

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > \frac{1}{2} > 0, \quad z \in \mathbb{U}.$$

Thus, the conditions (ii) and (iii) of Lemma 1.1 are satisfied. By applying Lemma 1.1 and using the substitutions given by (2.5), we observe that the subordination relation

$$\theta\,[p(z)] + zp'(z)\cdot\varphi\,[p(z)] \prec \theta\,[q(z)] + zq'(z)\cdot\varphi\,[q(z)] \tag{2.8}$$

becomes the subordination condition (2.2), which implies the subordination (2.4). This proves the desired result (2.3) of the theorem. □

Theorem 2.2. *Let* $q(z) = z + \sqrt{1+z^2}$ $(z \in \mathbb{U})$ *and* $p \in \mathcal{H}$. *Then*

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)} \Rightarrow p \in \Delta. \tag{2.9}$$

Proof. Let $\varphi\,[w] = \frac{1}{w}$ be analytic in the domain $\mathbb{D}$ such that $q(\mathbb{U}) \subset \mathbb{D}$. Then, for $q(z) = z + \sqrt{1+z^2}$, we find from (2.6) and (2.7) that $zq'(z)\cdot\varphi\,[q(z)]$ is starlike in $\mathbb{U}$, and the condition

$$zp'(z)\cdot\varphi\,[p(z)] \prec zq'(z)\cdot\varphi\,[q(z)]$$

yields the subordination $\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}$. Now, by Lemma 1.2, we have $p \prec q$, which in view of (1.3) establishes the desired result (2.9). □

In particular, choosing the substitutions: $p(z) = \frac{zf'(z)}{f(z)}$ and $p(z) = 1 + \frac{zf''(z)}{f'(z)}$, respectively, in Theorems 2.1 and 2.2, we get the following results for the classes Δ^* and Δ^e, defined above by (1.4) and (1.5), respectively.

Corollary 2.1. *If* $f \in \mathcal{A}$ *satisfies the condition:*

$$1 + \frac{zf''(z)}{f'(z)} \prec z + \sqrt{1+z^2} + \frac{z}{\sqrt{1+z^2}}, \quad z \in \mathbb{U},$$

then $f \in \Delta^*$.

Corollary 2.2. *If $f \in \mathcal{A}$ satisfies the condition:*

$$1+\frac{z\left(zf'(z)\right)''}{\left(zf'(z)\right)'} \prec z+\sqrt{1+z^2}+\frac{z}{\sqrt{1+z^2}}, \quad z \in \mathbb{U},$$

then $f \in \Delta^{\mathcal{C}}$.

Corollary 2.3. *If $f \in \mathcal{A}$ satisfies the condition:*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{z}{\sqrt{1+z^2}}, \quad z \in \mathbb{U},$$

then $f \in \Delta^{*}$.

Corollary 2.4. *If $f \in \mathcal{A}$ satisfies the condition:*

$$\frac{z\left(zf'(z)\right)''}{\left(zf'(z)\right)'}-\frac{zf''(z)}{f'(z)} \prec \frac{z}{\sqrt{1+z^2}}, \quad z \in \mathbb{U},$$

then $f \in \Delta^{\mathcal{C}}$.

3. Certain coefficient estimates

In this section we obtain some coefficient estimates for functions belonging to the classes Δ, Δ^* and $\Delta^{\mathcal{C}}$. To this end, we use certain inequalities stated in the following lemma (see [15, p. 108] and [11, p. 10]).

Lemma 3.1. *Let $w(z)$ be a Schwarz function given by*

$$w\left(z\right)=w_1 z+w_2 z^2+w_3 z^3+\ldots \; \left(z \in \mathbb{U}\right), \tag{3.1}$$

then$|w_1| \leq 1, \left|w_2-tw_1^2\right| \leq 1+\left(|t|-1\right)|w_1|^2 \leq \max\left\{1,|t|\right\}$, where $t \in \mathbb{C}$. The result is sharp for functions $w\left(z\right)=z$ or $w\left(z\right)=z^2$.

Theorem 3.1. *Let $p \in \mathcal{H}$ be of the form:*

$$p(z)=1+\sum_{n=1}^{\infty} p_n z^n. \tag{3.2}$$

If $p \in \Delta$, where Δ is defined by (1.3), then $|p_i| \leq 1$ for $i=1,2$.

Proof. Let $p \in \Delta$ be of the form (3.2), then there exists some Schwarz function $w(z)$ of the form (3.1) with $|w(z)|<1$ such that

$$\left(p(z)\right)^2-1=2p(z)\cdot w(z).$$

Hence, we have

$$\left(1+\sum_{n=1}^{\infty} p_n z^n\right)^2-1=2\left(1+\sum_{n=1}^{\infty} p_n z^n\right)\left(w_1 z+w_2 z^2+w_3 z^3+\ldots\right).$$

Comparing the coefficients of z and z^2 on both sides of the last equality, we get

$$p_1 = w_1 \text{ and } 2p_2 + p_1^2 = 2(w_2 + p_1 w_1).$$

Now, applying Lemma 3.1, we obtain

$$|p_1| = |w_1| \leq 1 \text{ and } |p_2| = \left|w_2 + \frac{w_1^2}{2}\right| \leq 1,$$

and the result follows. □

Using Theorem 3.1, we obtain the following result that contains coefficient estimates for functions belonging to the class Δ^*.

Corollary 3.1. *Let $f \in \Delta^*$ be defined by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{3.3}$$

then

$$|a_2| \leq 1 \text{ and } \left|a_3 - \frac{1}{2}a_2^2\right| \leq \frac{1}{2}. \tag{3.4}$$

Proof. Let

$$p(z) = \frac{zf'(z)}{f(z)}$$

be of the form (3.2), then $p \in \Delta$. Hence, using the series expansions (3.2) and (3.3), we obtain

$$1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = \left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\left(1 + \sum_{n=2}^{\infty} a_n z^{n-1}\right).$$

Comparing the coefficients of z and z^2 on both sides of the last equality, we get

$$2a_2 = p_1 + a_2 \text{ and } 3a_3 = p_2 + p_1 a_2 + a_3. \tag{3.5}$$

Now, applying Theorem 3.1, from (3.5) we obtain $|p_1| = |a_2| \leq 1$ and $|p_2| = |2a_3 - a_2^2| \leq 1$. This proves the estimates in (3.4). □

We observe that if $f \in \Delta^e$, then $zf' \in \Delta^*$. Hence, applying Corollary 3.1 to function zf', and replacing a_n by na_n in (3.4), we obtain coefficient estimates for functions $f \in \Delta^e$, which are given by the following corollary.

Corollary 3.2. *Let $f \in \Delta^e$ be of the form (3.3), then*

$$|a_2| \leq \frac{1}{2} \text{ and } \left|a_3 - \frac{2}{3}a_2^2\right| \leq \frac{1}{6}.$$

The next theorem contains coefficient estimates in the case where the function $g(z)$ has a specific form.

Theorem 3.2. *Let $g \in \mathcal{P}$ be of the form $g(z) = 1 + c_n z^n$ $(|c_n| < 1, n \in \mathbb{N})$, then $g \in \Delta$ if and only if*

$$|c_n| \le 2 - \sqrt{2}. \tag{3.6}$$

Proof. Define a function $\omega(z)$ to satisfy

$$g(z) = 1 + c_n z^n = \omega(z) + \sqrt{1 + (\omega(z))^2}, \quad z \in \mathbb{U}, \tag{3.7}$$

and observe that $\omega(z)$ is analytic in $\mathbb{U}$ with $\omega(0) = 0$. To show that $g \in \Delta$, we need to prove that $|\omega(z)| < 1$. By simple calculations, from (3.7) we find

$$\omega(z) = c_n z^n \left(1 + \frac{1}{1 + c_n z^n}\right), \quad z \in \mathbb{U},$$

showing that

$$|\omega(z)| < \frac{|c_n|}{2}\left(1 + \frac{1}{1 - |c_n|}\right) \le \frac{2 - \sqrt{2}}{2}\left(1 + \frac{1}{1 - (2 - \sqrt{2})}\right) = 1, \tag{3.8}$$

provided that the condition (3.6) holds. Conversely, if $g \in \Delta$, then in view of (3.7), we have $|\omega(z)| < 1$, $z \in \mathbb{U}$, which, in view of (3.8), shows that

$$\frac{|c_n|}{2}\left(1 + \frac{1}{1 - |c_n|}\right) \le 1,$$

or$(|c_n| - 2)^2 \ge 2$. This implies that either $|c_n| \ge 2 + \sqrt{2}$ or $|c_n| \le 2 - \sqrt{2}$. Since $g \in \mathcal{P}$, in view of Carathéodory condition $|c_n| \le 2$ (see [6, p. 41]), we obtain the inequality (3.6). □

Applying Theorem 3.2, we can determine certain functions belonging to the classes Δ^* and $\Delta^{\mathcal{C}}$ as follows.

Corollary 3.3. *Let $f \in \mathcal{A}$ be of the form:*

$$f(z) = z \exp(az) \quad (|a| < 1, \ z \in \mathbb{U}), \tag{3.9}$$

then $f \in \Delta^*$ if and only if $|a| \le 2 - \sqrt{2}$.

Proof. Observe that for $f \in \mathcal{A}$ of the form (3.9), we have

$$\frac{zf'(z)}{f(z)} = 1 + az =: g(z).$$

Hence, by Theorem 3.2, $g \in \Delta$ or $f \in \Delta^*$ if and only if $|a| \le 2 - \sqrt{2}$. □

Similarly, we can prove the following result.

Corollary 3.4. *Let $f \in \mathcal{A}$ be such that*

$$f'(z) = \exp(bz) \quad (|b| < 1, \ z \in \mathbb{U}), \tag{3.10}$$

then $f \in \Delta^{\mathcal{C}}$ if and only if $|b| \leq 2 - \sqrt{2}$.

Acknowledgement: The authors sincerely thank the referee for a number of useful comments and suggestions.

Список литературы

[1] M. K. Aouf, J. Dziok and J. Sokół, "On a subclass of strongly starlike functions", Appl. Math. Letters **24**, 27 – 32 (2011).

[2] J. Dziok, R. K. Raina and J. Sokół, "On alpha-convex functions related to shell-like functions connected with Fibonacci numbers", Appl. Math. Comput. **218**, 966 – 1002 (2011).

[3] J. Dziok, R. K. Raina and J. Sokół, "Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers", Comput. Math. Appl. **61** (9), 2605 – 2613 (2011).

[4] J. Dziok, R. K. Raina and J. Sokół, "On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers", Math. Comput. Modelling **57**, 1203 – 1211 (2013).

[5] J. Dziok, R. K. Raina and J. Sokół, "Differential subordinations and alpha-convex functions", Acta Math. Scientia **33B**, 609 – 620 (2013).

[6] P. L. Duren, Univalent Functions, New York, Springer-Verlag (1983).

[7] S. Kanas, "Alternative characterization of the class k-UCV and related classes of univalent functions", Serdica Math. J. **25**, 341 – 350 (1999).

[8] S. Kanas and A. Wiśniowska, "Conic regions and k-uniform convexity", J. Comput. Appl. Math., **105**, 327 – 336 (1999).

[9] S. Kanas and H. M. Srivastava, "Linear operators associated with k-uniformly convex functions", Integral Transform. Spec. Funct. **9**, 121 – 132 (2000).

[10] S. Kanas and A. Wiśniowska, "Conic domains and starlike functions", Rev. Roumaine Math. Pures Appl. **45**, 647 – 657 (2000).

[11] F. R. Keogh and E. P. Merkes, "A coefficient inequality for certain classes of analytic functions", Proc. Amer. Math. Soc. **20**, 8 – 12 (1969).

[12] W. Ma and D. A. Minda, "Unified treatment of some special classes of univalent functions", In: Proceedings of the Conference on Complex Analysis, Eds. Li, Z., Ren, F., Yang, L. and Zhang, S. Int. Press, 157 – 169 (1994).

[13] S. S. Miller and P. T. Mocanu, "On some classes of first order differential subordinations", Michigan Math. J. **32**, 185 – 195 (1985).

[14] S. S. Miller and P. T. Mocanu, Differential Subordinations, Theory and Applications, Series of Monographs and Textbooks in Pure and Applied Mathematics, **225**, Marcel Dekker Inc., New York / Basel (2000).

[15] Z. Nehari, Conformal Mapping, McGraw-Hill, New York (1952).

[16] R. K. Raina and J. Sokół, "Some properties related to a certain class of starlike functions", C. R. Acad. Sci. Paris, Ser. I **353**, 973 – 978 (2015).

[17] J. Sokół, "Coeffcient estimates in a class of strongly starlike functions", Kyungpook Math. J. **49**, 349 – 353 (2009).

Поступила 19 июня 2016