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SHARP NORM ESTIMATES FOR WEIGHTED BERGMAN PROJECTIONS IN THE MIXED NORM SPACES

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Abstract. In this paper, we show that the norm of the Bergman projection on $L^{p,q}$-spaces in the upper half-plane is comparable to $\csc(\pi/q)$. Then we extend this result to a more general class of domains, known as the homogeneous Siegel domains of type II.

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1. Introduction

In the recent paper [15], K. Zhu has obtained sharp norm estimates for Bergman projection on L^p-spaces in the unit ball of $\mathbb{C}^n$. In this paper, we first extend this result to $L^{p,q}$-spaces in the upper half-plane, that is, $\Pi_+ = \{z = x + iy,\ x \in \mathbb{R},\ y > 0\}$, and then, the obtained result we extend to a more general class of domains, known as the homogeneous Siegel domains of type II.

It will be convenient to introduce the mixed normed spaces for functions defined on Π_+. Let $0 < p, q \leq \infty$ and $\nu > 0$, and let $f(x+iy)$ be a measurable function on Π_+. Then, with the usual conventions if $p = \infty$ or $q = \infty$, we denote

$$\|f\|_{p,q,\nu} = \left(\int_0^{+\infty} \left(\int_{-\infty}^{+\infty} |f(x+iy)|^p dx \right)^{q/p} y^{\nu-1} dy \right)^{1/q}.$$

Definition 1.1. *For all $0 < p, q \leq \infty$, the mixed normed space $L^{p,q}_\nu$ is defined to be the set of measurable functions on Π_+ such that $\|f\|_{p,q,\nu} < \infty$. The space $A^{p,q}_\nu$ is defined to be the set of holomorphic functions on Π_+ such that $\|f\|_{p,q,\nu} < \infty$.*

It is worth to observe that these spaces were extensively studied in the literature (see [1, 2, 4] – [8]). For instance, in [2] it was proved that $A^{p,q}_\nu = \{0\}$ if and only if $\nu \leq 0$, and that the orthogonal projection P_ν from the Hilbert space $L^{2,2}_\nu = L^2_\nu$ onto

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the space $A_\nu^{2,2} = A_\nu^2$ can be extended to a bounded operator on $L_\nu^{p,q}$ if and only if $1 < p, q < \infty$. Also, the explicit expression of P_ν is given by the following formula:

$$P_\nu f(z) f = \int_0^{+\infty} \int_{-\infty}^{+\infty} B_\nu(z, u+iv) f(u+iv) v^{\nu-1} du dv,$$

where

$$B_\nu(z,w) = \frac{2^{\nu-1}\nu}{\pi} \left(\frac{z - \bar{w}}{i} \right)^{-\nu-1}.$$

The operator P_ν is called the weighted Bergman projection.

Our first main result is the following theorem.

Theorem 1.1. *Let $1 < p, q < \infty$ and $\nu > 0$. Then there exist positive constants C_1 and C_2 independent of p and q such that*

$$C_1 \csc(\pi/q) \leq \|P_\nu\| \leq C_2 \csc(\pi/q). \tag{1.1}$$

2. Proof of Theorem 1.1

The proof of the theorem is based on two estimates stated below. One of them is the refined version of the Schur lemma (see [16]). The other is an optimal pointwise estimate for functions from $A_\nu^{p,q}$ (see [8]).

Lemma 2.1. [16] *Suppose $H(x,y)$ is a positive kernel and*

$$Tf(x) = \int_X H(x,y) f(y) d\mu(y)$$

is the associated integral operator. Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. If there exists a positive function $\varphi(x)$ and two positive constants C_1 and C_2 such that

$$\int_X H(x,y)(\varphi(y))^q d\mu(y) \leq C_1 (\varphi(x))^q, \quad x \in X$$

$$\int_X H(x,y)(\varphi(x))^p d\mu(x) \leq C_2 (\varphi(y))^p, \quad y \in X,$$

then the operator T is bounded on $L^p(X, d\mu)$. Moreover, the norm of the operator T on $L^p(X, d\mu)$ does not exceed $C_1^q C_2^p$.

Proposition 2.1. [8] *Let $1 < p, q < \infty$ and $\nu > 0$. Then there exists a positive constant C independent of p and q such that $|f(x+iy)| \leq C y^{-\frac{\nu}{q} - \frac{1}{p}} \|f\|_{p,q,\nu}$ for all $f \in A_\nu^{p,q}$ and $x + iy \in \Pi_+$.*

In this section we prove our first main result - Theorem 1.1.

Proof of Theorem 1.1. Denote $f_y(x) = f(x+iy)$ and

$$P_\nu^+ f(z) = \int_0^{+\infty}\int_{-\infty}^{+\infty} |B_\nu(z, u+iv)| f(u+iv) v^{\nu-1} du dv.$$

Then we can write

$$\|P_\nu^+ f\|_{p,q,\nu} = \left(\int_0^{+\infty} \|(P_\nu^+ f)_y\|^q_{L^p(\mathbb{R})} y^{\nu-1} dy\right)^{1/q}$$

and

$$(P_\nu^+ f)_y(x) = c\int_0^{+\infty} (g_{y+v} * f_v(x)) v^{\nu-1} dv,$$

where $c = \frac{2^{\nu-1}\nu}{\pi}$ and $g_{y+v}(x) = \left(\frac{x+i(y+v)}{i}\right)^{-\nu-1}$. Using Minkowski and Young inequalities we obtain

$$\|(P_\nu^+ f)_y\|_{L^p(\mathbb{R})} \le c\int_0^{+\infty} \|g_{y+v}\|_{L^1(\mathbb{R})} \|f_v\|_{L^p(\mathbb{R})} v^{\nu-1} dv.$$

Moreover, simple calculations yield:

$$\|g_{y+v}\|_{L^1(\mathbb{R})} = \frac{\sqrt{\pi}\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})}(v+y)^{-\nu}.$$

Therefore

$$\|P_\nu^+ f\|^q_{p,q,\nu} \le C^q \int_0^{+\infty}\left(\int_0^{+\infty} (y+v)^{-\nu} \|f_v\|_{L^p(\mathbb{R})} v^{\nu-1} dv\right)^q y^{\nu-1} dy, \tag{2.1}$$

where $C = \frac{2^{\nu-1}\nu\Gamma(\frac{\nu}{2})}{\sqrt{\pi}\Gamma(\frac{\nu+1}{2})}$.

Now we introduce the ingredients for Schur's lemma:

$$Th(y) = \int_0^{+\infty} (y+v)^{-\nu} h(v) v^{\nu-1} dv$$

and $\varphi(v) = v^{-\frac{\nu}{qq'}}$. Then it is easy to see that

$$\begin{aligned}\int_0^{+\infty} (y+v)^{-\nu}\varphi^{q'}(v) v^{\nu-1} dv &= \frac{\Gamma(\frac{\nu}{q})\Gamma(\frac{\nu}{q'})}{\Gamma(\nu)}\varphi^{q'}(y),\\ \int_0^{+\infty} (y+v)^{-\nu}\varphi^{q}(y) y^{\nu-1} dy &= \frac{\Gamma(\frac{\nu}{q})\Gamma(\frac{\nu}{q'})}{\Gamma(\nu)}\varphi^{q}(v),\end{aligned}$$

and consequently

$$\|Th\|_{L^q(0,+\infty)} \le \frac{\Gamma(\frac{\nu}{q})\Gamma(\frac{\nu}{q'})}{\Gamma(\nu)} \|h\|_{L^q(0,+\infty)}. \tag{2.2}$$

We easily deduce from (2.1) and (2.2) that

$$\|P_\nu f\|_{p,q,\nu} \le C\Gamma\left(\frac{\nu}{q}\right)\Gamma\left(\frac{\nu}{q'}\right)\|f\|_{p,q,\nu}, \tag{2.3}$$

where

$$C = \frac{2^{\nu-1}\nu\Gamma(\frac{\nu}{2})}{\sqrt{\pi}\Gamma(\nu)\Gamma(\frac{\nu+1}{2})}.$$

Now the second inequality in (1.1), that is, $\|P_\nu\| \leq C\csc(\pi/q)$ follows from (2.3) and the inequality

$$\Gamma\left(\frac{\nu}{q}\right)\Gamma\left(\frac{\nu}{q'}\right) \leq C\csc(\pi/q). \tag{2.4}$$

The proof of (2.4) is given in the second part of the paper. Note that in the above inequalities, C is a positive constant independent of p and q. To prove the first inequality in (1.1), we first apply Lemma 2.1 to get

$$\|P_\nu\| \geq \frac{Cy^{\frac{\nu}{q}+\frac{1}{p}}|P_\nu f(x+iy)|}{\|f\|_{p,q,\nu}}. \tag{2.5}$$

Then taking $f(x+iy) = \frac{\pi}{2^{\nu-1}\nu}y^{1-\nu}\chi_{D(i,\frac{1}{2})}(x+iy)$ and $D\left(i,\frac{1}{2}\right) = \left\{z \in \mathbb{C} : |z-i| < \frac{1}{2}\right\}$, from the mean value property and some easy calculations, we obtain

$$P_\nu f(x+iy) = \frac{\pi}{4}\left(\frac{x+i(y+1)}{i}\right) \quad \text{and} \quad \|f\|_{p,q,\nu} \leq C \tag{2.6}$$

for all $x+iy \in \Pi_+$.

Finally, combining (2.5) and (2.6), and taking $x = e^{-q}$ and $\frac{1}{2} < y < \frac{3}{2}$, we obtain $\|P_\nu\| \geq Cq \geq C\csc(\pi/q)$ for all $q > 2$. In the case $1 < q \leq 2$ the result follows from duality argument. □

3. Bergman projection and Siegel domains

We fix a positive integer $n \geq 3$ and denote by D a domain in $\mathbb{C}^n$. We use dv to denote the Lebesgue measure defined in $\mathbb{C}^n$ and P to denote the orthogonal projection from the Hilbert space $L^2(D, dv)$ onto the space $A^2(D, dv)$, consisting of holomorphic functions on D. It is well-known that P is an integral operator defined on $L^2(D, dv)$. The orthogonal projection P is called *Bergman projection* and its kernel K is called *Bergman kernel*. In the following, D will be a homogeneous Siegel domain of type II. The goal of the second part of this paper is to extend the result obtained by K. Zhu [15], to the Siegel domains.

The main object in this part of the paper is the Siegel domain associated with a homogeneous cone. So, in this section, we recall the description of an open strictly convex homogeneous cone from T-algebra, introduce the notion of a homogeneous Siegel domain of type II, and state our second main result.

3.1. Homogeneous cone. We use the same notation as in [13]. We consider a (real) matrix algebra $\mathfrak{U}$ of rank k with canonical decomposition:

$$\mathfrak{U} = \bigoplus_{1\leq i,j\leq r} \mathfrak{U}_{i,j}$$

such that $\mathcal{U}_{i,j}\mathcal{U}_{j,k} \subset \mathcal{U}_{i,k}$ and $\mathcal{U}_{i,j}\mathcal{U}_{l,k} = 0$ for $j \neq l$. We assume that $\mathcal{U}$ has the structure of T-algebra (in the sense of [9]), in which an involution is given by $x \mapsto x^*$. This structure implies that the subspaces $\mathcal{U}_{i,j}$ satisfy the relation $\mathcal{U}_{i,i} = \mathbb{R}c_i$, where $c_i^2 = c_i$ and $\dim \mathcal{U}_{i,j} = n_{i,j} = n_{j,i}$. Also, the matrix $e = \sum_{j=1}^r c_j$ is a unit element for the algebra $\mathcal{U}$. Let ρ be the unique isomorphism from $\mathcal{U}_{i,i}$ onto $\mathbb{R}$ with $\rho(c_i) = 1$ for all $i = 1, \cdots, r$. We consider the subalgebra $\mathcal{T} \subset \mathcal{U}$ consisting of upper triangular matrices, and let

$$H = \{t \in \mathcal{T} : \rho(t_{i,i}) > 0,\ i = 1, \cdots, r\}$$

be the subgroup of upper triangular matrices whose diagonal element are positive.

Denote by V the vector space of Hermitian matrices in $\mathcal{U}$ $V = \{x \in \mathcal{U} : x^* = x\}$. We set $n_i = \sum_{j=1}^{i-1} n_{ji}$, $m_i = \sum_{j=i+1}^{r} n_{ij}$, and observe that

$$\dim V = n = r + \sum_{i=1}^{r} m_i = r + \sum_{i=1}^{r} n_i.$$

The vector space V becomes an Euclidean space with the inner product: $(x|y) = \mathrm{tr}(xy^*)$, where $\mathrm{tr}(x) = \sum_{i=1}^r \rho(x_{ii})$. Next, we define $\Omega = \{ss^* : s \in H\}$, and observe that, by a theorem of Vinberg (see [14], p. 384), Ω is an open convex homogeneous cone containing no entire straight lines, in which the group H acts simply transitively via the transformation:

$$\pi(w) : uu^* \mapsto \pi(w)[uu^*] = (wu)(u^*w^*) \qquad (w, u \in H).$$

Thus, to every element $y \in \Omega$ corresponds a unique $t \in H$ such that $y = \pi(t)[e] = t \cdot e$.

We assume that Ω is irreducible, and hence rank $(\Omega) = r$. Note that all homogeneous convex cones can be constructed in this way (see [14] p. 397). As in [13], we denote by Q_j the fundamental rational functions in Ω given by $Q_j(y) = \rho(t_{jj})^2$, when $y = t \cdot e \in \Omega$.

We consider the matrix algebra with involution $\mathcal{U}'$ which differs from $\mathcal{U}$ only on its grading, and put $\mathcal{U}'_{ij} = \mathcal{U}_{r+1-i,r+1-j}$ $(i, j = 1, \cdots, r)$. In [14], it was proved that $\mathcal{U}'$ is also a T-algebra and $V' = V$, where V' is the subspace of $\mathcal{U}'$ consisting of Hermitian matrices. We define the subalgebra

$$\mathcal{T}' = \bigoplus_{1 \leq i \leq j \leq r} \mathcal{U}'_{i,j}$$

of $\mathcal{U}'$, consisting of lower triangular matrices, and the subgroup H' of $\mathcal{T}'$ whose diagonal elements are positive. We have $\mathcal{T}' = \{t^* : t \in \mathcal{T}\}$ and $H' = \{t' : t \in H\}$. The corresponding homogeneous cone coincides with the dual cone of Ω, namely

$$\Omega^\star = \{\xi \in V' : (x|\xi) > 0, \qquad \forall x \in \bar{\Omega}\backslash 0\}.$$

One also has (see [14], p. 390) $\Omega^\star = \{t^\star t : t \in H\}$. For $\xi = t^\star t \in \Omega^\star$, we define $Q_j^\star(\xi) = \rho(t_{jj}^2)$, and observe that the following identity holds: $Q_j^\star(t^\star \cdot e) = Q_j(t \cdot e)$. We use the following notation: for all $x \in \Omega$, $\xi \in \Omega^\star$ and $\alpha = (\alpha_1, \cdots, \alpha_r) \in \mathbb{R}$, we set

$$Q^\alpha(x) = \prod_{j=1}^r Q_j^{\alpha_j}(x) \qquad \text{and} \qquad (Q^\star)^\alpha(\xi) = \prod_{j=1}^r (Q_j^\star)^{\alpha_j}(\xi).$$

We put $\tau = (\tau_1, \cdots, \tau_r) \in \mathbb{R}^r$ with $\tau_i = 1 + \frac{1}{2}(m_i + n_i)$. For $y \in \Omega$ and $j = 1, \cdots, r$, we have $Q_j(\pi(t)y) = Q_j(x)Q_j(y)$. Therefore, for any $s \in H$, we get $Q^\tau(\pi(s)x) = \det\pi(s)Q^\tau(x)$ since (see [14], p. 388) $\det\pi(s) = Q^\tau(s \cdot e)$. Note that the above properties are also valid if we replace Q_j by $Q_j^\star$ and $x \in \Omega$ by $\xi \in \Omega^\star$. In the following we call e_Ω the element e.

3.2. **Homogeneous Siegel domains.** Let $V^{\mathbb{C}} = V + iV$ be the complexification of V. Then each element of $V^{\mathbb{C}}$ is identified with a vector in $\mathbb{C}^n$. The coordinates of a point $z \in \mathbb{C}^n$ are arranged in the form: $z = (z_1, \cdots, z_n)$, where $z_j = (z_{1j}, \cdots, z_{j-1,j})$, $j = 2, \cdots, r$, and

$$z_{jj} \in \mathbb{C}, \; z_{ij} = (z_{ij}^{(1)}, \cdots, z_{ij}^{(n_{ij})}) \in \mathbb{C}^{n_{ij}}, \; 1 \le i < j < r.$$

For all $j = 1, \cdots, r$ we set $e_{jj} = z$, where $z_{jj} = 1$ and the other coordinates are equal to zero, and denote

$$e_\Omega = \sum_{j=1}^r e_{jj} = (1, 0, 1, \cdots, 0, 1).$$

Let $m \in \mathbb{N}$. For each row vector $u \in \mathbb{C}^m$, we denote by u' the transpose of u. Given $m \times m$ Hermitian matrices $\tilde{H}_1, \cdots, \tilde{H}_n$, we define an Ω-Hermitian homogeneous form $F : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ as $F(u, v) = (u\tilde{H}_1\bar{v}', \cdots, u\tilde{H}_n\bar{v}')$, $(u, v) \in \mathbb{C}^m \times \mathbb{C}^m$, such that

(i) $F(u, u) \in \bar{\Omega}$;

(ii) $F(u, u) = 0$ if only if $u = 0$;

(iii) for every $t \in H$, there exists $\tilde{t} \in GL(m, \mathbb{C})$ such that $t \cdot F(u, u) = F(\tilde{t}u, \tilde{t}u)$.

The homogeneous Siegel domain of type II associated with the cone Ω and with a V−Hermitian homogeneous form $F : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ is defined by

$$D(\Omega, F) = \{(z, u) \in \mathbb{C}^n \times \mathbb{C}^m : \Im m z - F(u, u) \in \Omega\}.$$

Using the above decomposition of an element in $\mathbb{C}^m$, we can write

$$F(u,u) = (F_{11}(u,u), F_2(u,u), F_{22}(u,u), \cdots, F_r(u,u), F_{rr}(u,u)),$$

where for $i = 1, \cdots, r$ and $j = 2, \cdots, r$,

$$F_{ii}(u,u) = u\tilde{H}_i\bar{u}', \; F_j(u,u) = (F_{1j}(u,u), \cdots, F_{j-1,j}(u,u))$$

and for $1 \le i < j \le r$ and $t = 1, \cdots, n_{ij}$,

$$F_{ij}^{(t)}(u,u) = u\tilde{H}_{ij}^{(t)}\bar{u}', \; F_{jj}(u,u) = (F_{ij}^{(1)}(u,u), \cdots, F_{ij}^{(n_{ij})}(u,u)).$$

We have the decomposition $\mathbb{C}^m = \prod_{i=1}^r \mathbb{C}_i$, where $\mathbb{C}_i$ is the subspace of $\mathbb{C}^n$ on which F_{ii} is positive definite. In what follows, we denote by b the vector $(b_1, \cdots, b_r) \in \mathbb{N}^r$ and by D the Siegel domain of second kind associated with the open convex homogeneous cone Ω and the Ω-Hermitian homogeneous form F.

3.3. Statement of the second main result. For each $(z,u) \in D$, we adopt the following notation:

$$dV_\nu(z,u) = Q^{\nu-\frac{b}{2}-\tau}(\Im m z - F(u,u))dv(z)dv(u)$$

with the convention that if $y = \Im m z$, then

$$dV_\nu(y,u) = Q^{\nu-\frac{b}{2}-\tau}(y - F(u,u))dy dv(u),$$

where dv is the Lebesgue measure on $\mathbb{C}^l$, and $l = n$ or $l = m$.

For $p, q \in [1, +\infty]$ and $\nu \in \mathbb{R}^r$, let $L_\nu^{p,q}$ denote the (Banach) space of measurable functions on D such that

$$\|f\|_{p,q,\nu} := \left(\int_{\mathbb{C}^m}\int_{\Omega+F(u,u)}\left(\int_V |f(x+iy,u)|^p dx\right)^{q/p} dV_\nu(y,u)\right)^{1/q} < +\infty.$$

We define the weighted Bergman space $A_\nu^{p,q}$ to be the subspace of $L_\nu^{p,q}$, formed by its holomorphic functions. Observe that $Q^{b-2\tau}(\Im m z - F(u,u))dv(z)dv(u)$ is the invariant measure with respect to the group of automorphism of D (see [9], p. 56). We denote by P_ν the integral operator on $L_\nu^{p,q}$ defined by

$$P_\nu f(z) = \int_D B_\nu((z,u),(w,t))f(w,t)dV_\nu(w,t),$$

and by P_ν^+ we denote the *weighted Bergman projection*, defined by

$$P_\nu^+ f(z) = \int_D |B_\nu((z,u),(w,t))|\, f(w,t)dV_\nu(w,t),$$

where

$$B_\nu((z,u),(\omega,v)) = d_{\nu,b}Q^{-\nu-\frac{b}{2}-\tau}\left(\frac{z-\bar{\omega}}{2i} - F(u,v)\right)$$

is the *weighted Bergman kernel*, that is, the reproducing kernel of $A^2_\nu(D)$.

Also, we denote by $\|P_\nu\|$ the norm of P_ν on $L^{p,q}_\nu$. It is well known (see, e.g., [12]) that P^+_ν and P_ν can be extended to bounded operators on $L^{p,q}_\nu$ for some $p, q \in [1, +\infty]$ and $\nu = (\nu_1, \cdots, \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j+n_j+b_j}{2}$, $j = 1, \cdots, r$.

The following theorem is the second main result of this paper. The proof is given in Section 5.

Theorem 3.1. *If P^+_ν is extended to a bounded operator on $L^{p,q}_\nu$, then there exist positive constants C_1 and C_2, depending only on ν, m and n but not on p and q, such that*

$$C_1 \csc^r(\pi/q) \le \|P_\nu\| \le C_2 \csc^r(\pi/q).$$

Now we state two lemmas and a proposition, proved in [13] and [12], which will play a key role in our analysis in the subsequent parts of this paper. We first adopt the following notation for the generalized gamma functions:

$$\Gamma_\Omega(\alpha) = \int_\Omega e^{-(e|x)} Q^{\alpha-\tau}(x)dx = \pi^{\frac{n-r}{2}} \prod_{i=1}^{r} \Gamma\left(\alpha_i - \frac{m_i}{2}\right), \qquad \alpha_i > \frac{m_i}{2},$$

$$\Gamma_{\Omega^*}(\alpha) = \int_\Omega e^{-(e|x)} (Q^*)^{\alpha-\tau}(\xi)d\xi = \pi^{\frac{n-r}{2}} \prod_{i=1}^{r} \Gamma\left(\alpha_i - \frac{n_i}{2}\right), \qquad \alpha_i > \frac{n_i}{2},$$

where Γ is the usual gamma function.

Lemma 3.1. [13, Lemma 4.20]. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r$. The integral*

$$J_\alpha(y) = \int_V \left| Q^{-\alpha}\left(\frac{x+iy}{i}\right)\right| dx \qquad (y \in \Omega)$$

converges if and only if $\alpha_j > 1 + n_j + \frac{m_j}{2}$, $j = 1, \ldots, r$. In this case, we have $J_\alpha(y) = c_\alpha Q^{-\alpha+\tau}(y)$, where

$$c_\alpha = \frac{(2\pi)^n 2^{-|\alpha|+|\tau|} \Gamma_{\Omega^*}(\alpha - \tau)}{\Gamma^2_{\Omega^*}(\alpha/2)}.$$

Lemma 3.2. [13, Lemma 4.19]. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathbb{R}^r$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{R}^r$. For all $y \in \Omega$, the integral*

$$J_{\mu\lambda}(y) = \int_\Omega Q^\mu(y+v) Q^{\lambda-\tau}(v)dv$$

is finite if and only if $\lambda_j > \frac{m_j}{2}$, $\mu_j + \lambda_j < -\frac{n_j}{2}$, $j = 1, \ldots, r$. In this case, we have $J_{\mu\lambda}(y) = M_{\lambda\mu} Q^{\mu+\lambda}(y)$, where

$$M_{\lambda\mu} = \frac{\Gamma_\Omega(\lambda)\Gamma_{\Omega^*}(-\mu-\lambda)}{\Gamma_{\Omega^*}(-\mu)}.$$

Proposition 3.1. [12, Proposition 5.2]. *Let $u, t \in \mathbb{C}^m$, $y \in \Omega + F(u,u)$ and $\tilde{y} \in \Omega + F(t,t)$. For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in \mathbb{R}^r$, the integral*

$$I(y,u,\tilde{y}) = \int_{\mathbb{C}^m} Q^{-\lambda}(y + \tilde{y} + F(t,t) - 2\Re e\, F(u,t))dv(t)$$

converges if $\lambda_j - b_j > \frac{n_j}{2}$, $j = 1, \ldots, r$. In this case, there is a positive constant C_λ such that $I(y,u,\tilde{y}) = C_\lambda Q^{-\lambda+b}(y - F(u,u) + \tilde{y})$.

4. An optimal pointwise estimate in Siegel domains

The proof of our second main result - Theorem 3.1, requires pointwise estimates in tubular (resp. Siegel) domains. So, we need more precise versions of pointwise estimates in the above domains.

Lemma 4.1. *Let $\nu = (\nu_1, \cdots, \nu_r) \in \mathbb{R}^r$ be such that $\nu_j > \frac{m_j + n_j}{2}$, $j = 1, \cdots, r$, and let f be a holomorphic function on tube domains $T_\Omega := V + i\Omega$ (the so-called Siegel domains of type I), such that*

$$\|f\|_{p,q,\nu} = \left(\int_\Omega \left(\int_V |f(x+iy)|^q dx\right)^{q/p} Q^{\nu-\tau}(y)dy\right)^{1/q} < +\infty.$$

Then there is a positive constant C independent of p and q such that

$$|f(x+iy)| \leq C Q^{-\frac{\nu}{q}-\frac{\tau}{p}}(y)\|f\|_{p,q,\nu}$$

for all $x + iy \in T_\Omega$.

Proof. Let $t^{\mathbb{C}}$ be an extension of $t = \pi(s)$ to $V^{\mathbb{C}} = V + iV$, defined as follows: $t^{\mathbb{C}}(x+iy) = tx + ity$ and $t^{\mathbb{C}}(iy) = ity$ for all $x + iy \in V^{\mathbb{C}}$. Then using the mean value property, the Hölder inequality and formulas (2.9) – (2.10) from [13] (p. 484), we can write

$$\begin{aligned} f(x+iy) &= f \circ t^{\mathbb{C}}(t^{-1}x + ie_\Omega) \\ |f(x+iy)| &\leq \int_{\tilde{y} \in B(e_\Omega,1)} \int_{|\tilde{x}-t^{-1}x|<1} |f \circ t^{\mathbb{C}}(\tilde{x} + \tilde{y})| d\tilde{x} d\tilde{y} \\ &\leq C\|f \circ t^{\mathbb{C}}\|_{p,q,\nu} \leq C Q^{-\frac{\nu}{q}-\frac{\tau}{p}}(y)\|f\|_{p,q,\nu}, \end{aligned}$$

where $B(e_\Omega, 1)$ is the Bergman ball of radius 1 centered at e_Ω, and

$$C = \pi^n \sup_{y \in B(e_\Omega,1)} Q^{\tau-\nu}(y)\max(1, Q^{\tau-\nu}(y)).$$

□

Proposition 4.1. *Let $\nu = (\nu_1, \cdots, \nu_r) \in \mathbb{R}^r$ be such that $\nu_j > \frac{m_j+n_j+b_j}{2}$, $j = 1, \cdots, r$, and let f be a holomorphic function on D. Then there exists a positive constant C independent of p and q such that*

$$|f(x+iy,u)| \leq CQ^{-\frac{\nu-\frac{b}{2}}{q}-\frac{\tau}{p}}(y-F(u,u))\|f\|_{p,q,\nu}$$

for all $(x+iy,u) \in D$.

Proof. Consider the following functions: $f_1(x+iy) = f(x+iy,u)$, $f_2(u) = f(x+iy,u)$, $t(x+iy,u) = (x+i(y+F(u,u)),u)$, and use the pointwise estimate in tube domains, to obtain

$$\begin{aligned} |f(x+iy,u)| &= |(f\circ t)_1(x+i(y-F(u,u))| \\ &\leq CQ^{-\frac{\nu-\frac{b}{2}}{q}-\frac{\tau}{p}}(y-F(u,u))\|(f\circ t)_1\|_{p,q,\nu}. \end{aligned}$$

Therefore

$$\begin{aligned} \|(f\circ t)_1\|_{p,q,\nu} &= \left(\int_\Omega \left(\int_V |f(\tilde{x}+i(\tilde{y}+F(u,u)),u)|^p d\tilde{x}\right)^{q/p} Q^{\nu-\frac{b}{2}-\tau}(\tilde{y})d\tilde{y}\right)^{1/q} \\ &= \left(\int_{\Omega+F(u,u)} \left(\int_V |f_2\circ h(0)|^p d\tilde{x}\right)^{q/p} Q^{\nu-\frac{b}{2}-\tau}(\tilde{y}-F(u,u))d\tilde{y}\right)^{1/q}, \end{aligned}$$

where $h(v) = v+u$. Finally, using the mean value property for holomorphic function $f_2\circ h$, the Minkowski and Hölder inequalities, and Fubini's theorem we conclude that

$$\|(f\circ t)_1\|_{p,q,\nu} \leq \pi^{\frac{m}{2}}\|f\|_{p,q,\nu}. \qquad \square$$

The next lemma play a key role to estimate the Bergman projection in the Siegel domains. We use the following notation:

$$K_\nu((y,u),(\tilde{y},t)) = Q^{-\nu-\frac{b}{2}}(y+\tilde{y}-2\Re eF(u,t)).$$

Lemma 4.2. *There exists a positive constant C independent of p such that*

$$\|(P_\nu f)_{y,u}\|_{L^p(V,dx)} \leq C\int_{\mathbb{C}^m}\int_{\Omega+F(t,t)} K_\nu((y,u),(\tilde{y},t))\|f_{\tilde{y},t}\|_{L^p(V,dx)}dV_\nu(\tilde{y},t)$$

where $f_{y,u}(x) = f(x+iy,u)$.

Proof. We set

$$g_{y+\tilde{y},u,t}(x) = Q^{-\nu-\frac{b}{2}-\tau}\left(\frac{x+i(y+\tilde{y}-2F(u,t))}{2i}\right),$$

and use Minkowski and Young inequalities, and Lemma 3.2 to obtain

$$\begin{aligned}
\|(P_\nu f)_{y,u}\|_{L^p(V,dx)} &\leq d_{\nu,b}\left(\int_V\left|\int_{\mathbb{C}^m}\int_{\Omega+F(t,t)} g_{y+\tilde{y},(u,t)} * f_{\tilde{y},t}(x)dV_\nu(\tilde{y},t)\right|^p dx\right)^{1/p} \\
&\leq d_{\nu,b}\int_{\mathbb{C}^m}\int_{\Omega+F(t,t)}\left(\int_V |g_{y+\tilde{y},(u,t)} * f_{\tilde{y},t}(x)|^p dx\right)^{1/p} dV_\nu(\tilde{y},t) \\
&= d_{\nu,b}\int_{\mathbb{C}^m}\int_{\Omega+F(t,t)} \|g_{y+\tilde{y},(u,t)} * f_{\tilde{y},t}\|_{L^p(V,dx)}dV_\nu(\tilde{y},t) \\
&\leq d_{\nu,b}\int_{\mathbb{C}^m}\int_{\Omega+F(t,t)} \|g_{y+\tilde{y},(u,t)}\|_{L^p(V,dx)}\|f_{\tilde{y},t}\|_{L^p(V,dx)}dV_\nu(\tilde{y},t) \\
&\leq C\int_{\mathbb{C}^m}\int_{\Omega+F(t,t)} K_\nu((y,u),(\tilde{y},t))\|f_{\tilde{y},t}\|_{L^p(V,dx)}dV_\nu(\tilde{y},t),
\end{aligned}$$

where the symbol $*$ stands for convolution. □

5. Proof of Theorem 3.1

Observe first that by Lemma 4.2 we have

$$\|P_\nu f\|_{p,q} \leq \left(\int_{\mathbb{C}^m}\int_{\Omega+F(u,u)} \left(T\|f_{\tilde{y},t}\|_{L^p(V,dx)}(y,u)\right)^q dV_\nu(y,u)\right)^{1/q},$$

where

$$Tg(y,u) = \int_{\mathbb{C}^m}\int_{\Omega+F(t,t)} K_\nu((y,u),(\tilde{y},t))g(\tilde{y},t)dV_\nu(\tilde{y},t).$$

Next, it is easy to see that

$$\int_{\mathbb{C}^m}\int_{\Omega+F(u,u)} K_\nu((y,u),(\tilde{y},t))\varphi^q(y,u)dV_\nu(y,u) \leq CM\varphi^q(\tilde{y},t) \tag{5.1}$$

and

$$\int_{\mathbb{C}^m}\int_{\Omega+F(t,t)} K_\nu((y,u),(\tilde{y},t))\varphi^{q'}(\tilde{y},t)dV_\nu(\tilde{y},t) \leq CM\varphi^{q'}(y,u), \tag{5.2}$$

where $\varphi(y,u) = Q^\gamma(y - F(u,u))$, $\gamma = (\gamma_1, \cdots, \gamma_r) \in \mathbb{R}^r$. Therefore, we have

$$\gamma_j = -\frac{\nu_j - \frac{b_j}{2} - \frac{m_j}{2}}{q^2} - \frac{\frac{n_j}{2}}{qq'}, \qquad j = 1, \cdots, r$$

and

$$M = \frac{\prod_{j=1}^r \Gamma\left(\frac{\nu_j - \frac{m_j}{2} - \frac{n_j}{2} - \frac{b_j}{2}}{q}\right)\Gamma\left(\frac{\nu_j - \frac{m_j}{2} - \frac{n_j}{2} - \frac{b_j}{2}}{q'}\right)}{\Gamma_{\Omega^*}(\nu - \frac{b}{2})} \text{ with } \frac{1}{q} + \frac{1}{q'} = 1.$$

So, using Schur's lemma, we get $\|P_\nu\| \leq CM$.

Next, taking into account the symmetry of sine function and the conjugacy between q and q', we only need to consider the case where q is very large. In this case, the function $\Gamma\left(\frac{\nu_j-\frac{m_j}{2}-\frac{n_j}{2}-\frac{b_j}{2}}{q'}\right)$ is bounded from above and below, and satisfies the relation:

$$\Gamma\left(\frac{\nu_j-\frac{m_j}{2}-\frac{n_j}{2}-\frac{b_j}{2}}{q}\right)\sim q\sim \csc\frac{\pi}{q}, \qquad j=1,\cdots,r,$$

because $x\Gamma(x)=\Gamma(x+1)\sim 1$ when x is a small positive number. Thus, there exists a positive constant C_1 independent of p and q, but depending on ν, m, n and r, such that $\|P_\nu\|\le C_1\csc^r\frac{\pi}{q}$. Now, we estimate $\|P_\nu\|$ from below, assuming that $q>2$. In the case $1<q\le 2$, the estimate will follow by duality argument and the symmetry of function $\sin\frac{\pi}{q}$. Noting that for $q>2$ the constant $\sin(\pi/q)$ is comparable to $1/q$, in view of the pointwise estimate, we obtain

$$\|P_\nu\|\ge \frac{C|P_\nu f(x+iy,u)|Q^{\frac{\nu-\frac{b}{2}}{q}+\frac{\tau}{p}}(y-F(u,u))}{\|f\|_{p,q,\nu}}$$

for all $(x+iy,u)\in D$. So, taking

$$f(x+iy,u)=d_{\nu,b}^{-1}Q^{-\nu+\frac{b}{2}+\tau}(y-F(u,u))\chi_{B((ie_\Omega,0),1)}(x+iy,u),$$

where $B((ie_\Omega,0),1)$ is the Euclidean ball of radius 1 centered at $(ie_\Omega,0)$, we obtain

$$\|P_\nu\|\ge C\left|Q^{-\nu-\frac{b}{2}-\tau}\left(\frac{x+i(y+e_\Omega)}{2i}\right)\right|Q^{\frac{\nu-\frac{b}{2}}{q}+\frac{\tau}{p}}(\dot{y})$$

for all $y\in(\Omega-e_\Omega)\cap\Omega$. Here

$$P_\nu f(x+iy,u)=\pi^{\frac{2n+m}{2}}Q^{-\nu-\frac{b}{2}-\tau}\left(\frac{x+i(y+e_\Omega)}{2i}\right)$$

and $\|f\|_{p,q,\nu}\le C$. Finally, for $x=e^{-q}e_\Omega$, $u=0$ and $y\in B(e_\Omega,1)$, we get $\|P_\nu\|\ge Cq^r$, and the result follows. □

As an immediate consequence of Theorem 3.1, we can state the following result.

Corollary 5.1. *Let $\nu=(\nu_1,\cdots,\nu_r)\in\mathbb{R}^r$ be such that $\nu_j>\frac{m_j+n_j}{2}$, $j=1,\cdots,r$. If P_ν^+ is extended to a bounded operator on $L_\nu^{p,q}$-spaces of tube domains T_Ω, then there exist positive constants C_1 and C_2, depending on ν, m and n, but not on p and q, such that*

$$C_1\csc^r(\pi/q)\le\|P_\nu\|\le C_2\csc^r(\pi/q).$$

Proof. The result follows from Theorem 3.1. It suffices to see that if $F\equiv 0$ and $m=0$, then $D=T_\Omega$. □

Remark 5.1. Our main results show how fast the norm of the weighted Bergman projection P_ν on $L^{p,q}$-spaces grows as q increases and ν is fixed. Moreover, the results do not depend on p. In this respect, it would be of interest to determine how the norm of P_ν grows when ν increases and q is fixed.

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Список литературы

[1] K. L. Avetisyan, "Fractional integro-differentiation in harmonic mixed norm spaces on a half-space", Comment. Math. Univ. Carolinae **42**, no. 4, 691 – 709 (2001).

[2] D. Békollé, A. Bonami, G. Garrigós, C. Nana, M. M. Peloso, F. Ricci, "Lecture notes on Bergman projectors in tube domains over cones: an analytic and geometric viewpoint", IMHOTEP J. Afr. Math. Pures Appl., **5**(1), (2004).

[3] D. Debertol, "Besov spaces and the boundedness of weighted Bergman projections over symmetric tube domains", Publi. Math., Barc., **49**(1), 21 – 72 (2005).

[4] M. M. Djrbashian, A. E. Djrbashian, "Integral representation for some classes of functions in a half-plane [in Russian]", Dokl. Akad. Nauk SSSR **285**, 547 – 550 (1985).

[5] M. M. Djrbashian, A. H. Karapetyan, "Integral representations for some classes of functions holomorphic in a Siegel domain", J. Math. Anal. Appl., **179**, no. 1, 91 – 109 (1993); See also: Soviet Journal of Contemporary Math ematical Analysis, **22**, no. 4, 86 – 93 (1987).

[6] M. M. Djrbashian, A. H. Karapetyan, "Integral representations in the generalized upper half-plane", Soviet Math. Dokl., **41**, no. 3, 430 – 433 (1990); See also: Journal of Contemporary Mathematical Analysis, **25**, no. 6, 1 – 28 (1990).

[7] T. M. Flett, "On the rate of growth of mean values of holomorphic and harmonic functions", Pros. London Math. Soc., **20**, 749 – 768 (1970).

[8] F. Ricci, M. Taibleson, "Boundary values of harmonic functions in mixed norm spaces and their atomic structure", Annali Scuola Nor. Sup. Pisa., **ser.4**, no. 10, 1 – 54 (1983).

[9] S. G. Gindikin, "Analysis on homogeneous domains", Russ. Math. Surv., **19**, , 1 – 83 (1964).

[10] J. Gonessa, "Espaces de type Bergman dans les domaines homogènes de Siegel de type II", Décomposition atomique et interpolation, Thèse de Doctorat-Ph. D, Université de Yaoundé I (2006).

[11] J. Gonessa, K. Zhu, "A sharp norm estimate for weighted Bergman projections on the minimal ball", New York. Math., **17A**, 101 – 112 (2011).

[12] C. Nana, "$L^{p,q}$–boundedness of Bergman projections in homogeneous Siegel domains of type II", J. Fourier Anal Appl., **19**, 997 – 1019 (2013).

[13] C. Nana, B. Trojan, "L^p–Boundedness of Bergman projections in tube domains over homogeneous cones", Ann. Sc. Norm. Super. Pisa, Cl. Sci., **X**(5), 477 – 511 (2011).

[14] E. B. Vinberg, "The theory of convex homogeneous cones", Tr. Moskov. Mat. Obsc., **12**, 359 – 388 (1963).

[15] K. Zhu, "A Sharp norm estimate of the Bergman projection on L^p spaces", Contemporary Math., **404**, 199 – 205 (2006).

[16] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Springer-Verlag, New York (2005).

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