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ON THE CONVERGENCE OF PARTIAL SUMS WITH RESPECT TO VILENKIN SYSTEM ON THE MARTINGALE HARDY SPACES

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Abstract. In this paper, we derive characterizations of boundedness of subsequences of partial sums with respect to Vilenkin system on the martingale Hardy spaces H_p when $0 < p < 1$. Moreover, we find necessary and sufficient conditions for the modulus of continuity of martingales $f \in H_p$, which provide convergence of subsequences of partial sums on the martingale Hardy spaces H_p. It is also proved that these results are the best possible in a special sense. As applications, some known and new results are pointed out.

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1. Introduction

The notation and definitions, used in this section, will be given in the next section of the paper. It is well-known that (for details see [14]):

$$\|S_n f\|_p \leq c_p \|f\|_p, \quad \text{when } p > 1,$$

where $S_n f$ is the n-th partial sum with respect to bounded Vilenkin system.

Moreover, the following more stronger result is also known (see [11]):

$$\|S^* f\|_p \leq c_p \|f\|_p, \quad \text{when } f \in L_p, \ p > 1,$$

where $S^* f = \sup_{n \in \mathbb{N}} |S_n f|$.

Lukomskii [13] obtained a two-sided estimate for Lebesgue constants L_n with respect to Vilenkin system. By using this result, we easily can show that for every

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integrable function f, the sequence $S_{n_k} f$ converges to f in L_1-norm if and only if

$$\sup_{k\in\mathbb{N}} L_{n_k} \leq c < \infty.$$

Pointwise and uniform convergence and some approximation properties of partial sums in L_1-norm were studied by a number of authors (see, e.g, the papers by Goginava [9], Goginava and Sahakian [10], Avdispahić and Memić [2], and references therein). Fine [4] obtained sufficient conditions for the uniform convergence which are in complete analogy with the Dini-Lipschitz conditions. Guličev [12] has estimated the rate of uniform convergence of a Walsh-Fourier series by using Lebesgue constants and modulus of continuity. Uniform convergence of a subsequence of partial sums with respect to Walsh system was investigated also in [8]. This problem for a Vilenkin group G_m was considered by Blahota [3], Fridli [5] and Gát [7].

It is known (for details see, e.g., [18]) that the Vilenkin system does not form a basis in the space $L_1(G_m)$. Moreover, there is a function f in the martingale Hardy space $H_1(G_m)$ such that the sequence of partial sums of f is not bounded in $L_1(G_m)$-norm, but a subsequence S_{M_n} of partial sums is bounded from the martingale Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$, for all $p > 0$.

In [21] it was proved that if $0 < p \leq 1$ and $\{\alpha_k : k \in \mathbb{N}\}$ is an increasing sequence of nonnegative integers such that

$$\sup_{k\in\mathbb{N}} \rho(\alpha_k) < \infty, \tag{1.1}$$

where $\rho(n) = |n| - \langle n \rangle$ and

$$\langle n \rangle = \min\{j \in \mathbb{N} : n_j \neq 0\}, \qquad |n| = \max\{j \in \mathbb{N} : n_j \neq 0\},$$

for $n = \sum_{k=0}^{\infty} n_j M_j, \quad n_j \in Z_{m_j} \quad (j \in \mathbb{N})$, then the restricted maximal operator

$$\widetilde{S}^{*,\Delta} f := \sup_{k\in\mathbb{N}} |S_{\alpha_k} f|$$

is bounded from the Hardy space H_p to the Lebesgue space L_p.

Moreover, if $0 < p < 1$ and $\{\alpha_k : k \in \mathbb{N}\}$ is an increasing sequence of nonnegative integers satisfying the condition

$$\sup_{k\in\mathbb{N}} \rho(\alpha_k) = \infty, \tag{1.2}$$

then there exists a martingale $f \in H_p$ such that

$$\sup_{k\in\mathbb{N}} \|S_{\alpha_k} f\|_{L_{p,\infty}} = \infty.$$

It immediately follows that for any $p > 0$ and $f \in H_p$, the following restricted maximal operator

$$\widetilde{S}_{\#}^{*} f := \sup_{n\in\mathbb{N}} |S_{M_n} f|,$$

where $M_0 := 1$, $M_{k+1} := \prod_{i=0}^{k} m_i$ and $m := (m_0, m_1, ...)$ is a sequence of positive integers not less than 2, which generates the Vilenkin system, is bounded from the Hardy space H_p to the space L_p :

$$\left\| \widetilde{S}_{\#}^{*} f \right\|_p \leq \|f\|_{H_p}, \quad f \in H_p. \tag{1.3}$$

For the Vilenkin system, Simon [15] proved that there is an absolute constant c_p, depending only on p, such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \tag{1.4}$$

for all $f \in H_p(G_m)$, where $0 < p < 1$. In [17] we proved that the sequence $\{1/k^{2-p} : k \in \mathbb{N}\}$ can not be improved.

A similar theorem for $p = 1$ with respect to the unbounded Vilenkin systems was proved in Gát [6].

In [18] we proved that if $0 < p < 1$, $f \in H_p(G_m)$ and

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = o\left(\frac{1}{M_n^{1/p-1}}\right) \quad \text{as } n \to \infty,$$

then

$$\|S_{n_k} f - f\|_{H_p} \to 0 \quad \text{as } k \to \infty. \tag{1.5}$$

Moreover, for every $p \in (0, 1)$ there exists a martingale $f \in H_p(G_m)$, for which

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = O\left(\frac{1}{M_n^{1/p-1}}\right) \quad \text{as } n \to \infty$$

and

$$\|S_k f - f\|_{L_{p,\infty}(G_m)} \nrightarrow 0 \quad \text{as } k \to \infty.$$

In [20] we investigated some (H_p, H_p), (H_p, L_p) and $(H_p, L_{p,\infty})$ type inequalities for subsequences of partial sums of Walsh-Fourier series for $0 < p \leq 1$.

In this paper, we derive characterizations of boundedness of subsequences of partial sums with respect to the Vilenkin system on the martingale Hardy spaces H_p when $0 < p < 1$. Moreover, we find necessary and sufficient conditions for the modulus of continuity of $f \in H_p$, which provide convergence of subsequences of partial sums on the martingale Hardy spaces H_p. It is also proved that these results are the best possible in a special sense. As applications, we point out some known and new results.

The paper is organized as follows: In Section 2 we present necessary notation and definitions, and state a number of auxiliary lemmas, needed in the proofs of the main results. Some of these lemmas are new and represent independent interest. The formulations and detailed proofs of the main results and some of their consequences are given in Sections 3 and 4.

2. Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, and $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$. Let $m = (m_0, m_1, ...)$ denote a sequence of positive integers not less than 2. By $Z_{m_k} = \{0, 1, ..., m_k - 1\}$ we denote the additive group of integers modulo m_k, and define the group G_m to be the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures $\mu_k(\{j\}) := 1/m_k$ $(j \in Z_{m_k})$ is the Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence $m := (m_0, m_1, ...)$ is bounded, then the group G_m is called a bounded Vilenkin group, else it is called an unbounded Vilenkin group. The elements of the group G_m are represented by sequences $x := (x_0, x_1, ..., x_j, ...)$ $(x_k \in Z_{m_k})$.

It is easy to give a base for the neighborhoods of G_m:

$$I_0(x) = G_m, \quad I_n(x) = \{y \in G_m \mid y_0 = x_0, ..., y_{n-1} = x_{n-1}\} \ (x \in G_m, \ n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$. It is clear that

$$\overline{I_N} = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1}. \tag{2.1}$$

If we define the so-called generalized number system based on m in the following way $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can uniquely be expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of n_j's differ from zero. For all $n \in \mathbb{N}$ we define

$$\langle n \rangle := \min\{j \in \mathbb{N} : n_j \neq 0\}, \ |n| := \max\{j \in \mathbb{N} : n_j \neq 0\}, \ \rho(n) = |n| - \langle n \rangle.$$

For a natural number $n = \sum_{j=1}^{\infty} n_j M_j$, we define the functions v and v^* as follows:

$$v(n) = \sum_{j=1}^{\infty} |\delta_{j+1} - \delta_j| + \delta_0, \quad v^*(n) = \sum_{j=1}^{\infty} \delta_j^*,$$

where $\delta_j = \operatorname{sign} n_j = \operatorname{sign}(\ominus n_j)$, $\delta_j^* = |\ominus n_j - 1|\,\delta_j$ and $\ominus$ is the inverse operation for $a_k \oplus b_k := (a_k + b_k) \bmod m_k$. The norms (or quasi-norms) of the spaces $L_p(G_m)$ and $L_{p,\infty}(G_m)$ $(0 < p < \infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p \, d\mu, \qquad \|f\|_{L_{p,\infty}} := \sup_{\lambda>0} \lambda\mu\,(f > \lambda)^{1/p}.$$

Next, on the group G_m we introduce an orthonormal system, which is called the Vilenkin system. To this end, we first define the complex-valued functions $r_k(x)$: $G_m \to \mathbb{C}$, called the generalized Rademacher functions, as follows:

$$r_k(x) := \exp\left(2\pi \imath x_k / m_k\right) \qquad (\imath^2 = -1,\ x \in G_m,\ k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in \mathbb{N}).$$

Notice that in the special case where $m \equiv 2$, that is, $m_k = 2$ for all $k \in \mathbb{N}$, the above defined system is called the Walsh-Paley system. Observe that the Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see, e.g. [1, 22]). If $f \in L_1(G_m)$, then we can define the Fourier coefficients, the partial sums of the Fourier series, and the Dirichlet kernel for the Vilenkin system ψ in the usual manner as follows:

$$\widehat{f}(k) \ : = \int_{G_m} f \overline{\psi}_k d\mu, \qquad (k \in \mathbb{N})$$

$$S_n f \ : = \sum_{k=0}^{n-1} \widehat{f}(k)\,\psi_k, \qquad D_n = \sum_{k=0}^{n-1} \psi_k\,, \ (n \in \mathbb{N}).$$

Recall that (see [1])

(2.2)
$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases}$$

and

(2.3)
$$D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{u=m_j-n_j}^{m_j-1} r_j^u \right).$$

Moreover, if $n \in \mathbb{N}$ and $x \in I_s \setminus I_{s+1}$, $0 \le s \le N-1$, then the following estimates hold (see Tephnadze [16, 19]):

(2.4)
$$|D_n(x)| = \left|D_{n-M_{|n|}}(x)\right| \ge M_{\langle n\rangle}, \qquad |n| \ne \langle n \rangle$$

and

(2.5)
$$\int_{I_N} |D_n(x-t)|\, d\mu(t) \le \frac{cM_s}{M_N}.$$

The n-th Lebesgue constant L_n for the Vilenkin system ψ is defined by

$$L_n := \|D_n\|_1 .$$

It is known that for every $n = \sum_{i=1}^{\infty} n_i M_i$, the following two-sided estimate is true (see Lukomskii [13]):

$$\frac{1}{4\lambda} v(n) + \frac{1}{\lambda} v^*(n) + \frac{1}{2\lambda} \leq L_n \leq \frac{3}{2} v(n) + 4v^*(n) - 1, \tag{2.6}$$

where $\lambda := \sup_{n\in\mathbb{N}} m_n$.

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ we denote by F_n $(n \in \mathbb{N})$, and by $f = (f_n, n \in \mathbb{N})$ we denote a martingale with respect to F_n $(n \in \mathbb{N})$ (for details see Weisz [23]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbb{N}} \left| f^{(n)} \right| .$$

In the case where $f \in L_1(G_m)$, the maximal function can also be given by the following formula:

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right| .$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales, for which $\|f\|_{H_p} := \|f^*\|_p < \infty$.

Let $X = X(G_m)$ denote either the space $L_1(G_m)$ or the space of continuous functions $C(G_m)$. The corresponding norm is denoted by $\|\cdot\|_X$. The modulus of continuity, when $X = C(G_m)$ and the integral modulus of continuity, when $X = L_1(G_m)$ are defined by

$$\omega\left(\frac{1}{M_n}, f\right)_X = \sup_{h\in I_n} \|f(\cdot + h) - f(\cdot)\|_X .$$

The modulus of continuity in the Hardy martingale spaces $H_p(G_m)$ $(0 < p \leq 1)$ can be defined as follows:

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} := \|f - S_{M_n} f\|_{H_p(G_m)} .$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n} f : n \in \mathbb{N})$ is a martingale. If $f = (f_n, n \in \mathbb{N})$ is a martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f_k(x)\, \overline{\psi}_i(x)\, d\mu(x) .$$

Notice that the Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as the martingale $(S_{M_n} f : n \in \mathbb{N})$ obtained from f . A bounded measurable function a is called a p-atom if there exists an interval I such that

$$\int_I a d\mu = 0, \qquad \|a\|_\infty \le \mu(I)^{-1/p}, \qquad \operatorname{supp}(a) \subset I.$$

Observe that for $0 < p \le 1$, the martingale Hardy spaces $H_p(G_m)$ have atomic characterizations (for details see, e.g., Weisz [23, 24]):

Lemma 2.1. *A martingale $f = (f_n, n \in \mathbb{N})$ is in $H_p\,(0 < p \le 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that, for every $n \in \mathbb{N}$*

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f_n \quad \text{a.e., where} \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{2.7}$$

Moreover,

$$\|f\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of form (2.7).

By using the atomic decomposition of martingales $f \in H_p$, we can construct a counterexample, which plays a central role to prove the sharpness of our main results, and it will be used several times in this paper (for details see Tephnadze [21], Section 1.7., Example 1.48).

Lemma 2.2. *Let $0 < p \le 1$, $\lambda = \sup_{n \in \mathbb{N}} m_n$, and $\{\lambda_k : k \in \mathbb{N}\}$ be a sequence of real numbers such that*

$$\sum_{k=0}^{\infty} |\lambda_k|^p \le c_p < \infty. \tag{2.8}$$

Let $\{a_k : k \in \mathbb{N}\}$ be a sequence of p-atoms defined by

$$a_k := \frac{M_{|\alpha_k|}^{1/p-1}}{\lambda} \left(D_{M_{|\alpha_k|+1}} - D_{M_{|\alpha_k|}} \right),$$

where $|\alpha_k| := \max\{j \in \mathbb{N} : (\alpha_k)_j \ne 0\}$ and $(\alpha_k)_j$ denotes the j-th binary coefficient of $\alpha_k \in \mathbb{N}$. Then $f = (f_n : n \in \mathbb{N})$, where

$$f_n := \sum_{\{k:\ |\alpha_k| < n\}} \lambda_k a_k,$$

is a martingale, $f \in H_p$ *for all* $0 < p \leq 1$, *and*

$$\widehat{f}(j) = \begin{cases} \frac{\lambda_k M_{|\alpha_k|}^{1/p-1}}{\lambda}, & j \in \{M_{|\alpha_k|}, ..., M_{|\alpha_k|+1} - 1\}, \ k \in \mathbb{N}, \\ 0, & j \notin \bigcup_{k=1}^{\infty} \{M_{|\alpha_k|}, ..., M_{|\alpha_k|+1} - 1\}. \end{cases} \tag{2.9}$$

Further, let $M_{|\alpha_l|} \leq j < M_{|\alpha_l|+1}$, $l \in \mathbb{N}$. *Then*

$$\begin{aligned} S_j f &= S_{M_{|\alpha_l|}} + \frac{\lambda_l M_{|\alpha_l|}^{1/p-1} \psi_{M_{|\alpha_l|}} D_{j-M_{|\alpha_l|}}}{\lambda} \\ &= \sum_{\eta=0}^{l-1} \frac{\lambda_\eta M_{|\alpha_\eta|}^{1/p-1}}{\lambda} \left(D_{M_{|\alpha_\eta|+1}} - D_{M_{|\alpha_\eta|}} \right) + \frac{\lambda_l M_{|\alpha_l|}^{1/p-1} \psi_{M_{|\alpha_l|}} D_{j-M_{|\alpha_l|}}}{\lambda}. \end{aligned} \tag{2.10}$$

Moreover, the following asymptotic relation holds:

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = O\left(\sum_{\{k: \ |\alpha_k| \geq n\}}^{\infty} |\lambda_k|^p \right)^{1/p} \quad as \quad n \to \infty, \tag{2.11}$$

There exists a close connection between the H_p and L_p norms of partial sums (see Tephnadze [21], Section 1.7., Example1.45):

Lemma 2.3. *Let* $M_k \leq n < M_{k+1}$ *and* $S_n f$ *be the n-th partial sum with respect to Vilenkin system, where* $f \in H_p$ *for some* $0 < p \leq 1$. *Then for every* $n \in \mathbb{N}$ *we have the following estimate:*

$$\|S_n f\|_p \leq \|S_n f\|_{H_p} \leq \left\| \sup_{0 \leq l \leq k} |S_{M_l} f| \right\|_p + \|S_n f\|_p \leq \left\| \widetilde{S}_{\#}^* f \right\|_p + \|S_n f\|_p.$$

3. Convergence of subsequences of partial sums on the martingale Hardy spaces

Our first main result in this paper is the following theorem.

Theorem 3.1. *The following assertions hold.*

a) Let $0 < p < 1$ *and* $f \in H_p$. *Then there exists an absolute constant* c_p, *depending only on p, such that*

$$\|S_n f\|_{H_p} \leq \frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n \rangle}^{1/p-1}} \|f\|_{H_p}.$$

b) Let $0 < p < 1$ *and* $\{n_k : k \in \mathbb{N}\}$ *be an increasing sequence of nonnegative integers such that condition (1.2) is satisfied, and let* $\{\Phi_n : n \in \mathbb{N}\}$ *be any nondecreasing* *sequence, satisfying the condition:*

$$\overline{\lim_{k\to\infty}} \frac{M_{|n_k|}^{1/p-1}}{M_{\langle n_k\rangle}^{1/p-1}\Phi_{n_k}} = \infty. \tag{3.1}$$

Then there exists a martingale $f \in H_p$ such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{S_{n_k} f}{\Phi_{n_k}} \right\|_{L_{p,\infty}} = \infty.$$

Proof. We first prove assertion a). Suppose that

$$\left\| \frac{M_{\langle n\rangle}^{1/p-1} S_n f}{M_{|n|}^{1/p-1}} \right\|_p \le c_p \|f\|_{H_p}. \tag{3.2}$$

Then according to Lemma 2.3 and estimates (1.3) and (3.2) we get

$$\left\| \frac{M_{\langle n\rangle}^{1/p-1} S_n f}{M_{|n|}^{1/p-1}} \right\|_{H_p} \le \left\| \widetilde{S}_{\#}^* f \right\|_p + \left\| \frac{M_{\langle n\rangle}^{1/p-1} S_n f}{M_{|n|}^{1/p-1}} \right\|_p \le c_p \|f\|_{H_p}. \tag{3.3}$$

In view of Lemma 2.1 and (3.3), the proof of part a) of the theorem will be completed, if we show that

$$\int_G \left| \frac{M_{\langle n\rangle}^{1/p-1} S_n a}{M_{|n|}^{1/p-1}} \right| d\mu \le c_p < \infty, \tag{3.4}$$

for every p-atom a, with support I and $\mu(I) = M_N^{-1}$.

We may assume that this arbitrary p-atom a has support $I = I_N$. It is easy to see that $S_n a = 0$, when $M_N \ge n$. Therefore, we can suppose that $M_N < n$. According to $\|a\|_\infty \le M_N^{1/p}$, we can write

$$\begin{aligned} \left| \frac{M_{\langle n\rangle}^{1/p-1} S_n a(x)}{M_{|n|}^{1/p-1}} \right| &\le \frac{M_{\langle n\rangle}^{1/p-1} \|a\|_\infty}{M_{|n|}^{1/p-1}} \int_{I_N} |D_n(x-t)|\, d\mu(t) \\ &\le \frac{M_{\langle n\rangle}^{1/p-1} M_N^{1/p}}{M_{|n|}^{1/p-1}} \int_{I_N} |D_n(x-t)|\, d\mu(t). \end{aligned} \tag{3.5}$$

Let $x \in I_N$. Since $x - t \in I_N$, $t \in I_N$ and $v(n) + v^*(n) \leq c(|n| - \langle n \rangle) = c\rho(n)$, we can apply (2.6) to obtain

$$\begin{aligned} \left| \frac{M_{\langle n \rangle}^{1/p-1} S_n a(x)}{M_{|n|}^{1/p-1}} \right| &\leq \frac{M_{\langle n \rangle}^{1/p-1} M_N^{1/p}}{M_{|n|}^{1/p-1}} \int_{I_N} |D_n(t)| \, d\mu(t) \\ &\leq \frac{M_{\langle n \rangle}^{1/p-1} M_N^{1/p} (v(n) + v^*(n))}{M_{|n|}^{1/p-1}} \qquad (3.6) \\ &\leq \frac{cM_{\langle n \rangle}^{1/p-1} M_N^{1/p} (|n| - \langle n \rangle)}{M_{|n|}^{1/p-1}} \leq \frac{cM_N^{1/p} \rho(n)}{2^{\rho(n)(1/p-1)}} \end{aligned}$$

and

$$\int_{I_N} \left| \frac{M_{\langle n \rangle}^{1/p-1} S_n a(x)}{M_{|n|}^{1/p-1}} \right|^p d\mu(x) \leq \frac{\rho^p(n)}{2^{\rho(n)(1-p)}} < c_p < \infty. \qquad (3.7)$$

Let $x \in I_s \setminus I_{s+1}$, $0 \leq s \leq N-1 < \langle n \rangle$ or $0 \leq s \leq \langle n \rangle \leq N-1$. Then $x - t \in I_s \setminus I_{s+1}$ for $t \in I_N$. Combining (2.2) and (2.3) we get $D_n(x-t) = 0$, and

$$\left| \frac{M_{\langle n \rangle}^{1/p-1} S_n a}{M_{|n|}^{1/p-1}} \right| = 0. \qquad (3.8)$$

Let $x \in I_s \setminus I_{s+1}$, $0 \leq \langle n \rangle < s \leq N-1$ or $0 \leq \langle n \rangle < s \leq N-1$. Then $x - t \in I_s \setminus I_{s+1}$ for $t \in I_N$. Hence, applying (2.5), we get

$$\left| \frac{M_{\langle n \rangle}^{1/p-1} S_n a(x)}{M_{|n|}^{1/p-1}} \right| \leq \frac{c_p M_{\langle n \rangle}^{1/p-1} M_N^{1/p}}{M_{|n|}^{1/p-1}} \frac{M_s}{M_N} = c_p M_{\langle n \rangle}^{1/p-1} M_s. \qquad (3.9)$$

Combining (2.1), (3.8) and (3.9), we obtain

$$\begin{aligned} \int_{\overline{I_N}} \left| \frac{M_{\langle n \rangle}^{1/p-1} S_n a}{M_{|n|}^{1/p-1}} \right|^p d\mu &= \sum_{s=0}^{N-1} \int_{I_s \setminus I_{s+1}} \left| \frac{M_{\langle n \rangle}^{1/p-1} S_n a}{M_{|n|}^{1/p-1}} \right|^p d\mu \qquad (3.10) \\ \leq c_p \sum_{s=\langle n \rangle}^{N-1} \int_{I_s \setminus I_{s+1}} \left| M_{\langle n \rangle}^{1/p-1} M_s \right|^p d\mu &= c_p \sum_{s=\langle n \rangle}^{N-1} \frac{c_p M_{\langle n \rangle}^{1-p}}{M_s^{1-p}} \leq c_p < \infty. \end{aligned}$$

This completes the proof of part a) of the theorem.

Now we proceed to prove part b) of the theorem. To this end, observe first that under the condition (3.1), there exists a sequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$ such that

$$\sum_{\eta=0}^{\infty} \frac{M_{\langle n_\eta \rangle}^{(1-p)/2} \Phi_{n_\eta}^{p/2}}{M_{|\alpha_\eta|}^{(1-p)/2}} < \infty. \qquad (3.11)$$

We note that such increasing sequence $\{\alpha_k : k \in \mathbb{N}\}$, which satisfies condition (3.11), can be constructed.

Let $f = (f_n, n \in \mathbb{N})$ be the martingale from Lemma 2.2, where

$$\lambda_k = \frac{M_{\langle\alpha_k\rangle}^{(1/p-1)/2}\Phi_{\alpha_k}^{1/2}}{M_{|\alpha_k|}^{(1/p-1)/2}}. \tag{3.12}$$

In view of (3.12) we conclude that (2.8) is satisfied, and hence, using Lemma 2.2, we obtain that $f \in H_p$.

Next, using (2.10) with λ_k defined by (3.12), we get

$$\begin{aligned}\frac{S_{\alpha_k} f}{\Phi_{\alpha_k}} &= \frac{1}{\Phi_{\alpha_k}} \sum_{\eta=0}^{k-1} M_{|\alpha_\eta|}^{(1/p-1)/2} M_{\langle\alpha_\eta\rangle}^{(1/p-1)/2} \Phi_{\alpha_\eta}^{1/2} \left(M_{|\alpha_\eta|+1} - D_{M_{|\alpha_\eta|}}\right)\\ &+ \frac{M_{|\alpha_k|}^{(1/p-1)/2} M_{\langle\alpha_k\rangle}^{(1/p-1)/2} D_{\alpha_k - M_{|\alpha_k|}}}{\Phi_{\alpha_k}^{1/2}} = I + II.\end{aligned}$$

Hence, according to (3.11), we can write

$$\begin{aligned}\|I\|_{L_{p,\infty}}^p &\leq \frac{1}{\Phi_{\alpha_k}^p} \sum_{\eta=0}^{\infty} \frac{M_{\langle\alpha_\eta\rangle}^{(1-p)/2}\Phi_{\alpha_\eta}^{p/2}}{M_{|\alpha_\eta|}^{(1-p)/2}} \left\| M_{|\alpha_\eta|}^{(1/p-1)} \left(M_{|\alpha_\eta|+1} - D_{M_{|\alpha_\eta|}}\right)\right\|_{L_{p,\infty}}^p\\ &\leq \frac{1}{\Phi_{\alpha_k}^p} \sum_{\eta=0}^{\infty} \frac{M_{\langle\alpha_\eta\rangle}^{(1-p)/2}\Phi_{\alpha_\eta}^{p/2}}{M_{|\alpha_\eta|}^{(1-p)/2}} < \frac{c}{\Phi_{\alpha_k}^p} \leq c < \infty.\end{aligned} \tag{3.13}$$

Let $x \in I_{\langle\alpha_k\rangle} \setminus I_{\langle\alpha_k\rangle+1}$. Then we can apply (2.4) to conclude that

$$\begin{aligned}|II| &= \frac{M_{|\alpha_k|}^{(1/p-1)/2} M_{\langle\alpha_k\rangle}^{(1/p-1)/2} D_{\alpha_k - M_{|\alpha_k|}}}{\Phi_{\alpha_k}^{1/2}}\\ &\geq \frac{M_{|\alpha_k|}^{(1/p-1)/2} M_{\langle\alpha_k\rangle}^{(1/p+1)/2}}{\Phi_{\alpha_k}^{1/2}}.\end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), for sufficiently large k, we can write

$$\begin{aligned}&\left\|\frac{S_{\alpha_k} f}{\Phi_{\alpha_k}}\right\|_{L_{p,\infty}}^p \geq \|II\|_{L_{p,\infty}}^p - \|I\|_{L_{p,\infty}}^p \geq \frac{1}{2}\|II\|_{L_{p,\infty}}^p\\ \geq\ & \frac{c M_{|\alpha_k|}^{(1-p)/2} M_{\langle\alpha_k\rangle}^{(1+p)/2}}{\Phi_{\alpha_k}^{p/2}} \mu\left\{x \in G_m :\ |II| \geq \frac{c M_{|\alpha_k|}^{(1/p-1)/2} M_{\langle\alpha_k\rangle}^{(1/p+1)/2}}{\Phi_{\alpha_k}^{1/2}}\right\}\\ \geq\ & \frac{c M_{|\alpha_k|}^{(1-p)/2} M_{\langle\alpha_k\rangle}^{(1+p)/2}}{\Phi_{\alpha_k}^{p/2}} \mu\left\{I_{\langle\alpha_k\rangle} \setminus I_{\langle\alpha_k\rangle+1}\right\} \geq \frac{c M_{|\alpha_k|}^{(1-p)/2}}{M_{\langle\alpha_k\rangle}^{(1-p)/2}\Phi_{\alpha_k}^{p/2}} \to \infty \text{ as } k \to \infty.\end{aligned}$$

This completes the proof of part b) of the theorem. □

The next corollary contains equivalent characterizations of boundedness of subsequences of partial sums with respect to the Vilenkin system of martingales $f \in H_p$ in terms of measurable properties of the Dirichlet kernel.

Corollary 3.1. *The following assertions hold.*

a) Let $0 < p < 1$ and $f \in H_p$. Then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f\|_{H_p} \leq c_p \left(n\mu\left\{supp\left(D_n\right)\right\}\right)^{1/p-1} \|f\|_{H_p}.$$

b) Let $0 < p < 1$ and $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers such that

$$\sup_{k\in\mathbb{N}} n_k\mu\left\{supp\left(D_{n_k}\right)\right\} = \infty, \tag{3.15}$$

and let $\{\Phi_n : n \in \mathbb{N}\}$ be any nondecreasing sequence, satisfying the condition

$$\overline{\lim_{k\to\infty}} \frac{\left(n_k\mu\left\{supp\left(D_{n_k}\right)\right\}\right)^{1/p-1}}{\Phi_{n_k}} = \infty.$$

Then there exists a martingale $f \in H_p$ such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{S_{n_k} f}{\Phi_{n_k}} \right\|_{L_{p,\infty}} = \infty.$$

Remark 3.1. Corollary 3.1 shows that when $0 < p < 1$; the main reason of divergence of partial sums of a Vilenkin-Fourier series is the unboundedness of Fourier coefficients, but in the case where the measure of the support of n_k-th Dirichlet kernel tends to zero, then the divergence rate drops and in the case when it is maximally small, that is,

$$\mu\left(suppD_{n_k}\right) = O\left(\frac{1}{M_{|n_k|}}\right) \quad \text{as} \quad k \to \infty, \quad \left(M_{|n_k|} < n_k \leq M_{|n_k|+1}\right),$$

then we have convergence.

Proof. Combining (2.2) and (2.3) we get $I_{\langle n\rangle} \backslash I_{\langle n\rangle+1} \subset \text{supp}D_n \subset I_{\langle n\rangle}$ and

$$\frac{1}{2M_{\langle n\rangle}} \leq \mu\left\{\text{supp}D_n\right\} \leq \frac{1}{M_{\langle n\rangle}}$$

Since $M_{|n|} \leq n < M_{|n|+1}$, we immediately get

$$\frac{M_{|n|}}{2M_{\langle n\rangle}} \leq n\mu\left\{\text{supp}\left(D_n\right)\right\} \leq \frac{\lambda M_{|n|}}{M_{\langle n\rangle}},$$

where $\lambda = \sup_{n\in\mathbb{N}} m_n$.

It follows that

$$\frac{M_{|n|}^{1/p-1}}{2M_{\langle n\rangle}^{1/p-1}} \le (n\mu\{\operatorname{supp}(D_n)\})^{1/p-1} \le \frac{\lambda^{1/p-1}M_{|n|}^{1/p-1}}{M_{\langle n\rangle}^{1/p-1}}.$$

The result follows by using these estimates in Theorem 3.1. □

As special cases of Theorem 3.1, we can infer a number of known and new results that are of particular interest. In Corollaries 3.2-3.4 that follow we list some of them.

Corollary 3.2. *Let* $0 < p < 1$, $f \in H_p$ *and* $\{n_k : k \in \mathbb{N}\}$ *be an increasing sequence of nonnegative integers. Then*

$$\|S_{n_k} f\|_{H_p} \le c_p \|f\|_{H_p}$$

if and only if condition (1.1) is satisfied.

Proof. It is easy to show that

$$2^{\rho(n_k)} \le \frac{M_{|n_k|}}{M_{\langle n_k\rangle}} \le \lambda^{\rho(n_k)},$$

where $\lambda = \sup_{n\in\mathbb{N}} m_n$. It follows that

$$\sup_{k\in\mathbb{N}} \frac{M_{|n_k|}^{1/p-1}}{M_{\langle n_k\rangle}^{1/p-1}} < c < \infty$$

if and only if (1.1) holds. Thus, the result follows from Theorem 3.1. □

Corollary 3.3. *Let* $n \in \mathbb{N}$ *and* $0 < p < 1$. *Then there exists a martingale* $f \in H_p$ *such that*

(3.16) $$\sup_{n\in\mathbb{N}} \|S_{M_n+1} f\|_{L_{p,\infty}} = \infty.$$

Proof. It is easy to check that

(3.17) $$|M_n + 1| = n, \ \ \langle M_n + 1\rangle = 0$$

and

(3.18) $$\rho(M_n + 1) = n.$$

By using Corollary 3.2 we obtain that there exists a martingale $f \in H_p$ $(0 < p < 1)$ such that (3.16) holds. The proof is complete. □

Corollary 3.4. *Let* $n \in \mathbb{N}$, $0 < p \le 1$ *and* $f \in H_p$. *Then*

(3.19) $$\|S_{M_n+M_{n-1}} f\|_{H_p} \le c_p \|f\|_{H_p}.$$

Proof. Similar to (3.17) and (3.18), we obtain

$$|M_n + M_{n-1}| = n, \quad \langle M_n + M_{n-1}\rangle = n - 1$$

and $\rho(M_n + M_{n-1}) = 1$. By using Corollary 3.2 we immediately get the inequality (3.19) for all $0 < p \leq 1$. The proof is complete. □

Corollary 3.5. *Let $n \in \mathbb{N}$, $0 < p \leq 1$ and $f \in H_p$. Then*

$$\|S_{M_n} f\|_{H_p} \leq c_p \|f\|_{H_p}. \tag{3.20}$$

Proof. Similar to (3.17) and (3.18) we obtain $|M_n| = n$, $\langle M_n\rangle = n$ and $\rho(M_n) = 0$. Using Corollary 3.2 we get the inequality (3.20) for all $0 < p \leq 1$. □

4. Necessary and sufficient condition for convergence of partial sums in terms of modulus of continuity

The main result of this section is the following theorem.

Theorem 4.1. *The following assertions hold.*

a) Let $0 < p < 1$, $f \in H_p$ and $M_k < n \leq M_{k+1}$. Then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f - f\|_{H_p} \leq \frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n\rangle}^{1/p-1}} \omega\left(\frac{1}{M_k}, f\right)_{H_p(G_m)}, \quad 0 < p < 1. \tag{4.1}$$

Moreover, if $\{n_k : k \in \mathbb{N}\}$ is an increasing sequence of nonnegative integers such that

$$\omega\left(\frac{1}{M_{|n_k|}}, f\right)_{H_p(G_m)} = o\left(\frac{M_{\langle n_k\rangle}^{1/p-1}}{M_{|n_k|}^{1/p-1}}\right) \quad as \ k \to \infty, \tag{4.2}$$

then

$$\|S_{n_k} f - f\|_{H_p} \to 0 \quad as \ k \to \infty. \tag{4.3}$$

b) Let $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers such that the condition (1.2) is satisfied. Then there exist a martingale $f \in H_p$ and a subsequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$, for which

$$\omega\left(\frac{1}{M_{|\alpha_k|}}, f\right)_{H_p(G_m)} = O\left(\frac{M_{\langle \alpha_k\rangle}^{1/p-1}}{M_{|\alpha_k|}^{1/p-1}}\right) \quad as \ k \to \infty \tag{4.4}$$

and

$$\overline{\lim_{k\to\infty}} \|S_{\alpha_k} f - f\|_{L_{p,\infty}} > c > 0 \quad as \ k \to \infty. \tag{4.5}$$

Proof. Let $0 < p < 1$. Then, by Theorem 3.1, we get

$$\begin{aligned} \|S_n f - f\|_{H_p}^p &\le \|S_n f - S_{M_k} f\|_{H_p}^p + \|S_{M_k} f - f\|_{H_p}^p \\ &= \|S_n (S_{M_k} f - f)\|_{H_p}^p + \|S_{M_k} f - f\|_{H_p}^p \le \left(\frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n \rangle}^{1/p-1}} + 1 \right) \omega_{H_p}^p \left(\frac{1}{M_k}, f \right). \end{aligned}$$

and

$$\|S_n f - f\|_{H_p} \le \frac{c_p M_{|n|}^{1/p-1}}{M_{\langle n \rangle}^{1/p-1}} \omega \left(\frac{1}{M_k}, f \right)_{H_p(G_m)}.$$

Next, it is easy to see that relation (4.3) immediately follows from (4.1) and (4.2). Thus, the assertion a) is proved. To prove part b) of the theorem, we first note that under the conditions of part b), there exists a subsequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$ such that

$$\frac{M_{|\alpha_k|}}{M_{\langle \alpha_k \rangle}} \uparrow \infty \quad \text{as} \quad k \to \infty \tag{4.6}$$

$$\frac{M_{|\alpha_k|}^{2(1/p-1)}}{M_{\langle \alpha_k \rangle}^{2(1/p-1)}} \le \frac{M_{|\alpha_{k+1}|}^{1/p-1}}{M_{\langle \alpha_{k+1} \rangle}^{1/p-1}}. \tag{4.7}$$

Let $f = (f_n, n \in \mathbb{N})$ be the martingale from Lemma 2.2, where

$$\lambda_k = \frac{\lambda M_{\langle \alpha_k \rangle}^{1/p-1}}{M_{|\alpha_k|}^{1/p-1}}. \tag{4.8}$$

Applying (4.6) and (4.7) with λ_k as in (4.8), we conclude that (2.8) is satisfied, and hence by Lemma 2.2, we obtain that $f \in H_p$.

Using (4.8) with λ_k as in (4.8), we get

$$\omega(\frac{1}{M_{|\alpha_k|}}, f)_{H_p(G_m)} \le \sum_{i=k}^{\infty} \frac{M_{\langle \alpha_i \rangle}^{1/p-1}}{M_{|\alpha_i|}^{1/p-1}} = O\left(\frac{M_{\langle \alpha_k \rangle}^{1/p-1}}{M_{|\alpha_k|}^{1/p-1}} \right) \quad \text{as} \quad k \to \infty. \tag{4.9}$$

Next, applying (2.10) with λ_k as in (4.8), we obtain

$$S_{\alpha_k} f = S_{M_{|\alpha_k|}} + M_{\langle \alpha_k \rangle}^{1/p-1} \psi_{M_{|\alpha_k|}} D_{j - M_{|\alpha_k|}}.$$

In view of (2.4) we conclude that $\left| D_{\alpha_k - M_{|\alpha_k|}} \right| \ge M_{\langle \alpha_k \rangle}$ for $I_{\langle \alpha_k \rangle} \setminus I_{\langle \alpha_k \rangle + 1}$, and

$$\begin{aligned} & M_{\langle \alpha_k \rangle} \mu \left\{ x \in G_m : \ \left| D_{\alpha_k - M_{|\alpha_k|}} \right| \ge M_{\langle \alpha_k \rangle} \right\} \\ & M_{\langle \alpha_k \rangle} \mu \left\{ I_{\langle \alpha_k \rangle} \setminus I_{\langle \alpha_k \rangle + 1} \right\} \ge M_{\langle \alpha_k \rangle}^{1-p} \end{aligned} \tag{4.10}$$

Finally, in view of Corollary 3.5 and formula (4.10), for sufficiently large k, we can write

$$\|S_{\alpha_k} f - f\|_{L_{p,\infty}} \geq M_{\langle\alpha_k\rangle}^{1/p-1} \|D_{\alpha_k}\|_{L_{p,\infty}} - \|S_{M_{|\alpha_k|}} f - f\|_{L_{p,\infty}}$$
$$\geq \frac{M_{\langle\alpha_k\rangle}^{1/p-1} \|D_{\alpha_k}\|_{L_{p,\infty}}}{2} \geq c.$$

This completes the proof of part b) of the theorem.

Theorem 4.1 is proved. □

Next, we present a simple consequence of Theorem 4.1, which was proved in Tephnadze [18]:

Corollary 4.1. *The following assertions hold.*

a) Let $0 < p < 1$, $f \in H_p$ and

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = o\left(\frac{1}{M_n^{1/p-1}}\right) \quad as \quad n \to \infty.$$

Then

$$\|S_k f - f\|_{H_p} \to 0 \quad as \quad k \to \infty.$$

b) For every $0 < p < 1$ there exists a martingale $f \in H_p$ for which

$$\omega\left(\frac{1}{M_n}, f\right) = O\left(\frac{1}{M_n^{1/p-1}}\right)_{H_p(G_m)} \quad as \quad n \to \infty$$

and

$$\|S_k f - f\|_{L_{p,\infty}} \nrightarrow 0 \quad as \quad k \to \infty.$$

Finally, we present a result that contains equivalent conditions for the modulus of continuity in terms of measurable properties of the Dirichlet kernel, which provide boundedness of the subsequences of partial sums with respect to the Vilenkin system of martingales $f \in H_p$.

Corollary 4.2. *The following assertions hold.*

a) Let $0 < p < 1$, $f \in H_p$ and $M_k < n \leq M_{k+1}$. Then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n f - f\|_{H_p} \leq c_p \left(n\mu\left(suppD_n\right)\right)^{1/p-1} \omega_{H_p}\left(\frac{1}{M_k}, f\right), \quad (0 < p < 1).$$

Moreover, if $\{n_k : k \in \mathbb{N}\}$ is a sequence of nonnegative integers such that

$$\omega\left(\frac{1}{M_{|n_k|}}, f\right)_{H_p(G_m)} = o\left(\frac{1}{\left(n_k\mu\left(suppD_{n_k}\right)\right)^{1/p-1}}\right) \quad as \quad k \to \infty,$$

then (4.3) holds.

b) Let $\{n_k : k \in \mathbb{N}\}$ be an increasing sequence of nonnegative integers such that the condition (1.2) is satisfied. Then there exist a martingale $f \in H_p$ and a subsequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$, for which

$$\omega\left(\frac{1}{M_{|\alpha_k|}}, f\right)_{H_p(G_m)} = O\left(\frac{1}{\left(\alpha_k \mu\left(suppD_{\alpha_k}\right)\right)^{1/p-1}}\right) \quad as\ k \to \infty$$

and

$$\overline{\lim_{k\to\infty}} \|S_{\alpha_k} f - f\|_{L_{p,\infty}} > c > 0 \quad as\ k \to \infty.$$

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