Известия НАН Армении, Математика, том 53, н. 5, 2018, стр. 61 – 76.

EIGENFUNCTIONS OF COMPOSITION OPERATORS ON BLOCH-TYPE SPACES

B. PAUDYAL

Central State University, Wilberforce, Ohio, USA
E-mail: bpaudyal@centralstate.edu

Abstract. Suppose φ is a holomorphic self map of the unit disk and C_φ is a composition operator with symbol φ that fixes the origin and $0 < |\varphi'(0)| < 1$. This paper explores sufficient conditions that ensure all the holomorphic solutions of Schröder equation for the composition operator C_φ to belong to a Bloch-type space $\mathcal{B}_\alpha$ for some $\alpha > 0$. In the second part of the paper, the results obtained for composition operators are extended to the case of weighted composition operators.

MSC2010 numbers: 47B33.
Keywords: Composition operator; Schröder equation; Bloch space.

1. Introduction

Let $\mathcal{D}$ be the unit disk of the complex plane $\mathbb{C}$, and let $\mathcal{H}(\mathcal{D})$ denote the space of holomorphic functions defined on the unit disk $\mathcal{D}$. Recall that a holomorphic function f defined on $\mathcal{D}$ is said to be in the Bloch-type space $\mathcal{B}_\alpha$ for some $\alpha > 0$ if

$$\sup_{z\in\mathcal{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

Notice that under the Bloch-type norm:

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \sup_{z\in\mathcal{D}}(1-|z|^2)^\alpha|f'(z)|, \tag{1.1}$$

the space $\mathcal{B}_\alpha$ becomes a Banach space. From the definition of Bloch-type spaces, it immediately follows that $\mathcal{B}_\alpha \subset \mathcal{B}_\beta$ for $\alpha \le \beta$ and $\mathcal{B}_\alpha \subset H^\infty$ for $\alpha < 1$.

The Bloch type spaces have been studied extensively by many authors (see [1], [8], and references therein). In [8], it has been shown that the Bloch-type norm for $\alpha > 1$ is equivalent to the $\alpha - 1$ Lipschitz-type norm:

$$\|f\|_{\mathcal{B}_\alpha} \approx \sup_{z\in\mathcal{D}}(1-|z|^2)^{\alpha-1}|f(z)|, \qquad f\in\mathcal{B}_\alpha,\ \alpha > 1. \tag{1.2}$$

Composing functions f in $\mathcal{H}(\mathcal{D})$ with any holomorphic self-map φ of $\mathcal{D}$, induces a linear transformation, denoted by C_φ and called a ***composition operator*** on $\mathcal{H}(\mathcal{D})$:

$$C_\varphi f = f \circ \varphi.$$

For any $u \in \mathcal{H}(\mathcal{D})$ we define the ***weighted composition operator*** uC_φ on $\mathcal{H}(\mathcal{D})$ as follows:

$$uC_\varphi(f) = (u)(f \circ \varphi).$$

In this paper, we study holomorphic solutions f of the following Schröder's equation:

$$(C_\varphi)f(z) = \lambda f(z), \tag{1.3}$$

and of the corresponding weighted Schröder's equation:

$$uC_\varphi f = \lambda f, \tag{1.4}$$

where λ is a complex constant.

Assuming that φ fixes the origin and satisfies $0 < |\varphi'(0)| < 1$, Königs [5] showed that the set of all holomorphic solutions of equation (1.3) (the eigenfunctions of the operator C_φ acting on $\mathcal{H}(\mathcal{D})$) is exactly $\{\sigma^n\}_{n=0}^{\infty}$, where σ, the principal eigenfunction of C_φ, is called ***Königs function of*** φ.

Following the Königs work, Hosokawa and Nguyen [4] showed that the set of all eigenfunctions of the weighted operator uC_φ acting on $\mathcal{H}(\mathcal{D})$ is exactly $\{v\sigma^n\}_{n=0}^{\infty}$, where v is the principal eigenfunction of uC_φ and σ is the Königs function.
According to a general result of Hammond [2], if uC_φ is compact on any Banach space of holomorphic functions on $\mathcal{D}$ containing polynomials, then all the eigenfunctions $v\sigma^n$ belong to a Banach space. Under somewhat strong restrictions on the growths of u and φ near the boundary of the unit disk, Hosokawa and Nguyen [4] showed that all the eigenfunctions $v\sigma^n$ are eigenfunctions of uC_φ acting on the Bloch space $\mathcal{B}$.

Our goal in this paper is to obtain conditions under which all the eigenfunctions $v\sigma^n$ belong to a Bloch-type space $\mathcal{B}_\alpha$.

The rest of the paper is organized as follows. Section 2 contains some preliminary results. In Section 3 we present our main results concerning composition operators. Theorem 3.1 provides sufficient conditions ensuring all the eigenfunctions σ^n to belong to Bloch type spaces $\mathcal{B}_\alpha$ for $\alpha < 1$. Similar results for $\alpha = 1$ and $\alpha > 1$ are presented in Theorems 3.2 and 3.3, respectively. In Section 4 we prove results concerning the weighted composition operators.

2. Preliminaries

We recall the following criterion for boundedness of the operator uC_φ on the Bloch-type spaces $\mathcal{B}_\alpha$ (see [6, Theorem 2.1]).

Theorem 2.1. *Let u be an analytic function on $\mathcal{D}$, φ be an analytic self-map of $\mathcal{D}$, and let α be a positive real number. Then the following assertions hold.*

1. *If $0 < \alpha < 1$, then uC_φ is bounded on $\mathcal{B}_\alpha$ if and only if $u \in \mathcal{B}_\alpha$ and*
$$\sup_{z\in\mathcal{D}} |u(z)| \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty.$$
2. *The operator uC_φ is bounded on $\mathcal{B}$ if and only if the following conditions are satisfied.*
   (a) $\sup_{z\in\mathcal{D}} |u'(z)|(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2} < \infty$,
   (b) $\sup_{z\in\mathcal{D}} |u(z)|\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)}|\varphi'(z)| < \infty$.
3. *If $\alpha > 1$, then uC_φ is bounded on $\mathcal{B}_\alpha$ if and only if the following conditions are satisfied.*
   (a) $\sup_{z\in\mathcal{D}} |u'(z)|\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha-1}} < \infty$,
   (b) $\sup_{z\in\mathcal{D}} |u(z)|\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)| < \infty$.

The following theorem provides a compactness criterion for the operator uC_φ acting on $\mathcal{B}_\alpha$ (see [6, Theorem 3.1]).

Theorem 2.2. *Let u be a holomorphic function on $\mathcal{D}$ and let φ be a holomorphic self-map of $\mathcal{D}$. Let α be a positive real number, and let uC_φ be bounded on $\mathcal{B}_\alpha$. Then the following assertions hold.*

1. *If $0 < \alpha < 1$, then uC_φ is compact on $\mathcal{B}_\alpha$ if and only if*
$$\lim_{|\varphi(z)|\to 1^-} |u(z)| \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0.$$
2. *The operator uC_φ is compact on $\mathcal{B}$ if and only if the following conditions are satisfied.*
   (a) $\lim_{|\varphi(z)|\to 1^-} |u'(z)|(1-|z|^2)\log\frac{1}{(1-|\varphi(z)|^2)} = 0$,
   (b) $\lim_{|\varphi(z)|\to 1^-} |u(z)|\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)}|\varphi'(z)| = 0$.
3. *If $\alpha > 1$, then uC_φ is compact on $\mathcal{B}_\alpha$ if and only if the following conditions are satisfied.*
   (a) $\lim_{|\varphi(z)|\to 1^-} |u'(z)|\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha-1}} = 0$,

(b) $\lim_{|\varphi(z)|\to 1^-} |u(z)| \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| = 0.$

Remark 2.1. *If in Theorems 2.1 and 2.2 we assume $u \equiv 1$, then they provide a criterion for boundedness and compactness of composition operators C_φ acting on the Bloch-type spaces $\mathcal{B}_\alpha$.*

The following two theorems are fundamental for our work. Theorem 2.3 is the famous Königs theorem about the solutions of Schröder equations (see [5] and [7, Chapter 6]).

Theorem 2.3 (Königs theorem (1884)). *Assume that φ is a holomorphic self-map of $\mathbb{D}$ such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Then the following assertions hold.*

(i) The sequence of functions

$$\sigma_k(z) := \frac{\varphi_k(z)}{\varphi'(0)^k},$$

where φ_k is the k^{th} iteration of φ, converges uniformly on a compact subset of $\mathbb{D}$ to a non-constant function σ that satisfies (1.3) with $\lambda = \varphi'(0)$.

(ii) f and λ satisfy (1.3) if and only if there is a positive integer n such that $\lambda = \varphi'(0)^n$ and f is a constant multiple of σ^n.

The next theorem characterizes all the eigenfunctions of a weighted composition operator under some restriction on the symbol (see [4]).

Theorem 2.4. *Assume that φ is a holomorphic self-map of $\mathbb{D}$ and u is a holomorphic map of $\mathbb{D}$ such that $u(0) \neq 0$, $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Then the following statements hold.*

(i) The sequence of functions

$$v_k(z) = \frac{u(z)u(\varphi(z))...u(\varphi_{k-1}(z))}{u(0)^k},$$

where φ_k is the k^{th} iteration of φ, converges to a non-constant holomorphic function v of $\mathbb{D}$ that satisfies (1.4) with $\lambda = u(0)$.

(ii) f and λ satisfy (1.4) if and only if $f = v\sigma^n$ and $\lambda = u(0)\varphi'(0)^n$, where n is a nonnegative integer and σ is a solution of the Schröder equation (1.3) $\sigma \circ \varphi = \varphi'(0)\sigma$.

3. Composition operators

In this section, we obtain sufficient conditions that ensure all the eigenfunctions σ^n of a composition operator to belong to $\mathcal{B}_\alpha$ for some positive number α and for all positive integers n.

Definition 3.1. Given a number $\alpha > 0$, the *Hyperbolic α-derivative* of a function φ at $z \in \mathcal{D}$ is defined by

$$\varphi^{(h_\alpha)}(z) = \frac{(1-|z|^2)^\alpha \, \varphi'(z)}{(1-|\varphi(z)|^2)^\alpha}.$$

For $\alpha = 1$, it simply is called the Hyperbolic derivative of φ at z, and is denoted by $\varphi^{(h)}(z)$.

Definition 3.2. Let φ be a holomorphic self-map of $\mathcal{D}$ such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let φ_m be the m^{th} iteration of φ for some fixed nonnegative integer m. Then we say that φ satisfies condition (A) if there exists a nonnegative integer m such that

$$\text{(A)} \qquad |\varphi^{(h_\alpha)}(\varphi_m(z))| = \frac{(1-|\varphi_m(z)|^2)^\alpha \, |\varphi'(\varphi_m(z))|}{(1-|\varphi_{m+1}(z)|^2)^\alpha} \le |\varphi'(0)|,$$

for all $z \in \mathcal{D}$ and for some fixed $\alpha > 0$.

Remark 3.1. *If condition* (A) *is satisfied for some m, then it also is satisfied for all nonnegative integers greater than m.*

The following example provides a family of maps that satisfies condition (A). The example is borrowed from [3].

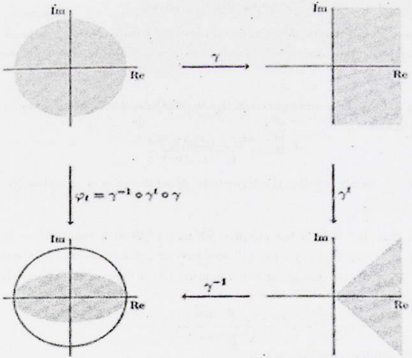
Example 3.1. Consider a map γ that maps the unit disk univalently to the right half plane. This map is given by formula:

$$\gamma(z) = \frac{1+z}{1-z}.$$

For any $t \in (0,1)$, define

$$\varphi_t(z) = \frac{\gamma(z)^t - 1}{\gamma(z)^t + 1}.$$

It is well known that φ_t maps the unit disk into itself for each $t \in (0,1)$ (see [7]). These maps are known as *lens maps*.



Claim 3.1. The map φ_t satisfies the condition (A) for $\alpha = 1$ and $m = 0$, that is, $|\varphi_t^{(h)}(z)| \leq |\varphi_t'(0)|$ for all $t \in (0,1)$ and for all $z \in \mathcal{D}$.

Proof. Clearly, we have $\varphi_t(0) = 0$ and

$$|\varphi_t'(z)| = \frac{2t\ |\gamma(z)^{t-1}|\ |\gamma'(z)|}{|\gamma(z)^t + 1|^2}.$$

Since $\gamma'(z) = \dfrac{2}{(1-z)^2}$, we see that $|\varphi_t'(0)| = t$. It is known that the image of φ_t touches the boundary of the unit disk non-tangentially at 1 and -1. Now we put $w = \gamma(z) = re^{i\theta}$ to obtain

$$|\varphi_t^{(h)}(z)| = \frac{1-|z|^2}{1-\left|\frac{w^t-1}{w^t+1}\right|^2}\ \frac{2t\ |w^{t-1}|\ |w'|}{|w^t+1|^2}$$
$$= \frac{1-|z|^2}{|w^t+1|^2 - |w^t-1|^2}\ 2t\ |w^{t-1}|\ |w'|.$$

On the other hand, we have

$$|w^t+1|^2-|w^t-1|^2=(w^t+1)\overline{(w^t+1)}-(w^t-1)\overline{(w^t-1)}=$$
$$=(w^t+1)(\overline{w}^t+1)-(w^t-1)(\overline{w}^t-1)=2(w^t+\overline{w}^t)=2\,r^t(e^{it\theta}+e^{-it\theta})=4\,r^t\cos t\theta.$$

Also, we have $w'=\gamma'(z)=\dfrac{2}{(1-z)^2}$, and

$$|\varphi_t^{(h)}(z)|=\frac{1-|z|^2}{|1-z|^2}\,\frac{t\,r^{t-1}|e^{i(t-1)\theta}|}{r^t\cos t\theta}.$$

Using $z=\dfrac{w-1}{w+1}$, we get

$$|\varphi_t^{(h)}(z)|=\frac{1-\left|\frac{w-1}{w+1}\right|^2}{\left|1-\frac{w-1}{w+1}\right|^2}\,\frac{t\,r^{t-1}}{r^t\cos t\theta}=$$
$$=\frac{|w+1|^2-|w-1|^2}{4}\,\frac{t\,r^{t-1}}{r^t\cos t\theta}=\frac{4\,r\cos\theta}{4}\,\frac{t\,r^{t-1}}{r^t\cos t\theta}=\frac{t\cos\theta}{\cos t\theta}.$$

If $z\in(-1,1)$, then $\gamma(z)\in\mathbb{R}_+$. Therefore $\theta=0$ and so $|\varphi_t^{(h)}(z)|=t$. On the other hand, if $z\in\mathcal{D}\setminus(-1,1)$, then $|\theta|\in(0,\pi/2)$. Hence $\cos t\theta>\cos\theta>0$, and so $|\varphi_t^{(h)}(z)|<t$. This completes the proof. □

Remark 3.2. *From the proof of Claim 3.1, we see that $|\varphi_t^{(h)}(z)|\nrightarrow 0$ as z approaches the boundary of the unit disk along the real-axis. Hence the composition operator with symbol φ_t is a non-compact operator on $\mathcal{B}$.*

The following proposition, which provides a sufficient condition for Königs function to belong to Bloch-type spaces, plays an important role in the proofs of our main results.

Proposition 3.1. *Assume that the operator C_φ is bounded on $\mathcal{B}_\alpha$, and φ satisfies condition (A) for some $\alpha>0$ and for some fixed nonnegative integer m. Then σ belongs to $\mathcal{B}_\alpha$.*

Proof. Since the operator C_φ is bounded on $\mathcal{B}_\alpha$, there exists a positive number M such that

(3.1) $$(1-|z|^2)^\alpha|\varphi'(z)|\le M(1-|\varphi(z)|^2)^\alpha \qquad \text{for } z\in\mathcal{D}.$$

For m given by the assumption, choose a nonnegative integer k such that $k > m$. For $z \in \mathcal{D}$, we have

$$(1-|z|^2)^\alpha|\varphi_k'(z)| = (1-|z|^2)^\alpha\ |\varphi'(\varphi_{k-1}(z))\varphi'(\varphi_{k-2}(z))...\varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z))...\varphi'(z)|$$
$$= (1-|z|^2)^\alpha\ |\varphi'(z)\varphi'(\varphi(z))\ ...\varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z))...\varphi'(\varphi_{k-2}(z))\ \varphi'(\varphi_{k-1}(z)|.$$

By using (3.1), we obtain

$$(1-|z|^2)^\alpha|\varphi_k'(z)| \leq$$
$$\leq M(1-|\varphi(z)|^2)^\alpha\ |\varphi'(\varphi(z))\ ...\varphi'(\varphi_{m-1}(z))\varphi'(\varphi_m(z))...\varphi'(\varphi_{k-2}(z))\ \varphi'(\varphi_{k-1}(z)|.$$

Again using (3.1) repeatedly, we get

$$(1-|z|^2)^\alpha|\varphi_k'(z)| \leq M^m\ (1-|\varphi_m(z)|^2)^\alpha|\varphi'(\varphi_m(z))...\varphi'(\varphi_{k-1}(z)|$$

Now using condition (A) repeatedly, we get

$$(1-|z|^2)^\alpha|\varphi_k'(z)| \leq M^m|\varphi'(0)^{k-m}|\ (1-|\varphi_k(z)|^2)^\alpha.$$

Thus, we have

$$\lim_{k\to\infty}(1-|z|^2)^\alpha\ \left|\frac{\varphi_k'(z)}{\varphi'(0)^k}\right| \leq \frac{M^m}{|\varphi'(0)^m|}\overline{\lim}_{k\to\infty}\ (1-|\varphi_k(z)|^2)^\alpha \leq \frac{M^m}{|\varphi'(0)^m|},$$

implying that $(1-|z|^2)^\alpha|\sigma'(z)| \leq \dfrac{M^m}{|\varphi'(0)^m|}$. Hence, $\sigma \in \mathcal{B}_\alpha$. Proposition 3.1 is proved. □

The following corollary provides a sufficient condition that ensures all the integer powers of the Königs function to belong to Bloch-type spaces $\mathcal{B}_\alpha$ for $\alpha < 1$.

Theorem 3.1. *Suppose $\alpha < 1$. If operator C_φ is bounded on $\mathcal{B}_\alpha$ and φ satisfies the condition* (A), *then $\sigma^n \in \mathcal{B}_\alpha$ for all positive integers n.*

Proof. From Proposition 3.1, we see that $\sigma \in \mathcal{B}_\alpha$. Let $\mathbb{H}^\infty$ denote the space of bounded holomorphic functions on the unit disk $\mathcal{D}$. Since $\mathcal{B}_\alpha \subset \mathbb{H}^\infty$ for $\alpha < 1$, there exists a positive constant C such that $\|\sigma\|_{\mathbb{H}^\infty} \leq C$, and

$$\begin{aligned}(1-|z|^2)^\alpha|(\sigma^n(z))'| &= (1-|z|^2)^\alpha\ |n\ \sigma^{n-1}(z)\ \sigma'(z)|\\ &\leq \|\sigma\|_{\mathcal{B}_\alpha}\ n\ |\sigma^{n-1}(z)|\\ &\leq n\ \|\sigma\|_{\mathcal{B}_\alpha}\ C^{n-1}.\end{aligned}$$

Hence, $\sigma^n \in \mathcal{B}_\alpha$ for all positive integers n. □

The following theorem gives a sufficient condition that ensures all the integer powers of Königs function to belong to the Bloch space.

Theorem 3.2. *Let φ be a holomorphic self-map of $\mathcal{D}$ such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$. Also, assume that*

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}\frac{\log\frac{2}{1-|z|}}{\log\frac{2}{1-|\varphi(z)|}}|\varphi'(z)| \leq |\varphi'(0)| \quad \text{for all } z \in \mathcal{D}. \tag{3.2}$$

Then operator C_φ is bounded on $\mathcal{B}$ and $\sigma^n \in \mathcal{B}$ for all positive integers n.

Proof. The boundedness of C_φ on the Bloch space follows from Schwarz-Pick theorem. From the hypothesis of the theorem, we have

$$(1-|z|^2)\log\frac{2}{1-|z|}|\varphi'(z)| \leq |\varphi'(0)|(1-|\varphi(z)|^2)\log\frac{2}{1-|\varphi(z)|} \quad \text{for all } z \in \mathcal{D}. \tag{3.3}$$

Let k be a positive integer, then we have

$$\begin{aligned}(1-|z|^2)|\varphi_k'(z)|\log\frac{2}{1-|z|} =& (1-|z|^2)|\varphi'(z)\varphi'(\varphi(z))......\varphi'(\varphi_{k-1}(z))|\log\frac{2}{1-|z|}\\ =& (1-|z|^2)\log\frac{2}{1-|z|}|\varphi'(z)\varphi'(\varphi(z))......\varphi'(\varphi_{k-1}(z))|.\end{aligned}$$

By using (3.3), we see that

$$(1-|z|^2)|\varphi_k'(z)|\log\frac{2}{1-|z|} \leq |\varphi'(0)|(1-|\varphi(z)|^2)\log\frac{2}{1-|\varphi(z)|}|\varphi'(\varphi(z))......\varphi'(\varphi_{k-1}(z))|.$$

And using (3.3) repeatedly, we get

$$\begin{aligned}(1-|z|^2)|\varphi_k'(z)|\log\frac{2}{1-|z|} =& |\varphi'(0)|^k(1-|\varphi_k(z)|^2)\log\frac{2}{1-|\varphi_k(z)|}\\ \leq& 2|\varphi'(0)|^k(1-|\varphi_k(z)|)\log\frac{2}{1-|\varphi_k(z)|}.\end{aligned}$$

Since $\log x \leq x$ for $x > 1$, we have

$$(1-|z|^2)|\varphi_k'(z)|\log\frac{2}{1-|z|} \leq 4|\varphi'(0)|^k.$$

Hence,

$$\lim_{k\to\infty}(1-|z|^2)\left|\frac{\varphi_k'(z)}{\varphi'(0)^k}\right|\log\frac{2}{1-|z|} = (1-|z|^2)|\sigma'(z)|\log\frac{2}{1-|z|} \leq 4, \quad z \in \mathcal{D},$$

showing that

$$|\sigma'(z)| \leq \frac{4}{(1-|z|^2)\log\frac{2}{1-|z|}}. \tag{3.4}$$

Recall that $\sigma(0) = 0$. Now we obtain an estimate for σ. We have

$$|\sigma(z)| = \left|\int_0^1 \sigma'(tz)d(tz)\right| \leq \int_0^1 |\sigma'(tz)d(t|z|) \leq \int_0^1 \frac{4}{\log\frac{2}{1-|tz|}}\frac{1}{1-|tz|^2}d(t|z|) \leq$$

$$(3.5) \qquad \leq 4\left[\log\left(\log\frac{2}{1-t|z|}\right)\right]_0^1 = 4\left[\log\left(\log\frac{2}{1-|z|}\right) - \log(\log 2)\right].$$

Next, by using (3.4) and the above obtained estimate for σ, we get

$$\begin{aligned}(1-|z|^2)(\sigma^n(z))' &= (1-|z|^2)\; n\; |\sigma^{n-1}(z)\; \sigma'(z)| \\ &\leq 4^n n\left(\log\log\frac{2}{1-|z|} - \log\log 2\right)^{n-1} \frac{1}{\log\frac{2}{1-|z|}}.\end{aligned}$$

Finally, it is easy to see that the right-hand side of the last expression teds to zero as $|z| \to 1$. Hence $\sigma^n \in \mathcal{B}$ for all positive integers n. □

Let us recall the Lipschitz-type norm, which is equivalent to the usual norm, defined for function $f \in \mathcal{B}_\alpha$, $\alpha > 1$ by

$$\|f\|_{\mathcal{B}_\alpha} \equiv \sup_{z\in\mathcal{D}}(1-|z|^2)^{\alpha-1}|f(z)|.$$

Next, we present results for the Bloch-type spaces $\mathcal{B}_\alpha$ for $\alpha > 1$. We start with the following definition.

Definition 3.3. Suppose $f \in \mathcal{B}_\alpha$ for some $\alpha > 0$, then we define the *Bloch number* of f by $b_f = \inf_\alpha \{\alpha : f \in \mathcal{B}_\alpha\}$.

Proposition 3.2. *Suppose $\beta > 0$. Then $f^n \in \mathcal{B}_{\beta+1}$ for all positive integers n if and only if b_f is at most 1.*

Proof. Suppose $f^n \in \mathcal{B}_{\beta+1}$ for all positive integers n. We have to show that $b_f \leq 1$. On the contrary, assume $b_f > 1$. Then there exists a positive integer n_0 such that $1 < 1 + \frac{\beta}{n_0} < b_f$. Now, in view of definition of Lipschitz-type norm, we see that for any fixed positive integer M there exists $z \in \mathcal{D}$ such that

$$M \leq (1-|z|^2)^{\beta/n_0}|f(z)| \leq \{(1-|z|^2)^{\beta/n_0}|f(z)|\}^{n_0} = (1-|z|^2)^{\beta}|f(z)|^{n_0},$$

showing that

$$M \leq \sup_{z\in\mathcal{D}}(1-|z|^2)^{\beta}|f(z)|^{n_0} = \|f^{n_0}\|_{\mathcal{B}_{\beta+1}}.$$

Since M is an arbitrary positive integer, we have $f^{n_0} \notin \mathcal{B}_{\beta+1}$. Which is a contradiction.

Conversely, suppose that $b_f \leq 1$. Since $\mathcal{B}_\alpha \subset \mathcal{B}$ for all $\alpha \leq 1$, then clearly $f \in \mathcal{B}$. For any fixed $\beta > 0$ and for any fixed positive integer n, we have

$$\begin{aligned}(1-|z|^2)^{\beta+1}|(f^n)'(z)| =&(1-|z|^2)^{\beta+1}|nf^{n-1}(z)f'(z)|\\ =&n(1-|z|^2)|f'(z)|(1-|z|^2)^{\beta}|f^{n-1}(z)|\\ \leq& n\|f\|_{\mathcal{B}}(1-|z|^2)^{\beta}\left(\|f\|_{\mathcal{B}}\log\frac{1}{1-|z|}\right)^{n-1}\\ =&n(\|f\|_{\mathcal{B}})^n(1-|z|^2)^{\beta}\left(\log\frac{1}{1-|z|}\right)^{n-1}.\end{aligned}$$

The last expression goes to zero as $|z| \to 1$, showing that $f^n \in \mathcal{B}_{\beta+1}$ for all positive integers n. □

Theorem 3.3. *Let φ be a holomorphic self-map of $\mathcal{D}$ such that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let $\alpha > 1$. If $|\varphi^{(h)}(z)| \leq |\varphi'(0)|$ for all $z \in \mathcal{D}$, then operator C_φ is bounded on $\mathcal{B}_\alpha$ and $\sigma^n \in \mathcal{B}_\alpha$ for all positive integers n.*

Proof. Since $|\varphi^{(h)}(z)| \leq |\varphi'(0)|$ for all $z \in \mathcal{D}$, by Proposition 3.1 we have $\sigma \in \mathcal{B}$. So $b_f \leq 1$. Therefore the result follows from Proposition 3.2. □

4. Weighted Composition operators

Recall that if u is a holomorphic function of the unit disk, and φ is a holomorphic self-map of the unit disk, then the Schröder equation for weighted composition operator is given by

$$u(z)f(\varphi(z)) = \lambda f(z), \tag{4.1}$$

where $f \in \mathcal{H}(\mathcal{D})$ and λ is a complex constant.

Also, recall that if $u(0) \neq 0$, $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, then the solutions of equation (4.1) are given by Theorem 2.4. The principal eigenfunction corresponding to the eigenvalue $u(0)$ we denote by v, and observe that all the other eigenfunctions are of the form $v\sigma^n$, where σ is the Königs function of φ and n is a positive integer. Hosokawa and Nguyen [4] studied the equation (4.1) in the Bloch space and obtained the following result.

Theorem 4.1. *Let φ be a holomorphic self-map of $\mathcal{D}$ with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let u be a holomorphic map of $\mathcal{D}$ such that $u(0) \neq 0$. Assume that operator*

uC_φ is bounded on $\mathcal{B}$. Further, for $0 < r < 1$, we set

$$M_r(\varphi) = \sup_{|z|=r} |\varphi(z)|, \qquad a_r = \sup_{|z|=r} \left(|u'(z)\varphi(z)| + |u(z)\varphi'(z)|\right),$$

and assume that the following conditions are satisfied:

(i) $\lim_{r\to 1} \log(1-r)\log M_r(\varphi) = \infty$.

(ii) $\log|a_r| < \epsilon \log(1-r) \log M_r(\varphi)$,

where $\epsilon > 0$ is a constant satisfying $\epsilon \log \|\varphi\|_\infty > -1$.

Then $u\sigma^n \in \mathcal{B}$ for all nonnegative integers n.

Now we proceed to obtain conditions on the weight u and on the symbol φ of the weighted composition operators uC_φ that ensure $u\sigma^n$ to belong to Bloch-type spaces $\mathcal{B}_\alpha$ for some $\alpha > 0$ and for all nonnegative integers n. We begin with the following remark.

Remark 4.1. *Let f be a holomorphic function defined on $\mathcal{D}$. If $\|f'\|_\infty < M$ for some $M > 0$, then we have*

$$|f(z) - f(0)| = \left|\int_0^1 zf'(tz)dt\right| \le \int_0^1 |z\ f'(tz)|dt \le M\int_0^1 |z|dt.$$

If, in addition, f also satisfies $f(0) = 0$, then $\|f\|_\infty \le M$.

Proposition 4.1. *Let φ be a univalent holomorphic self-map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let σ be the Königs function of φ. Then σ is bounded if and only if there is a positive integer k such that $\|\varphi_k\|_\infty < 1$.*

Proof. Suppose that σ is bounded. Since φ is univalent, σ is also univalent (see [7], p. 91). Since σ is bounded univalent map, there is a positive integer k such that $\|\varphi_k\|_\infty < 1$ (see [7]).

Conversely, suppose there is a positive integer k such that $\|\varphi_k\|_\infty < 1$. Since $\sigma(\varphi(z)) = \varphi'(0)\sigma(z)$, we have

$$\sigma(\varphi_k(z)) = \sigma(\varphi(\varphi_{k-1}(z)) = \varphi'(0)\sigma(\varphi_{k-1}(z)) = \varphi'(0)^k\sigma(z).$$

Clearly the left-hand side of the last relation is bounded, and therefore σ is also bounded, which completes the proof. □

Theorem 4.2. *Let φ be a univalent holomorphic self-map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$ satisfying $|\varphi^{(h_\alpha)}(z)| \le |\varphi'(0)|$ for all $z \in \mathcal{D}$ and for some fixed*

$\alpha < 1$. If u is a holomorphic map of $\mathcal{D}$ such that $u(0) \neq 0$ and $\|u'\|_\infty < \infty$, then operator uC_φ is bounded on $\mathcal{B}_\alpha$ and $v\sigma^n \in \mathcal{B}_\alpha$ for all nonnegative integers n.

Proof. Since $\|u\|_\infty < \|u'\|_\infty + |u(0)| < \infty$ and $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$, the operator uC_φ is bounded on $\mathcal{B}_\alpha$ for some $\alpha < 1$.

Since $|\varphi^{(h_\alpha)}(z)| \leq |\varphi'(0)|$ for some $\alpha < 1$, in view of Proposition 3.1, we see that $\sigma \in \mathcal{B}_\alpha$ for $\alpha < 1$, and hence is bounded. Next, since φ is univalent, σ is also univalent. Consequently, there exists a nonnegative integer k such that $\|\varphi_k\|_\infty < 1$. Composing φ_{k-1} on both sides of the Schröder equation (4.1) from right, we get

$$u(\varphi_{k-1}(z))f(\varphi_k(z)) = \lambda f(\varphi_{k-1}(z)). \tag{4.2}$$

The left-hand side of the above equation is bounded, and so is $f \circ \phi_{k-1}$. Hence, differentiating both side of (4.2), we get

$$u'(\varphi_{k-1}(z))\, \varphi'_{k-1}(z)\, f(\varphi_k(z)) + u(\varphi_{k-1}(z))\, f'(\varphi_k(z))\, \varphi'_k(z) = \lambda f'(\varphi_{k-1}(z))\, \varphi'_{k-1}(z).$$

Next, multiplying both sides of the last equation by $(1-|z|^2)^\alpha$, and using boundedness of $\|u'\|_\infty$, $\|u\|_\infty$, $f \circ \varphi_k$ and $f' \circ \varphi_k$, we see that there exists a constant M such that

$$(1-|z|^2)^\alpha |\lambda f'(\varphi_{k-1}(z))\varphi'_{k-1}(z)| \leq M(1-|z|^2)^\alpha \left(|\varphi'_{k-1}(z)| + |\varphi'_k(z)|\right). \tag{4.3}$$

The right-hand side of the above inequality is uniformly bounded, and therefore the left-hand side is bounded. Again, we compose φ_{k-2} on (4.1), to get

$$u(\varphi_{k-2}(z))f(\varphi_{k-1}(z)) = \lambda f(\varphi_{k-2}(z)).$$

Now we differentiate the above equation, then multiply by both sides by $(1-|z|^2)^\alpha$, and use (4.2) and (4.3) to show that $(1-|z|^2)^\alpha |f'(\varphi_{k-2}(z))\varphi'_{k-2}(z)|$ is bounded.

Continuing this process, we see that that $\sup_{z\in\mathcal{D}}(1-|z|^2)^\alpha|f'(z)|$ is bounded, and hence $f \in \mathcal{B}_\alpha$. By Theorem 2.4, any holomorphic f satisfying (4.1) is of the form $v\sigma^n$ for some positive integer n, implying that $v\sigma^n \in \mathcal{B}_\alpha$ for all nonnegative integers n. This completes the proof. Theorem 4.2 is proved. □

The following two theorems give sufficient conditions that ensure $v\sigma^n$ to belong to Bloch-type spaces $\mathcal{B}_\alpha$ for some $\alpha > 1$ and for all nonnegative integers n.

Theorem 4.3. *Let φ be a holomorphic self-map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let u be a holomorphic map of $\mathcal{D}$ such that $u(0) \neq 0$. Assume* *that for a fixed positive number β*

$$|u(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta} \le |u(0)| \quad \text{for all } z \in \mathcal{D}.$$

Then the following statements hold.

(i) *If $|\varphi^{(h_\alpha)}(z)| \le |\varphi'(0)|$ for all $z \in \mathcal{D}$ and for some $\alpha < 1$, then $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all nonnegative integers n.*

(ii) *If $|\varphi^{(h)}(z)| \le |\varphi'(0)|$ for all $z \in \mathcal{D}$, then $v\sigma^n \in \mathcal{B}_{p+1}$ for some $p > \beta$ and for all nonnegative integers n.*

Proof. We first prove the assertion (i). From the definition of v_k (see Theorem 2.4), we have

$$\begin{aligned}(1-|z|^2)^\beta|v_k(z)| &= (1-|z|^2)^\beta\frac{|u(z)u(\varphi(z))\ldots\ldots u(\varphi_{k-1}(z))|}{|u(0)|^k}\\ &\le (1-|\varphi(z)|^2)^\beta\frac{|u(\varphi(z))\ldots\ldots u(\varphi_{k-1}(z))|}{|u(0)|^{k-1}}\ldots \le 1.\end{aligned}$$

Hence $(1-|z|^2)^\beta|v(z)| = \lim_{k\to\infty}(1-|z|^2)^\beta|v_k(z)| \le 1$. Since z is arbitrary, we have

$$\sup_{z\in\mathcal{D}}(1-|z|^2)^\beta|v(z)| < \infty.$$

On the other hand, the assumption $|\varphi^{(h_\alpha)}(z)| \le |\varphi'(0)|$ and Proposition 3.1 imply that $\sigma^n \in \mathcal{B}_\alpha \subset \mathbb{H}^\infty$ for all nonnegative integer n. Therefore,

$$\sup_{z\in\mathcal{D}}(1-|z|^2)^\beta|v(z)\sigma^n(z)| < \infty$$

for all nonnegative integers n. Considering the equivalent norm (see (1.2)), we conclude that $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all nonnegative integers n. This completes the proof of assertion (i).

To prove the assertion (ii), observe first that from the proof of part (*i*), we have

$$\sup_{z\in\mathcal{D}}(1-|z|^2)^\beta|v(z)| < \infty. \tag{4.4}$$

On the other hand, since $|\varphi^{(h)}(z)| \le |\varphi'(0)|$, Proposition 3.1 implies that $\sigma \in \mathcal{B}$, and hence there exists a number $M > 0$ such that

$$|\sigma(z)| \le M\log\frac{2}{1-|z|^2}. \tag{4.5}$$

Next, using equations (4.4) and (4.5), with some constant $C > 0$ we have

$$\begin{aligned}(1-|z|^2)^p|v(z)\sigma^n(z)| =&\{(1-|z|^2)^\beta|v(z)|\}\{(1-|z|^2)^{p-\beta}|\sigma^n(z)|\}\\ \leq& CM(1-|z|^2)^{p-\beta}\left(\log\frac{2}{1-|z|^2}\right)^n.\end{aligned}$$

Finally, it is easy to see that the last expression goes to zero as $|z| \to 1$. Hence, $v\sigma^n \in \mathcal{B}_{p+1}$ for all nonnegative integers n. Theorem 4.3 is proved. □

Theorem 4.4. *Let φ be a holomorphic self-map of the unit disk with $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and let u be a holomorphic map of $\mathcal{D}$ such that $u(0) \neq 0$. Suppose that β is a positive integer and the following conditions are satisfied:*

(i) $|u(z)|\dfrac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}\dfrac{\log\frac{2}{(1-|z|)^\beta}}{\log\frac{2}{(1-|\varphi(z)|)^\beta}} \leq |u(0)|$ *for all* $z \in \mathcal{D}$

(ii) $|\varphi^{(h)}(z)|\dfrac{\log\frac{2}{1-|z|}}{\log\frac{2}{1-|\varphi(z)|}} \leq |\varphi'(0)|$ *for all* $z \in \mathcal{D}$.

Then $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all nonnegative integers n.

Proof. In view of the definition of v_k (see Theorem 2.4) and the condition (i), we can write

$$\begin{aligned}(1-|z|^2)^\beta\log\frac{2}{(1-|z|)^\beta}|v_k(z)| &= (1-|z|^2)^\beta\log\frac{2}{(1-|z|)^\beta}\frac{|u(z)u(\varphi(z))......u(\varphi_{k-1}(z))|}{|u(0)|^k}\\ &\leq (1-|\varphi(z)|^2)^\beta\log\frac{2}{(1-|\varphi(z)|)^\beta}\frac{|u(\varphi(z))......u(\varphi_{k-1}(z))|}{|u(0)|^{k-1}}\\ &\leq (1-|\varphi_k(z)|^2)^\beta\log\frac{2}{(1-|\varphi_k(z)|)^\beta} \leq 2^\beta(1-|\varphi_k(z)|)^\beta\log\frac{2}{(1-|\varphi_k(z)|)^\beta}.\end{aligned}$$

Since $\log x \leq x$ for $x > 1$, we have

$$(1-|z|^2)^\beta\log\frac{2}{(1-|z|)^\beta}|v_k(z)| \leq 2^{\beta+1}.$$

So taking limit as k approaches to ∞, we see that

$$(1-|z|^2)^\beta|v(z)| \leq \frac{2^{\beta+1}}{\log\frac{2}{1-|z|}}. \tag{4.6}$$

On the other hand, since φ satisfies condition (ii), in view of equation (3.5), there exists $K > 0$ such that

$$|\sigma(z)| \leq K\log\log\frac{2}{1-|z|}. \tag{4.7}$$

Now using (4.6) and (4.7), we get

$$(1-|z|^2)^\beta|v(z)\sigma^n(z)| \leq \frac{2^{\beta+1}K^n}{\log\frac{2}{1-|z|}}\left(\log\log\frac{2}{1-|z|}\right)^n.$$

Clearly the right-hand side of the above equation goes to 0 as $|z| \to 1$. Using the norm defined in (1.2), we conclude that $v\sigma^n \in \mathcal{B}_{\beta+1}$ for all nonnegative integers n. Theorem 4.4 is proved. □

This paper is based on a research which forms a part of the author's Ph.D. dissertation from University of Toledo. The author wishes to express his deep gratitude to his dissertation adviser Professor Željko Čučković.

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Поступила 29 мая 2016