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MEROMORPHIC SOLUTIONS FOR A CLASS OF DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

WEIRAN LÜ, FENG LÜ, LINLIN WU AND JING YANG

China University of Petroleum, P.R. China

E-mails: *lvfeng18@126.com; 2528307075@qq.com; 1310231370@qq.com*

Abstract. In this note, we study the admissible meromorphic solutions for algebraic differential equation $f^n f' + P_{n-1}(f) = R(z)e^{\alpha(z)}$, where $P_{n-1}(f)$ is a differential polynomial in f of degree $\leq n-1$ with small function coefficients, R is a non-vanishing small function of f, and α is an entire function. We show that this equation does not possess any meromorphic solution $f(z)$ satisfying $N(r,f) = S(r,f)$ unless $P_{n-1}(f) \equiv 0$. Using this result, we generalize a well-known result by Hayman.

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1. Introduction and main results

Let f denote a transcendental meromorphic function. We assume that the reader is familiar with the fundamental results of Nevanlinna theory and its standard notation such as $m(r,f)$, $N(r,f)$, $T(r,f)$, $S(r,f)$, etc. (see [8] and [24]). Recall that a nonconstant meromorphic function α is said to be a small function of f if $T(r,\alpha) = S(r,f)(= o(1)T(r,f))$ as $r \to \infty$, possibly outside a set of r values of finite linear measure. Also, a polynomial in f and its derivatives with small functions of f being the coefficients is called a differential polynomial in f. By $P_n(f)$ we will denote a differential polynomial in f with the total degree in f and its derivatives $\leq n$. By $\rho(f)$ and $\lambda(f)$ we will denote the order and the exponent of convergence of zeros of f, respectively. We will need the following concept of admissibility (see, e.g., [14], [15]).

Definition 1.1. *Let $R(z,\omega)$ be rational in ω with meromorphic coefficients. A meromorphic solution ω of equation $(\omega')^n = R(z,\omega)$ is called admissible if $T(r,a) = S(r,\omega)$ for all coefficients $a(z)$ of $R(z,\omega)$.*

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It is clear that admissibility makes sense relative to any family of meromorphic functions, without any reference to differential equations.

In 1980, Gackstatter and Laine [6] conjectured that the following algebraic differential equation:

$$(f')^n = p_m(f),$$

where $p_m(f)$ is a polynomial in f and n is a positive integer, does not possess any admissible solution when $m \leq n-1$. In 1990, He and Laine [12] gave a positive answer to this conjecture. Recently, Zhang and Liao [25] proved that if the following algebraic differential equation with polynomial coefficients:

$$P_n(f) = 0 \tag{1.1}$$

has only one dominant term (highest-degree term), then the equation (1.1) has no admissible transcendental meromorphic solutions with a few poles. Liu et al. [18] considered the possible admissible solutions for the following algebraic differential equation:

$$f^n f^{(k)} + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0 = Re^{\alpha}, \tag{1.2}$$

where a_j $(j = 0, 1, \cdots, n-1)$ are small functions of f, R is a nonzero small function and α is an entire function. They have obtained a simple expression for meromorphic solutions of equation (1.2) provided that the solutions satisfy $N(r, f) = S(r, f)$. This also means that the solutions have finitely many zeros determined by the term Re^{α} in the differential equation. Further, this result can be viewed as a generalization of the following well-known result due to Hayman [9] in 1959, which is a prototype of the studies of the zeros of certain special type of differential polynomials.

Theorem A. *Let f be a transcendental meromorphic function, and $n \geq 3$ be an integer. Then $f^n f'$ assumes all finite values, except possibly zero, infinitely many times.*

Later, Hayman [10] conjectured that Theorem A remains valid for $n = 1$ and 2. Then, Hayman's conjecture was confirmed by Mues [20] in the case $n = 2$, and independently by Bergweiler and Eremenko [2] and Chen and Fang [3] in the case $n = 1$. For the related results we refer to [1], [5], [7], [13], [16], [21], [22], and references therein.

It is clear now that distributions of zeros of differential polynomials $P(f)$ of the form $P(f) = f^n f^{(k)} - b$, with $n \geq 1$, $k = 1$ and b a nonzero constant, have been dealt with. In this paper, we study similar problems for such differential polynomials when $n = 1$ and $k \geq 2$, as well as for more general differential polynomials when $n \geq 2$.

Before proceeding further, we recall two known results from [17] and [18].

Theorem B ([17]). *Let $Q_d(z, f)$ be a differential polynomial in f of degree d with rational function coefficients. Suppose that u is a nonzero rational function and v is a nonconstant polynomial. If $n \geq d + 1$ and the differential equation*

$$f^n f' + Q_d(z, f) = u(z)e^{v(z)} \tag{1.3}$$

has a meromorphic solution f with finitely many poles, then f has the following form:

$$f(z) = s(z)e^{v(z)/(n+1)} \text{ and } Q_d(z, f) \equiv 0,$$

where $s(z)$ is a rational function satisfying $s^n((n+1)s' + v's) = (n+1)u$.

Theorem C ([18]). *Let f be a transcendental meromorphic function and α be an entire function, and let q and R be small functions of f with $q \not\equiv 0$.. Then the differential equation $ff' - q = Re^{\alpha}$ has no transcendental meromorphic solutions.*

Remark 1.1. *In* [19], *the authors of the present paper proved the following result. Let α and β be entire functions, and let p, q, R_1 and R_2 be non-vanishing rational functions. Then the system of equations: $pff^{(k)} - q - R_1e^{\alpha}$, $pff^{(l)} - q - R_2e^{\beta}$ has no transcendental solutions for integers l and k with $l > k \geq 2$.*

Now we are in position to state our first main result, which extends Theorem B, proved in [17]. Note that our proof is different and much simple than that of applied [17]. For related recent results we refer the papers [17] – [19]).

Theorem 1.1. *Let $P_{n-1}(f)$ be a differential polynomial in f with coefficients being small functions, and let $\deg P_{n-1}(f) \leq n - 1$. Then for any positive integer n, any entire function α and any small function R, the equation*

$$f^n f' + P_{n-1}(f) = Re^{\alpha} \tag{1.4}$$

does not possess any transcendental meromorphic solution $f(z)$ with $N(r, f) = S(r, f)$ unless $P_{n-1}(f) \equiv 0$. Moreover, if the equation (1.4) possesses a meromorphic solution f with $N(r, f) = S(r, f)$, then (1.4) will become $f^n f' = Re^{\alpha}$ and $f(z)$ has the form $f(z) = u\exp(\alpha/(n+1))$ as the only possible admissible solution of (1.4), where u is a small function of f.

Corollary 1.1. *Let f be a transcendental meromorphic function with $N(r, f) = S(r, f)$, and let $P_{n-1}(f)$ be a differential polynomial in f with small functions as its* *coefficients, such that $P_{n-1}(0) \not\equiv 0$ and $\deg P_{n-1}(f) \leq n-1$. Then for any positive integer n, the differential form $f^n f' + P_{n-1}(f)$ has infinitely many zeros.*

Based on Corollary 1.1, we pose the following more general conjecture.

Conjecture 1.1. *Let f be a transcendental meromorphic function with $N(r, f) = S(r, f)$, and let $P_{n-1}(f)$ be a differential polynomial in f with small functions as its coefficients, such that $\deg P_{n-1}(f) \leq n-1$ and $P_{n-1}(0) \not\equiv 0$. Then for any positive integers n and k, the differential form $f^n f^{(k)} + P_{n-1}(f)$ has infinitely many zeros.*

Remark 1.2. *The condition $N(r, f) = S(r, f)$ in Corollary 1.1 is necessary. For example, let $f(z) = \frac{e^z}{e^z - 1}$. Then $f^2 f' + \frac{3}{2} f'' + \frac{3}{2} f' + f - 1 = -\frac{1}{(e^z-1)^4}$ has no zeros.*

Also, the condition $P_{n-1}(0) \not\equiv 0$ is necessary. For instance, if $f(z) = z^2 e^z$, then $z^2 f^3 f' + z^2 f f' - (2+z) z f^2 = (2+z) z^9 e^{4z}$ has finitely many zeros. The conclusion of Corollary 1.1 becomes invalid, if we replace the condition $\deg P_{n-1}(f) \leq n-1$ by the condition $\deg P_n(f) \leq n$. Indeed, to see this, take $f(z) = e^z - 1$, and observe that $P_2(f) = 2f^2 + 3f + 1$ and $f^2 f' + P_2(f) = e^{3z}$ has no zeros.

Remark 1.3. *(see [18]). Let f be an admissible meromorphic solution of equation (1.2), and let $a_0 \equiv 0$. Then for $n \geq 2$ and $k \geq 1$, the other coefficients $a_1, \cdots, a_{n-1}$ must be identically zero. In this case, (1.2) becomes $f^n f^{(k)} = Re^{\alpha}$ and f has the form $f(z) = u \exp(\alpha/(n+1))$ as the only possible admissible solution of the equation (1.2), where u is a small function of f.*

In view of Theorem 1.1 and Remark 1.3, we obtain the following result, which improves the corresponding result from [17].

Theorem 1.2. *Let f be a transcendental meromorphic function with $N(r, f) = S(r, f)$, and $q_m(f) = b_m f^m + \cdots + b_1 f + b_0$ be a polynomial of degree m with coefficients being small functions of f, and let n be an integer with $n \geq m+1$. Then the differential form $f' f^n + q_m(f)$ assumes every small function γ infinitely many times, except for a possible small function $b_0 = q_m(0)$. On the other hand, if $f' f^n + q_m(f)$ assumes the small function $b_0 = q_m(0)$ finitely many times, then $q_m(z) \equiv b_0$.*

2. Proof of Theorem 1.1

The following lemma is crucial in the proof of our theorem (see [4, 23]).

Lemma 2.1. *(see [4, 23]). Let f be a transcendental meromorphic solution of the equation:*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda | \lambda \in I\}$ such that $m(r, a_\lambda) = S(r, f)$ for all $r \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is at most n, then

$$m(r, P(r, f)) = S(r, f).$$

Proof of Theorem 1.1. We first show that $f^n f' + P_{n-1}(f)$ can not be a small function of f. Indeed, assuming the opposite, from $N(r, f) = S(r, f)$ and Lemma 2.1, we get $m(r, f') = S(r, f)$, and then $T(r, f') = S(r, f)$. A contradiction $T(r, f) = S(r, f)$ now follows by relying to a Theorem from [11] and combining it with the proof of Proposition E from [12]. Thus, for any transcendental meromorphic function f under the condition $N(r, f) = S(r, f)$, we have

$$T(r, f^n f' + P_{n-1}(f)) \neq S(r, f), \tag{2.1}$$

showing that Re^α is not a small function of f.

In view of Theorem C, without loss of generality, we can assume that $n \geq 2$. Let $P_{n-1}(f) \not\equiv 0$. From (1.4) and a result of Milloux (see, e.g., [8]), we obtain

$$T(r, e^\alpha) \leq (n+1)T(r, f) + S(r, f),$$

which and the equality $T(r, \alpha) + T(r, \alpha') = S(r, e^\alpha)$ lead to $T(r, \alpha) + T(r, \alpha') = S(r, f)$.

By taking the logarithmic derivative on both sides of (1.4), we get

$$\frac{nf^{n-1}(f')^2 + f^n f'' + P'_{n-1}(f)}{f^n f' + P_{n-1}(f)} = \frac{R'}{R} + \alpha',$$

implying that

$$-(\frac{R'}{R} + \alpha')f^n f' + nf^{n-1}(f')^2 + f^n f''$$
$$= (\frac{R'}{R} + \alpha')P_{n-1}(f) - P'_{n-1}(f). \tag{2.2}$$

Next, we set

$$\varphi = -(\frac{R'}{R} + \alpha')ff' + n(f')^2 + ff'', \tag{2.3}$$

and use (2.2) to obtain

$$f^{n-1}\varphi = (\frac{R'}{R} + \alpha')P_{n-1}(f) - P'_{n-1}(f) := Q_{n-1}(f). \tag{2.4}$$

Clearly, $Q_{n-1}(f)$ is a differential polynomial in f with $\deg Q_{n-1}(f) \le n-1$. We claim $\varphi \not\equiv 0$. Indeed, if $\varphi \equiv 0$, then in view of $Q_{n-1}(f) \equiv 0$, and (2.4), with some constant B we have $BP_{n-1}(f) \equiv Re^{\alpha}$. Since f is a transcendental meromorphic function, (1.4) shows that $B \neq 1$, and

$$f^n f' = (B-1)P_{n-1}(f),$$

which together with Lemma 2.1 implies $m(r, f') = S(r, f)$. Thus, by $N(r, f) = S(r, f)$ we have $T(r, f') = S(r, f)$, yielding a contradiction. Hence $\varphi \not\equiv 0$. Moreover, applying Lemma 2.1 to (2.4) again, we can conclude that $m(r, \varphi) = S(r, f)$ and $T(r, \varphi) = S(r, f)$.

From (2.3), we get $m(r, \frac{\varphi}{f^2}) = S(r, f)$, and hence

$$m(r, \frac{1}{f}) = S(r, f). \tag{2.5}$$

It follows from (2.3) that

$$N_{(2}(r, \frac{1}{f}) \le N(r, \frac{1}{\varphi}) + S(r, f)$$
$$\le T(r, \varphi) + S(r, f) = S(r, f),$$

implying that the zeros of f are mainly simple zeros. Thus, by (2.5), we obtain

$$T(r, f) = N(r, \frac{1}{f}) + S(r, f) = N_{1)}(r, \frac{1}{f}) + S(r, f), \tag{2.6}$$

where $N_{1)}(r, 1/f)$ involves only the simple zeros of f.

Let z_0 be a simple zero of f such that $R(z_0) \neq 0$. Then in view of (2.3) we have

$$n(f')^2(z_0) = \varphi(z_0). \tag{2.7}$$

Now, we show that $\varphi' \not\equiv 0$. Suppose, contrary to our assertion, that $\varphi' \equiv 0$, that is, φ is a constant. If z_0 is a zero of $f'(z) - \sqrt{\varphi/n}$, then we set

$$h(z) = \frac{f'(z) - \sqrt{\frac{\varphi}{n}}}{f(z)}, \tag{2.8}$$

and observe that $h \not\equiv 0$. It follows by (2.5), (2.7) and (2.8) that

$$m(r, h) = S(r, f). \tag{2.9}$$

From (2.6) and (2.8), we get $N(r, h) = S(r, f)$, which together with (2.9) show that $T(r, h) = S(r, f)$, and

$$f' = hf + \sqrt{\frac{\varphi}{n}}, \quad f'' = (h^2 + h')f + h\sqrt{\frac{\varphi}{n}}. \tag{2.10}$$

By (2.10) and (2.3), we obtain

$$[(n+1)h^2 + h' - h(\frac{R'}{R} + \alpha')]f + [(2n+1)h - (\frac{R'}{R} + \alpha')]\sqrt{\frac{\varphi}{n}} = 0.$$

Therefore, we must have

$$(n+1)h^2 + h' - h(\frac{R'}{R} + \alpha') \equiv 0, \quad (2n+1)h - (\frac{R'}{R} + \alpha') \equiv 0,$$

which implies $(2n+1)\frac{h'}{h} = n(\frac{R'}{R} + \alpha')$, and thus $(Re^{\alpha})^n = Ch^{2n+1}$ with a constant C. This, however, contradicts (2.1) and $T(r, h) = S(r, f)$, and thus $\varphi' \not\equiv 0$.

Using the above arguments, it can be shown that $\varphi' \not\equiv 0$. In this case we set

$$\hbar(z) = \frac{f'(z) + \sqrt{\varphi/6}}{f(z)}$$

and assume that $f'(z_0) + \sqrt{\varphi/n} = 0$.

Again, from (2.3), we get

$$\varphi' = -t'ff' - t(f')^2 - tff'' + (2n+1)f'f'' + ff''', \tag{2.11}$$

where $t = \frac{R'}{R} + \alpha'$. In view of (2.11) and (2.7), we see that a simple zero z_0 of $f(z)$ such that $R(z_0) \neq 0$, is a zero of $(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z)$.

If $(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z) \not\equiv 0$, we set

$$g(z) = \frac{(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z)}{f(z)}.$$

It is clear that g is a small function of f. Therefore, we have

$$f'' = \frac{g}{(2n+1)\varphi}f + \frac{t\varphi + n\varphi'}{(2n+1)\varphi}f'$$

$$:= s_1 f + s_2 f', \tag{2.12}$$

and

$$f''' = (s_1' + s_1 s_2)f + (s_1 + s_2' + s_2^2)f'. \tag{2.13}$$

Next, it follows from (2.13), (2.12), (2.11) and (2.3) that

$$(2n + 1 - t' - ts_2 + s_1 + s_2' + s_2^2 + t\frac{\varphi'}{\varphi} - s_2\frac{\varphi'}{\varphi})f'$$

$$+(s_1' + s_1 s_2 - ts_1 - s_1\frac{\varphi'}{\varphi})f = 0. \tag{2.14}$$

In this case, (2.14) and (2.6) imply

$$s_1' + s_1 s_2 - ts_1 - s_1\frac{\varphi'}{\varphi} \equiv 0.$$

Therefore, we have $(2n+1)\log s_1 = 2n(\log R+\alpha)+(3n+1)\log\varphi+B$ with a constant B, which implies that $(Re^{\alpha})^{2n}e^{B}\varphi^{3n+1} = s_1^{2n+1}$. Thus, Re^{α} is a small function of f, which contradicts (2.1). Therefore, $(2n+1)\varphi f''(z)-(t\varphi+n\varphi')f'(z)\equiv 0$, and we have

$$f'' = \beta f' \tag{2.15}$$

with $\beta = \frac{n\varphi'}{(2n+1)\varphi}+\frac{t}{2n+1}$. From (2.15) we obtain

$$f''' = (\beta'+\beta^2)f'. \tag{2.16}$$

It follows from (2.16), (2.15) and (2.11) that

$$(\beta'+\beta^2)f' = (t'-t\frac{\varphi'}{\varphi})f'+(t+\frac{\varphi'}{\varphi})\beta f'.$$

Therefore, we have

$$\beta'-t' \equiv -\beta(\beta-t)+(\beta-t)\frac{\varphi'}{\varphi}. \tag{2.17}$$

If $\beta-t\equiv 0$, then by the definitions of t and β, we see that $(Re^{\alpha})^2 = C\varphi$, where C is a constant. So, Re^{α} is a small function of f, which contradicts (2.1). Hence, we have $\beta-t\not\equiv 0$. In this case, again, by (2.17), we obtain $(2n+1)\log(\beta-t) = n\log\varphi+\log R+\alpha+D$ with a constant D, showing that Re^{α} is a small function of f, which also contradicts (2.1).

This completes the proof of the theorem, namely the equation $f^nf'+P_{n-1}(f) = Re^{\alpha}$ does not possess any meromorphic solution f with $N(r,f) = S(r,f)$ unless $P_{n-1}(f)\equiv 0$.

3. Conclusions

Using different and much simpler proofs, this paper provides two main results, extending the main results of the paper [17] to more general differential polynomials. Some examples are discussed showing that the imposed conditions are necessary. For further study, a general conjecture is posed.

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