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CONJUGATE FUNCTIONS AND THE MODULUS OF SMOOTHNESS OF FRACTIONAL ORDER

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Abstract. In the present paper, estimates of the partial moduli of smoothness of fractional order of the conjugate functions of several variables are obtained in the space $C(\mathbb{T}^n)$. The accuracy of the obtained estimates is established by appropriate examples.

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1. Introduction

Let $\mathbb{R}^n$ $(n \geq 1;\ \mathbb{R}^1 \equiv \mathbb{R})$ be the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$ with real coordinates. Let B be an arbitrary non-empty subset of the set $M = \{1, \dots, n\}$. Denote by $|B|$ the cardinality of B. Let x_B be such a point in $\mathbb{R}^n$ whose coordinates with indices in $M \backslash B$ are zero.

As usual, let $\mathbb{T} = [-\pi, \pi]$ and let $\mathbb{C}(\mathbb{T}^n)$ $(\mathbb{C}(\mathbb{T}^1) \equiv \mathbb{C}(\mathbb{T}))$ denote the space of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ that are 2π-periodic in each variable, endowed with the norm

$$\|f\| = \max_{x \in \mathbb{T}^n} |f(x)|.$$

If $f \in L(\mathbb{T}^n)$, then following Zhizhiashvili [14 , p. 182], the function

$$\widetilde{f}_B(x) = \Big(-\frac{1}{2\pi}\Big)^{|B|} \int_{\mathbb{T}^{|B|}} f(x + s_B) \prod_{i \in B} \cot \frac{s_i}{2}\, ds_B$$

we call the conjugate function of n variables with respect to those variables whose indices form the set B (with $\widetilde{f}_B \equiv \widetilde{f}$ for $n = 1$).

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Suppose that $f \in \mathbb{C}(\mathbb{T}^n)$, $1 \le i \le n$, and $h \in \mathbb{R}$. For each $x \in \mathbb{R}^n$, we consider the difference of fractional order α ($\alpha > 0$):

$$\Delta_i^\alpha(h)\, f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x_1, \dots, x_{i-1}, x_i + j\,h, x_{i+1}, \dots, x_n),$$

where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$ for $j \ge 1$, and $\binom{\alpha}{j} = 1$ for $j = 0$.

Then we define the partial modulus of smoothness of order α of a function f with respect to the variable x_i by the equality (see ([2], [10]):

$$\omega_{\alpha,i}(f;\delta) = \sup_{|h| \le \delta} \|\Delta_i^\alpha(h)\, f\|\,.$$

For $n = 1$ we write $\Delta_i^\alpha(h)\, f(x) \equiv \Delta^\alpha(h)\, f(x)$ and $\omega_{\alpha,i}(f;\delta) \equiv \omega_\alpha(f;\delta)$.

Definition 1.1. We say that a function φ is almost decreasing in $[a, b]$ if there exists a positive constant A such that $\varphi(t_1) \ge A\,\varphi(t_2)$ for $a \le t_1 \le t_2 \le b$.

Definition 1.2. If for $f \in \mathbb{C}(\mathbb{T})$ there exists a function $g \in \mathbb{C}(\mathbb{T})$ such that $\lim_{h \to 0+} \|h^{-\alpha}\Delta^\alpha(h) f - g\| = 0$, then g is called the Liouville-Grunwald derivative of order $\alpha > 0$ of f in the $\mathbb{C}(\mathbb{T})$-norm, and is denoted by $D^\alpha f$.

Let Φ_α ($\alpha > 0$) be the set of nonnegative, continuous functions $\varphi(\delta)$ defined on $[0, 1)$ and satisfying the following conditions:

1. $\varphi(\delta) = 0$,
2. $\varphi(\delta)$ is nondecreasing,
3. $\int_0^\delta \frac{\varphi(t)}{t} dt = O(\varphi(\delta))$,
4. $\delta^\alpha \int_\delta^1 \frac{\varphi(t)}{t^{\alpha+1}} dt = O(\varphi(\delta))$.

Note that when $\alpha = k$ is an integer number, then the class Φ_α coincides with the well-known class of Bari-Stechkin of order k (see [1]).

Let φ be a nonnegative, nondecreasing continuous function defined on $[0, 1)$ with $\varphi(\delta) = 0$. Then by $H_i^\alpha(\varphi; \mathbb{C}(\mathbb{T}^n))$ ($i = 1, \dots, n$) we denote the set of all functions $f \in \mathbb{C}(\mathbb{T}^n)$ such that

$$\omega_{\alpha,i}(f;\delta) = O\big(\varphi(\delta)\big), \quad \delta \to 0+, \quad i = 1, \dots, n,$$

and define

$$H^\alpha(\varphi; \mathbb{C}(\mathbb{T}^n)) = \bigcap_{i=1}^{n} H_i^\alpha(\varphi; \mathbb{C}(\mathbb{T}^n)).$$

In the theory of real-valued functions there is a well-known theorem by Privalov on the invariance of Lipschitz classes under the conjugate operator $\widetilde{f}$. An analogous result, in terms of modulus of smoothness of fractional order, has been obtained by Samko and Yakubov in [8], where they proved that the generalized Hölder class $H^{\alpha}(\varphi;\mathbb{C}(\mathbb{T}))$ $(\varphi \in \Phi_{\alpha}, \alpha > 0)$ is invariant under the operator $\widetilde{f}$. In the paper [9] by Simonov and Tikhonov, embedding theorems for generalized Weyl-Nikol'skii classes and for generalized Lipschitz classes are obtained. In the paper [12] by Simonov, Besov-Nikol'skii classes are considered and embedding theorems for some classes of functions are established.

In the present paper, we obtain exact estimates of the partial moduli of smoothness of fractional order of the conjugate functions of several variables in the space $H(\varphi;\mathbb{C}(\mathbb{T}^n))$, provided that $\varphi \in \Phi_{\alpha}$, $\alpha > 0$. Notice that similar results for classical moduli of smoothness (that is, when the moduli of continuity of different orders satisfy Zygmund's condition) were obtained in the papers [3], [5] – [7], [13].

Now we state some auxiliary results that will be used in the proof of the main result of this paper.

Lemma 1.1 (see [4]). *Let $f \subset \mathbb{C}(\mathbb{T})$, and let $\omega_k(f;t)$ and $\omega_{k+1}(f;t)$ be the moduli of continuity of f of k-th and $(k+1)$-th orders, respectively. Then for all $t \in [0,1)$ the following inequality holds:*

$$\omega_k(f;t^2) \le A\omega_{k+1}(f;t),$$

where A is a constant, which is independent of f.

Lemma 1.2. *Let $f \in H^{\alpha}(\varphi;\mathbb{C}(\mathbb{T}^n))$ and $\varphi \in \Phi_{\alpha}$, $\alpha > 0$. Then the following asymptotic relations hold:*

$$(1)\quad \omega_{\alpha,i}(\widetilde{f}_{\{i\}};\delta) = O(\varphi(\delta)),\ \ i=1,...,n,\ \ \delta \to 0+,$$

$$(2)\quad \omega_{\alpha,k}(\widetilde{f}_{\{i\}};\delta) = O(\varphi(\delta)|\ln\delta|),\ \ i,k=1,...,n, i\neq k,\ \ \delta \to 0+.$$

Proof. The statement (1) of the lemma is a multivariate version of Theorem 2 from [8] and can be proved exactly in the same way with some minor changes. So, we have to prove only the statement (2) of the lemma.

Let $h_{\{k\}} = (\underbrace{0,\dots,0}_{k-1},h,0,...0)$. For a given α, there exists a natural number p such that $p-1 < \alpha \le p$.

By the definitions of the difference of fractional order and the conjugate function, we can write

$$(-2\pi)\Delta_k^{\alpha}(h)\,\widetilde{f}_i(x) =$$

$$= \sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} \int_0^{h^{2^{p-1}}} [f(x - jh_{\{k\}} + s_{\{i\}}) - f(x - jh_{\{k\}} - s_{\{i\}})] \cot \frac{s_i}{2} ds_i +$$

$$+ \int_{h^{2^{p-1}}}^{\pi} \Delta_k^{\alpha}(h)\, f(x + s_{\{i\}}) \cot \frac{s_i}{2} ds_i - \int_{h^{2^{p-1}}}^{\pi} \Delta_k^{\alpha}(h)\, f(x - s_{\{i\}}) \cot \frac{s_i}{2} ds_i =$$

$$= \sum_{j=0}^{\infty}(-1)^j \binom{\alpha}{j} I_j(x, h_{\{k\}}) + J_1(x, h_{\{k\}}) + J_2(x, h_{\{k\}}).$$

For each j $(j = 1, ..., \infty)$ we have

$$\|I_j(x, h_{\{k\}})\| \le A \int_0^{h^{2^{p-1}}} \frac{\omega_{1,i}(f; s_i)}{s_i} ds_i,$$

where A is a constant independent of f.

Taking into account Lemma 1 and substituting s_i by $s_i^{2^{-p+1}}$, we get

$$\|I_j(x, h_{\{k\}})\| \le A_1 \int_0^{h^{2^{p-1}}} \frac{\omega_{p,i}(f; s_i^{2^{-p+1}})}{s_i} ds_i \le A_2 \int_0^{h} \frac{\omega_{p,i}(f; s_i)}{s_i} ds_i,$$

where A_1 and A_2 are constants independent of f.

Now using the inequality $\omega_{p,i}(f; s_i) \le C\omega_{\alpha,i}(f; s_i)$ (see [11]), where C is a constant independent of f, we obtain

$$\|I_j(x, h_{\{k\}})\| \le A_3 \int_0^{h} \frac{\omega_{\alpha,i}(f; s_i)}{s_i} ds_i, \; j = 1, ..., \infty,$$

where A_3 is a constant independent of f.

It is easy to see that

$$\|J_i(x, h_{\{k\}})\| \le A_4 \omega_{\alpha,k}(f, h) |\ln h|, \; i = 1, 2,$$

where A_4 is a constant independent of f.

In view of the above estimates for $I_j(x, h_{\{k\}})$ $(j = 1, ..., \infty)$ and $J_i(x, h_{\{k\}})$ $(i = 1, 2)$, and the condition $\varphi \in \Phi_\alpha$, we complete the proof of the statement (2). Lemma 1.2 is proved. □

The next two lemmas can be proved in the same way as the statement (Lemma 3) given in [1, pp. 498-499].

Lemma 1.3. *If $\varphi \in \Phi_\alpha (\alpha > 0)$, then the function $\frac{\varphi(t)}{t^\alpha}$ is almost decreasing in* $[0, 1]$.

Lemma 1.4. *If $\varphi \in \Phi_\alpha (\alpha > 0)$, then there exists a real number β $(0 < \beta < \alpha)$ such that the function $\frac{\varphi(t)}{t^\beta}$ is almost decreasing in* $[0, 1]$.

Notice that Lemma 1.4 actually implies Lemma 1.3.

2. Estimates for the partial moduli of smoothness of fractional order of the conjugate functions

The following theorem is the main result of this paper.

Theorem 2.1. *The following assertions hold:*

(a) *Let $f \in H^{\alpha}(\varphi; \mathbb{C}(\mathbb{T}^n))$ and $\varphi \in \Phi_\alpha$, $\alpha > 0$. Then*

$$\omega_{\alpha,i}(\widetilde{f}_B;\delta) = O(\varphi(\delta)|\ln\delta|^{|B|-1}), \quad i \in B,\ \delta \to 0+, \tag{2.1}$$

$$\omega_{\alpha,i}(\widetilde{f}_B;\delta) = O(\varphi(\delta)|\ln\delta|^{|B|}), \quad i \in M\backslash B,\ \delta \to 0+. \tag{2.2}$$

(b) *For each $B \subseteq M$ there exists a function G such that $G \in H(\varphi; \mathbb{C}(\mathbb{T}^n))$ and*

$$\omega_{\alpha,i}(\widetilde{G}_B;\delta) \geq C\varphi(\delta)|\ln\delta|^{|B|-1} \quad i \in B,\ 0 \leq \delta \leq \delta_0, \tag{2.3}$$

$$\omega_{\alpha,i}(\widetilde{G},\delta_B,\delta) \geq C\varphi(\delta)|\ln\delta|^{|B|}, \quad i \in M \setminus B,\ 0 \leq \delta \leq \delta_0, \tag{2.4}$$

where C and δ_0 are positive constants.

It should be noted that, for the case of modulus of continuity of first order, the theorem was proved in [7].

Proof. Part (a) of the theorem follows from Lemma 1.2. So, we have to prove only part (b). Without loss of generality, we carry out the proof of part (b) for the case $B = \{1, \dots, n-1\}$.

We consider a strictly decreasing sequence of positive numbers $(b_l)_{l\geq 1}$ satisfying the following conditions:

1. $\sum_{l=0}^{\infty} b_l \leq 1 \quad (b_0 = 0)$;
2. $\sum_{i=l+1}^{\infty} b_i < b_l$;
3. $\varphi^{-1}(b_{l+1}) < (\varphi^{-1}(b_l))^{\frac{\alpha}{\alpha-\beta}}$, where $\varphi^{-1}(b_l)$ $(l = 1, 2, \dots)$ is a certain element of the set $\{t : \varphi(t = b_l)\}$ and β $(0 < \beta < \alpha)$ satisfies the condition of Lemma 1.4.

We set

$$\tau_l = 2\sum_{j=0}^{l-1} \varphi^{-1}(b_j), \quad \tau_l^* = \tau_l + \varphi^{-1}(b_l).$$

For any $l = 1, 2, ..$, let us consider the functions g_l and h_l in $\mathbb{T}$, defined as follows:

$$g_l(u) = \begin{cases} 0, & -\pi \leq u \leq 0, \\ \dfrac{u^\alpha}{(\tau_l^* - \tau_l)^\alpha}, & 0 < u \leq \tau_l^* - \tau_l, \\ 1, & \tau_l^* - \tau_l < u \leq \pi - \tau_l^* + \tau_l, \\ \dfrac{(\pi - u)^\alpha}{(\tau_l^* - \tau_l)^\alpha}, & \pi - \tau_l^* + \tau_l < u \leq \pi. \end{cases}$$

and

$$h_l(u) = \begin{cases} \dfrac{(u-\tau_l)^\alpha(\tau_l^*-u)^\alpha}{(\tau_l^*-\tau_l)^{2\alpha}} & ,\tau_l \le u \le \tau_l^*, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we define the functions G_l in $\mathbb{T}^n$ as follows:

$$G_l(x_1,\dots,x_n) = b_l \prod_{i=1}^{n-1} g_l(x_i)h_l(x_n), \quad l = 1,2,\dots$$

and consider the function G defined by the series

$$G(x_1,\dots,x_n) = \sum_{l=1}^{\infty} G_l(x_1,\dots,x_n).$$

We extend the function G 2π-periodically in each variable to the whole space $\mathbb{R}^n$.

We claim that

$$G \in H^\alpha(\varphi; \mathbb{C}(\mathbb{T}^n)).$$

Let $0 < h < \varphi^{-1}(b_1)$. Then we have

$$\|\Delta_n^\alpha(h)G\| \le \sum_{l=1}^{\infty} \|\Delta_n^\alpha(h)G_l\| = \sum_{l=1}^{\infty} I_l(h).$$

Let us estimate each $I_l(h)$ $(l = 1,2,\dots)$ from above.

For given h, there exists a number N such that $\tau_{N+1}^* - \tau_{N+1} \le h < \tau_N^* - \tau_N$.

Let $l = 1,\dots,N$. It is known (see [2]) that if a function of one variable $f \in \mathbb{C}(\mathbb{T})$ has fractional derivative of order α $(\alpha > 0)$, then

$$\omega_\alpha(f;\delta) \le C\delta^\alpha \|D^\alpha f\| (\delta > 0), \; C = const > 0.$$

In our case, using the definition of the function G_l and this fact for the variable x_n, we can conclude that

$$I_l(h) \le A_1 h^\alpha \frac{b_l}{(\tau_l^* - \tau_l)^\alpha}, \quad A_1 = const.$$

If $l = N+1,\dots,$ then we have

$$I_l(h) \le A_2 b_l, \quad A_2 = const.$$

Therefore

$$\|\Delta_n^\alpha(h)G\| \le A_1 \sum_{l=1}^{N} h^\alpha \frac{b_l}{(\tau_l^* - \tau_l)^\alpha} + A_2 \sum_{l=N+1}^{\infty} b_l.$$

If $\tau_{N+1}^* - \tau_{N+1} \le h \le (\tau_N^* - \tau_N)^{\frac{\alpha}{\alpha-\beta}}$, then by Lemma 4 and by the construction of the sequence $(b_l)_{l\ge1}$, with some constant A_3, we obtain

$$\|\Delta_n^\alpha(h)G\| \le A_1 \sum_{l=1}^{N} \frac{b_l}{(\tau_l^* - \tau_l)^\alpha} h^\beta h^{\alpha-\beta} + A_2 \sum_{l=N+1}^{\infty} b_l \le A_3\varphi(h).$$

If $(\tau_N^* - \tau_N)^{\frac{\alpha}{\alpha-\beta}} \le h \le \tau_N^* - \tau_N$, then by Lemmas 3 and 4, and by the construction of the sequence $(b_l)_{l\ge1}$, we get

$$\|\Delta_n^\alpha(h)G\| \le A_1 \sum_{l=1}^{N-1} \frac{b_l}{(\tau_l^* - \tau_l)^\alpha} h^\beta h^{\alpha-\beta}$$

$$+A_1 \frac{b_N}{(\tau_N^* - \tau_N)^\alpha} h^\alpha + A_2 \sum_{l=N+1}^{\infty} b_l \le A_3\varphi(h), \quad A_3 = const.$$

Hence, we have

$$\omega_{\alpha,n}(G;\delta)(h)G = O(\varphi(\delta)), \quad \delta \to 0+ .$$

Analogously, we can show that

$$\omega_{\alpha,i}(G;\delta)(h)G = O(\varphi(\delta)), \delta \to 0+, \quad i = 1, \dots, n-1.$$

Hence

$$G \in H^\alpha(\varphi; C(\mathbb{T}^n)).$$

Now we proceed to prove the inequalities (2.3) and (2.4).

Let $h = \tau_l^* - \tau_l$. According to the definition of the conjugate function and the function G, we obtain

$$\Delta_n^\alpha(h)\,\widetilde{G}_{\{1,\dots,n-1\}}(0,\dots,0,\frac{\tau_l^* + \tau_l}{2}) =$$

$$= \left(-\frac{1}{2\pi}\right)^{n-1} \sum_{j=1}^{\infty} \int_{\mathbb{T}^{n-1}} \Delta_n^\alpha(h) G_j(s_1,\dots,s_{n-1},\frac{\tau_l^* + \tau_l}{2}) \prod_{i=1}^{n-1} \cot\frac{s_i}{2}\, ds_i =$$

$$= \left(-\frac{1}{2\pi}\right)^{n-1} \int_{\mathbb{T}^{n-1}} \Big[\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} G_j(s_1,\dots,s_{n-1},\frac{\tau_l^* + \tau_l}{2} + kh)\Big] \prod_{i=1}^{n-1} \cot\frac{s_i}{2}\, ds_i.$$

Now using the inequality $\left|\binom{\alpha}{k}\right| \le C_1 k^{-\alpha-1}$ $(k = 1, 2, \dots)$ (see [9]), the construction of the sequence $(b_l)_{l\ge1}$ and the definition of the function G_j, we can write

$$\Big|\sum_{j=l+1}^{\infty}\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} G_j(s_1,\dots,s_{n-1},\frac{\tau_l^* + \tau_l}{2} + kh)\Big| \le C_2 \sum_{j=l+1}^{\infty} b_j \prod_{i=1}^{n-1} g_j(s_i),$$

$$\Big|\sum_{j=1}^{l-1}\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} G_j(s_1,\dots,s_{n-1},\frac{\tau_l^* + \tau_l}{2} + kh)\Big| \le$$

$$\le \sum_{j=1}^{l-1} b_j \prod_{i=1}^{n-1} g_j(s_i) \sum_{k=[\frac{2\pi+\tau_l - \frac{\tau_l^*+\tau_l}{2}}{h}]+1}^{\infty} \left|\binom{\alpha}{k}\right| \le C_3 h^\alpha \sum_{j=1}^{l-1} b_j \prod_{i=1}^{n-1} g_j(s_i),$$

$$\Big|\sum_{k=1}^{\infty}(-1)^k\binom{\alpha}{k}G_j(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2}+kh)\Big|\le b_l\prod_{i=1}^{n-1}g_l(s_i)\sum_{k=[\frac{2\pi}{h}-\frac{1}{2}]+1}^{\infty}\Big|\binom{\alpha}{k}\Big|\le$$

$$\le C_4h^{\alpha}b_l\prod_{i=1}^{n-1}g_l(s_i),$$

where C_i $(i=1,\dots,4)$ are positive constants and the symbol $[a]$ denotes the integer part of a real number a.

Hence, we can conclude that with some constants C_5 and C_6

$$|\Delta_n^{\alpha}(h)\widetilde{G}_{\{1,\dots,n-1\}}(0,\dots,0,\frac{\tau_l^*+\tau_l}{2})|\ge C_5\int_{[0,\pi]^{n-1}}G_l(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2})\prod_{i=1}^{n-1}s_i^{-1}\,ds_i$$

$$\ge C_6b_l|\ln(\tau_l^*-\tau_l)|^{n-1}.$$

Thus, the inequality (4) is proved. Now prove the inequality (3). Without loss of generality, we can take $i=n-1$.

Let $h=\tau_l^*-\tau_l$. Then in view of the definition of conjugate function, we can write

$$\Delta_{n-1}^{\alpha}(-h)\widetilde{G}_{\{1,\dots,n-1\}}(0,\dots,0,\frac{\tau_l^*+\tau_l}{2})=$$

$$=\Big(-\frac{1}{2\pi}\Big)^{n-1}\int_{\mathbb{T}^{n-1}}\Delta_{n-1}^{\alpha}(-h)G(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2})\prod_{i=1}^{n-1}\cot\frac{s_i}{2}\,ds_i=$$

$$=\Big(-\frac{1}{2\pi}\Big)^{n-1}\int_{[0,\pi]^{n-2}}\int_{\mathbb{T}}\Delta_{n-1}^{\alpha}(-h)G(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2})\prod_{i=1}^{n-1}\cot\frac{s_i}{2}\,ds_i=$$

$$=\Big(-\frac{1}{2\pi}\Big)^{n-1}\int_{[0,\pi]^{n-2}}\int_0^{\pi}G(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2})(\Delta^{\alpha}(h)\cot\frac{s_{n-1}}{2})ds_{n-1}\prod_{i=1}^{n-2}\cot\frac{s_i}{2}\,ds_i.$$

Next, using the definition of the function G, we obtain

$$|\Delta_{n-1}^{\alpha}(-h)\widetilde{G}_{\{1,\dots,n-1\}}(0,\dots,0,\frac{\tau_l^*+\tau_l}{2})|=$$

$$=\Big(\frac{1}{2\pi}\Big)^{n-1}\Big|\int_{[0,\pi]^{n-2}}\Big[\int_0^{\pi}G_l(s_1,\dots,s_{n-1},\frac{\tau_l^*+\tau_l}{2})(\Delta^{\alpha}(h)\cot\frac{s_{n-1}}{2})ds_{n-1}\Big]\prod_{i=1}^{n-2}\cot\frac{s_i}{2}\,ds_i\Big|$$

$$\ge C_7b_l\int_{\tau_l*-\tau_l}^{1}\cdots\int_{\tau_l*-\tau_l}^{1}\prod_{i=1}^{n-2}s_i^{-1}\frac{h^{\alpha}}{s_{n-1}^{\alpha+1}}\,ds_i$$

$$\ge C_8b_l|\ln(\tau_l^*-\tau_l)|^{n-2},$$

where C_7 and C_8 are positive constants. Thus, the inequality (2.3) is proved. □

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