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ALMOST EVERYWHERE CONVERGENCE OF STRONG NÖRLUND LOGARITHMIC MEANS OF WALSH-FOURIER SERIES

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Abstract. In this paper we study the maximal operator for a class of subsequences of strong Nörlund logarithmic means of Walsh-Fourier series. For such a class we prove the almost everywhere strong summability for every integrable function f.

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1. Introduction

We denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \le p \le 2^n$. Denote $I_N := [0, 2^{-N})$ and $I_N(x) := I_N + x$.

Let $r_0(x)$ be the function defined by

$$r_0(x) = \begin{cases} 1, \text{ if } & x \in [0,1/2) \\ -1, \text{ if } & x \in [1/2,1) \end{cases}, \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \ge 1.$$

Let $w_0, w_1, ...$ denote the Walsh functions, that is, $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a nonnegative integer with $n_1 > n_2 > \cdots > n_s$, then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

Given $x \in \mathbb{I}$, the expansion

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \tag{1.1}$$

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where each $x_k = 0$ or 1, will be called a dyadic expansion of x. If $x \in \mathbb{I}\backslash\mathbb{Q}$, then (1.1) is uniquely determined. For the dyadic expansion $x \subset \mathbb{Q}$ we choose the one for which $\lim_{k\to\infty} x_k = 0$.

The dyadic addition of $x, y \in \mathbb{I}$ in terms of the dyadic expansion of x and y is defined by

$$\rho(x, y) := x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

If $f \in L^1(\mathbb{I})$, then by

$$\hat{f}(n) = \int_{\mathbb{I}} f(x) \, w_n(x) dx$$

we denote the n-th Fourier coefficient of f.

The partial sums of Fourier series with respect to the Walsh system are defined by

$$S_M(x; f) = \sum_{m=0}^{M-1} \hat{f}(m) \, w_m(x).$$

For $n \in \mathbb{N}$ let us introduce the projections

$$E_n(x; f) := S_{2^n}(x; f) = 2^n \int_{I_n(x)} f(s) \, ds \quad (f \in L_1(\mathbb{I}), x \in \mathbb{I}),$$

and

$$E^*(x; f) := \sup_{n \in \mathbb{N}} E_n(x; |f|).$$

The question of almost everywhere convergence is one of the important questions in the theory of Fourier series. It is well known that for Walsh and trigonometric Fourier series the logarithmic means defined by

$$\frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k(f)}{k}, \quad l_n = \sum_{k=1}^{n} \frac{1}{k}$$

have a nice behavior, in the sense that, for each integrable on the unit interval function f, these means converge to f almost everywhere. Thus, to examine the logarithmic means is a good idea, because for the partial sums there are divergence results. For instance, for Walsh system it is known that for each measurable function ϕ satisfying $\phi(u) = o(u\sqrt{\log u})$ there exists an integrable function f such that

$$\int_{\mathbb{I}} \phi(|f(x)|) dx < \infty,$$

and the Walsh-Fourier series of f diverges everywhere (see [1]).

The notion of Nörlund logarithmic means is similar to that of logarithmic means, the difference is that the denominators are taken in the reversed order. More precisely, the Nörlund logarithmic means are defined by

$$t_n(f) := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{n-k}.$$

In [5, 6] it is proved that these means are much more closer to the partial sums than the logarithmic means. More precisely, we proved that in the function class above (see the result of Bochkarev [1]), there exist a function and a set with positive measure, such that the Walsh-Nörlund logarithmic means of the function diverge on that set. This also says that, in this point of view, not all classical summation methods improve the convergence properties of the partial sums. On the other hand, in [9], the author studied the maximal operator for a class of Nörlund logarithmic means of Walsh-Fourier series, where only the logarithmic means of order 2^n was considered. For such subsequence we have proved the almost everywhere convergence for every integrable function f. In [22], Memić enlarged the convergence class of subsequences given in [9].

The strong summability problem, that is, the convergence of the strong means

$$H_n^{T,p}(x;f) := \frac{1}{n+1}\sum_{k=0}^{n}\left|S_k^T(x,f)-f(x)\right|^p, \quad x\in\mathbb{T}, \quad p>0, \tag{1.2}$$

was first considered by Hardy and Littlewood in [18], where by $S_k^T(x,f)$ we denote the partial sums of Fourier series with respect to trigonometric system. They showed that for any $f\in L_r(\mathbb{T})$ $(1<r<\infty)$ the strong means tend to 0 a.e., as $n\to\infty$. The Fourier series of $f\in L_1(\mathbb{T})$ is said to be (H,p)-summable at $x\in T$, if the strong means (1.2) converge to 0 as $n\to\infty$. The (H,p)-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [21] for $p=2$, and later by Zygmund [31] for the general case $1\le p<\infty$.

In [25], Schipp investigated the strong (H,p)- and BMO-summability of Walsh-Fourier series. Among others, he gave a characterization of points at which the Walsh-Fourier series of an integrable function is (H,p)- and BMO-summable. This result is an analogue of Gabisonia's result, obtained in [4], that characterizes the points of strong summability with respect to the trigonometric system.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems, see Schipp [31, 34], Fridli and Schipp [2, 3], Leindler [20], Totik [29], Rodin [24], Weisz [40], Goginava, Gogoladze

[13, 12], Gogoladze [15, 16], Glukhov [17], Goginava [10, 11], Goginava, Gogoladze, Karagulyan [14] Gát, Goginava, Karagulyan [7, 8], Karagulyan [19], Oskolkov [23].

In this paper we study the maximal opearator for a class of subsequences of strong Nörlund logarithmic means of Walsh-Fourier series. For such a class we prove the almost everywhere strong summability for every integrable function f.

2. Main results

The strong logarithmic means are defined by

$$L_n^{(p)}(x;f) := \left(\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{|S_k(x;f)|^p}{n-k}\right)^{1/p}.$$

Let

(2.1) $$m_n := 2^{\alpha_1(n)} + 2^{\alpha_2(n)} + \cdots + 2^{\alpha_r(n)},$$

where

$$\alpha_1(n) > \alpha_2(n) > \cdots > \alpha_r(n) \geq 0, \quad r = r(n).$$

and

(2.2) $$m_n^{(i)} := 2^{\alpha_{i+1}(n)} + 2^{\alpha_{i+2}(n)} + \cdots + 2^{\alpha_r(n)}, \quad i = 0, 1, ..., r-1.$$

The following are the main results of this paper.

Theorem 2.1. *Let $p > 0$ and*

(2.3) $$\overline{\lim_{n\to\infty}}\frac{1}{l_{m_n}^{1/p}}\sum_{s=0}^{r-1} l_{m_n^{(s)}}^{1/p} < \infty.$$

Then

$$\lambda\left|\left\{x : \sup_n L_{m_n}^{(p)}(f) > \lambda\right\}\right| \leq c(p)\,\|f\|_1, \quad f \in L_1(\mathbb{I}).$$

By making use the well-known density argument due to Marcinkiewicz and Zygmund we can show that Corollary 2.1 follows from Theorem 2.1.

Corollary 2.1. *Let the condition (2.3) be satisfied and $f \in L_1(\mathbb{I})$. Then*

$$\frac{1}{l_{m_n}}\sum_{j=0}^{m_n-1}\frac{|S_j(x;f) - f(x)|^p}{m_n - j} \to 0, \quad n \to \infty$$

for a. e. $x \in \mathbb{I}$ and for any $p > 0$.

Corollary 2.2. *Let $f \in L_1(\mathbb{I})$, $m_n := 2^n + \gamma_n$, $\gamma_n \leq 2^{n^{1/(1+p)}}$ and $p > 0$. Then*

$$\frac{1}{l_{m_n}} \sum_{j=0}^{m_n - 1} \frac{|S_j(x; f) - f(x)|^p}{m_n - j} \to 0, \quad n \to \infty$$

for a. e. $x \in \mathbb{I}$.

Corollary 2.3. *Let $f \in L_1(\mathbb{I})$ and $p > 0$. Then*

$$\frac{1}{l_{2^n}} \sum_{j=0}^{2^n - 1} \frac{|S_j(x; f) - f(x)|^p}{2^n - j} \to 0, \quad n \to \infty$$

for a. e. $x \in \mathbb{I}$.

3. Auxiliary propositions

In [25], Schipp introduced the following operator ($p > 1$)

$$V_n^{(p)}(x; f) := \left(\sum_{l=0}^{2^n - 1} \left(\int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) f(x + t + e_j)\, dt \right)^q \right)^{1/q},$$

$$\frac{1}{p} + \frac{1}{q} = 1, e_j := 2^{-j-1}.$$

Set

$$V_*^{(p)}(x; f) := \sup_n \left| V_n^{(p)}(x; f) \right|.$$

The proof of the next lemma can be found in [25] (for $p = 2$) and in [7] (for $p > 2$).

Lemma 3.1. *Let $p \geq 2$. Then*

$$\sup_\lambda \lambda \left| \left\{ x \in \mathbb{I} : V_*^{(p)}(x; |f|) > \lambda \right\} \right| \leq c(p) \|f\|_1 .$$

Set

$$H_n^{(p)}(x; f) := \left(\frac{1}{n} \sum_{m=0}^{n-1} |S_m(x; f)|^p \right)^{1/p}.$$

Lemma 3.2. *Let $p \geq 2$. The following inequality holds:*

$$H_{2^n}^{(p)}(x; f) \leq c V_n^{(p)}(x; |f|).$$

Proof of Lemma 3.2. Observe first that for $p = 2$ the lemma was proved in [25]. Let

$$\varepsilon_{ij} := \begin{cases} -1, & \text{if } j = 0, 1, ..., i-1 \\ 1, & \text{if } j = i. \end{cases}$$

In [25], Schipp proved that

$$
\begin{aligned}
(3.1)\qquad D_m(t) &= \sum_{k=0}^{n-1} \mathbb{I}_{I_k\setminus I_{k+1}}(t) \sum_{j=0}^{k} \varepsilon_{kj} 2^{j-1} w_m(t+e_j) \\
&\quad -\frac{1}{2} w_m(t) + (m+1/2)\,\mathbb{I}_{I_n}(t), \quad m < 2^n.
\end{aligned}
$$

We can write

$$
(3.2)\qquad 2^{n/p} H_{2^n}^{(p)}(x;f) = \left\{ \sum_{m=0}^{2^n-1} |S_m(x;f)|^p \right\}^{1/p} = \sup_{\{\alpha_n\}} \left| \sum_{m=0}^{2^n-1} \alpha_m(x) S_m(x;f) \right|,
$$

by taking the supremum over all $\{\alpha_n\}$ for which

$$
\left(\sum_{m=0}^{2^n-1} |\alpha_m(x)|^q \right)^{1/q} \leq 1, \quad 1/p + 1/q = 1.
$$

Let us assume that $p \geq 2$. From (3.1) we have

$$
\begin{aligned}
&\left| \sum_{m=0}^{2^n-1} \alpha_m(x) S_m(x;f) \right| \\
&\leq \left| \sum_{m=0}^{2^n-1} \alpha_m(x) \int_{\mathbb{I}} f(x+t) \sum_{k=0}^{n-1} \mathbb{I}_{I_k\setminus I_{k+1}}(t) \sum_{j=0}^{k} \varepsilon_{kj} 2^{j-1} w_m(t+e_j)\, dt \right| \\
&\quad + \left| \sum_{m=0}^{2^n-1} \alpha_m(x) \int_{\mathbb{I}} f(x+t) \frac{w_m(t)}{2} dt \right| \\
&\quad + \left| \sum_{m=0}^{2^n-1} \alpha_m(x) \int_{\mathbb{I}} f(x+t)(m+1/2)\,\mathbb{I}_{I_n}(t)\, dt \right| \\
&:= J_1 + J_2 + J_3.
\end{aligned}
$$

Since

$$
\left| \sum_{k=j}^{n-1} \varepsilon_{kj} \mathbb{I}_{I_k\setminus I_{k+1}}(t) \right| \leq \mathbb{I}_{I_j}(t),
$$

for J_1 we get

$$
\begin{aligned}
(3.3)\qquad J_1 &\leq \int_{\mathbb{I}} |f(x+t)| \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) \\
&\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_m(x) w_m(t+e_j) \right| dt.
\end{aligned}
$$

Set

$$
P_n(x;t) := \sum_{m=0}^{2^n-1} \alpha_m(x) w_m(t).
$$

It is easy to see that $\mathbb{I}_{I_j}(t) = \mathbb{I}_{I_j}(t + e_j)$. Then from (3.3) we have

$$J_1 \le \int_{\mathbb{I}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) \left| f(x+t+e_j) \right| \left| P_n(x;t) \right| dt \tag{3.4}$$

$$= \sum_{l=0}^{2^n-1} \int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) \left| f(x+t+e_j) \right| \left| P_n(x;t) \right| dt$$

$$= \sum_{l=0}^{2^n-1} \left| P_n\left(x; \frac{l}{2^n}\right) \right| \int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) \left| f(x+t+e_j) \right| dt$$

$$\le \left(\sum_{l=0}^{2^n-1} \left| P_n\left(x; \frac{l}{2^n}\right) \right|^p \right)^{1/p}$$

$$\times \left(\sum_{l=0}^{2^n-1} \left(\int_{l2^{-n}}^{(l+1)2^{-n}} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) \left| f(x+t+e_j) \right| dt \right)^q \right)^{1/q}.$$

First use Hölder's inequality and Hausdorf-Young inequality to obtain ($p \ge 2, 1/p + 1/q = 1$)

$$\left(\sum_{l=0}^{2^n-1} \left| P_n\left(x; \frac{l}{2^n}\right) \right|^p \right)^{1/p} = 2^{n/p} \left(\sum_{l=0}^{2^n-1} \int_{l2^{-n}}^{(l+1)2^{-n}} \left| P_n(x;t) \right|^p dt \right)^{1/p}$$

$$= 2^{n/p} \left(\int_{\mathbb{I}} \left| P_n(x;t) \right|^p dt \right)^{1/p} = 2^{n/p} \sup_{\|h\|_q \le 1} \int_{\mathbb{I}} P_n(x;t)\, h(t)\, dt$$

$$= 2^{n/p} \sup_{\|h\|_q \le 1} \sum_{m=0}^{2^n-1} \alpha_m(x)\, \widehat{h}(m)$$

$$\le 2^{n/p} \sup_{\|h\|_q \le 1} \left(\sum_{m=0}^{2^n-1} \left| \alpha_m(x) \right|^q \right)^{1/q} \left(\sum_{m=0}^{2^n-1} \left| \widehat{h}(m) \right|^p \right)^{1/p}$$

$$\le c 2^{n/p} \sup_{\|h\|_q \le 1} \|h\|_q = c 2^{n/p}.$$

Consequently, from (3.4) we obtain the estimate

$$J_1 \le c 2^{n/p} V_n^{(p)}(x; |f|), \quad p \ge 2. \tag{3.5}$$

For J_2 we can write

$$J_2 \le c \int_{\mathbb{I}} \left| f(x+t+e_0) \right| \left| \sum_{m=0}^{2^n-1} \alpha_m(x)\, w_m(e_0)\, w_m(t) \right| dt \tag{3.6}$$

$$\leq \int_{I} \sum_{j=0}^{n-1} 2^{j-1} \mathbb{I}_{I_j}(t) \left|f\left(x+t+e_j\right)\right| \left|P_n'(x;t)\right| dt \leq c 2^{n/p} V_n(x;|f|), \quad p \geq 2,$$

where

$$P_n'(x;t) := \sum_{m=0}^{2^n-1} \alpha_m(x) w_m(e_0) w_m(t).$$

Analogously, we can write

$$J_3 \leq c 2^{(1+1/p)n} \int_{I_n} |f(x+t)| \, dt \leq c 2^{n/p} V_n(x;|f|), \quad p \geq 2. \tag{3.7}$$

Combining (3.2) and (3.5)-(3.7) we complete the proof of the lemma. □

4. Proof of Theorem 2.1

Observe first that in view of (2.1) and (2.2) we can write

$$\begin{aligned} L_{m_n}^{(p)}(x;f) \leq & \left(\frac{1}{l_{m_n}} \sum_{j=0}^{2^{\alpha_1(n)}-1} \frac{|S_j(x;f)|^p}{2^{\alpha_1(n)}-j}\right)^{1/p} \\ & + \left(\frac{1}{l_{m_n}} \sum_{j=0}^{m_n^{(1)}-1} \frac{\left|S_{j+2^{\alpha_1(n)}}(x;f)\right|^p}{m_n^{(1)}-j}\right)^{1/p}. \end{aligned}$$

Since for $j = 0, 1, ..., 2^{\alpha_1(n)} - 1$

$$S_{j+2^{\alpha_1(n)}}(x;f) = S_{2^{\alpha_1(n)}}(x;f) + w_{2^{\alpha_1(n)}}(x) S_j\left(x; f w_{2^{\alpha_1(n)}}\right),$$

we obtain

$$\begin{aligned} L_{m_n}^{(p)}(x;f) \leq & \left(\frac{1}{l_{m_n}} \sum_{j=0}^{2^{\alpha_1(n)}-1} \frac{|S_j(x;f)|^p}{2^{\alpha_1(n)}-j}\right)^{1/p} + \left(\frac{l_{m_n^{(1)}}}{l_{m_n}}\right)^{1/p} \left|S_{2^{\alpha_1(n)}}(x;f)\right| \\ & + \left(\frac{l_{m_n^{(1)}}}{l_{m_n}}\right)^{1/p} \left(\frac{1}{l_{m_n}^{(1)}} \sum_{j=0}^{m_n^{(1)}-1} \frac{\left|S_j\left(x; f w_{2^{\alpha_1(n)}}\right)\right|^p}{m_n^{(1)}-j}\right)^{1/p}. \end{aligned}$$

Iterating the last inequality we obtain

$$\begin{aligned} L_{m_n}^{(p)}(x;f) \leq & \sum_{s=0}^{r-1} \left(\frac{l_{m_n^{(s)}}}{l_{m_n}}\right)^{1/p} \left(\frac{1}{l_{m_n}^{(s)}} \sum_{j=0}^{2^{\alpha_{s+1}(n)}-1} \frac{\left|S_j\left(x; f w_{2^{\alpha_1(n)}} \cdots w_{2^{\alpha_s(n)}}\right)\right|^p}{2^{\alpha_{s+1}(n)}-j}\right)^{1/p} \\ & + \sum_{s=0}^{r-2} \left(\frac{l_{m_n^{(s+1)}}}{l_{m_n}}\right)^{1/p} \left|S_{2^{\alpha_{s+1}(n)}}\left(x; f w_{2^{\alpha_1(n)}} \cdots w_{2^{\alpha_s(n)}}\right)\right|. \end{aligned}$$

Next, since

$$D_{2^k-j} = D_{2^k} - w_{2^k-1} D_j, j = 1, 2, ..., 2^k - 1,$$

we can write

$$L_{m_n}^{(p)}(x;f) \leq \sum_{s=0}^{r-1}\left(\frac{l_{m_n^{(s)}}}{l_{m_n}}\right)^{1/p}\left(\frac{1}{l_{m_n}^{(s)}}\sum_{j=1}^{2^{\alpha_{s+1}(n)}}\frac{\left|S_{2^{\alpha_{s+1}(n)}-j}\left(x;fw_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}}\right)\right|^p}{j}\right)^{1/p}$$

$$+\sum_{s=0}^{r-2}\left(\frac{l_{m_n^{(s+1)}}}{l_{m_n}}\right)^{1/p}\left|S_{2^{\alpha_{s+1}(n)}}(x;|f|)\right| \leq 2\sum_{s=0}^{r-1}\left(\frac{l_{m_n^{(s+1)}}}{l_{m_n}}\right)^{1/p}\left|S_{2^{\alpha_{s+1}(n)}}(x;|f|)\right|$$

$$(4.1)\qquad +\sum_{s=0}^{r-1}\left(\frac{l_{m_n^{(s)}}}{l_{m_n}}\right)^{1/p}\left(\frac{1}{l_{m_n}^{(s)}}\sum_{j=1}^{2^{\alpha_{s+1}(n)}}\frac{\left|S_j\left(x;fw_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}}w_{2^{\alpha_{s+1}(n)}-1}\right)\right|^p}{j}\right)^{1/p}.$$

Let $p \geq 2$. Then using Lemma 3.2, we can write

$$(4.2)\qquad \sum_{j=1}^{2^{\alpha_{s+1}(n)}-1}\frac{\left|S_j\left(x;fw_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}}w_{2^{\alpha_{s+1}(n)}-1}\right)\right|^p}{j}$$

$$= \sum_{l=0}^{\alpha_{s+1}(n)-1}\sum_{j=2^l}^{2^{l+1}-1}\frac{\left|S_j\left(x;fw_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}}w_{2^{\alpha_{s+1}(n)}-1}\right)\right|^p}{j}$$

$$\leq \sum_{l=0}^{\alpha_{s+1}(n)-1}2^{-l}\sum_{j=2^l}^{2^{l+1}-1}\left|S_j\left(x;fw_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}}w_{2^{\alpha_{s+1}(n)}-1}\right)\right|^p$$

$$\leq 2\sum_{l=0}^{\alpha_{s+1}(n)-1}\left(H_{2^{l+1}}^{(p)}\left(x;fw_{2^{\alpha_1(n)}}\cdots w_{2^{\alpha_s(n)}}w_{2^{\alpha_{s+1}(n)}-1}\right)\right)^p$$

$$\leq 2\sum_{l=0}^{\alpha_{s+1}(n)-1}\left(V_{l+1}^{(p)}(x;|f|)\right)^p \leq 2\alpha_{s+1}(n)\left(V_*^{(p)}(x;|f|)\right)^p.$$

Combining (4.1) and (4.2), and taking into account the condition (2.3) of the theorem, we obtain

$$(4.3)\qquad L_{m_n}^{(p)}(x;f) \leq c\left\{E^*(x;|f|) + V_*^{(p)}(x;|f|)\right\}, \quad p \geq 2.$$

Now let $0 < p < 2$. Since

$$H_{2^{l+1}}^{(p)}(x;f) \leq H_{2^{l+1}}^{(2)}(x;f),$$

we can write

$$(4.4)\qquad L_{m_n}^{(p)}(x;f) \leq c\left\{E^*(x;|f|) + V_*^{(2)}(x;|f|)\right\}, \quad 0 < p < 2.$$

Finally, taking into account the inequality

$$\lambda\left|\{x \in \mathbb{I} : E^*(x;|f|) > \lambda\}\right| \leq c\|f\|_1, \quad f \in L_1(\mathbb{I}),$$

from estimates (4.3), (4.4) and Lemma 3.1, we conclude the proof of the theorem. Theorem 2.1 is proved.

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