Известия НАН Армении, Математика, том 53, н. 4, 2018, стр. 66 – 71.

FIRST PASSAGE TIME DISTRIBUTION FOR LINEAR FUNCTIONS OF A RANDOM WALK

B. FATHI-VAJARGAH, M. NAVIDI

University of Guilan, Rasht, Iran
E-mails: *fathi@guilan.ac.ir, navidimahmod@yahoo.com*

Abstract. In this paper, theorems about asymptotic behavior of the local probabilities of crossing the linear boundaries by a perturbed random walk are proved.

MSC2010 numbers: 60Fxx, 60Hxx
Keywords: First passage time; local limit theorem; perturbed random walk.

1. Introduction

The paper investigates the asymptotic behavior of local probabilities of crossing the linear boundaries by a perturbed random walk. This problem was studied by M. Woodroofe in [1]. The goal is to extend some results from [1].

Let $\{\varepsilon_n \ n = 1, 2, \ldots\}$ be a sequence of independent identically distributed random variables defined on some probability space (Ω, F, P) with $E|\varepsilon_i| < \infty$ and $\sigma^2 = \operatorname{Var} \varepsilon_i < \infty$. Let $\Delta(x)$ be a strictly convex and continuously differentiable in R function, and let $V = E\varepsilon_i < \infty$. In [1], M. Woodroofe described the asymptotic distribution of the first passage time in the case where the function $\Delta(x)$ satisfies the condition $\Delta'(v) > 0$. In this paper, we examine the case $\Delta'(v) < 0$. Denote

$$S_n = \sum_{k=1}^{n} \varepsilon_n, \quad \overline{S_n} = \frac{1}{n} S_n, \quad T_n = n\Delta(\overline{S_n}), \qquad n = 1, 2 \ldots$$

Also, define the stopping times:

$$\tau_a = \inf\{n \geq 1 : T_n > a\}, \quad R_a = T_{\tau_a} - a, \quad a > 0.$$

Note that the family of the first passage times was investigated in the papers [2]-[5].

2. Assumptions and formulation of the main results

We assume that $\Delta(x)$ is a strictly convex and continuously differentiable function in R with $V = E\varepsilon_1, \mu = \Delta(v)$ and $\delta = \Delta'(v)$. For sequence $\{\varepsilon_n, \ n = 1, 2, \ldots\}$ we assume that $\int_{-\infty}^{+\infty} |\Psi(t)|^m dt < \infty$ for some m, where Ψ is the characteristic function of ε_n. With this assumption, for n big enough, by a local limit theorem, the sum

S_n has a continuous probability density function $p_{S_n}(s)$, such that

$$p_{S_n}(s) = (\frac{1}{\sigma\sqrt{n}})\varphi\left[\frac{s-nv}{\sigma\sqrt{n}}\right] + O(\frac{1}{\sqrt{n}}) \quad \text{as} \quad n \to \infty. \tag{2.1}$$

Here, and in what follows, φ denotes the density function of standard normal distribution. According to the definition of T_n, it can always be represented as $T_n = n\Delta(\overline{S}_n) = Z_n + e_n$, where

$$Z_n = \sum_{k=1}^{n} X_k, \quad X_k = \Delta(v) + \Delta'(v)(\varepsilon_k - v),$$
$$e_n = n\tau(\overline{S}_n), \quad \tau(x) = \Delta(x) - \Delta(v) - \Delta'(v)(x - v).$$

The following two lemmas from [1] will be used in the proofs of the main results of this paper.

Lemma A (see [1]). *The following relations hold:*

1) $P(\tau_a < \infty) = 1$ *for all* $a \geq a_0$,
2) $\tau_a \xrightarrow{a.s.} \infty$ *as* $a \to \infty$,
3) $\frac{\tau_a}{a} \xrightarrow{a.s.} \frac{1}{\mu}$ *as* $a \to \infty$.

Lemma B (see [1]). *Under the above stated conditions, the random variable* R_a *has a limit distribution with density function given by*

$$h(r) = \frac{1}{\mu}P(S_k \geq r, k \geq 1), \quad r > 0.$$

The main results of this paper are the following theorems.

Theorem 2.1. *Let* $\{\varepsilon_n, n = 1, 2, \ldots\}$ *be a sequence of independent and identically distributed random variables satisfying the condition (2.1), and let the function* Δ *be strictly convex and continuous differentiable in a vicinity of the point* $v = E\varepsilon_1$. *If there is*

$$n = n_a = \frac{a}{\mu} + Z_a\sqrt{\frac{a}{\mu}}, \quad \text{where} \quad Z_a \to z \in R \quad \text{as} \quad a \to \infty,$$

then the following asymptotic relation holds:

$$q_a(n, r) \sim \frac{\mu}{|\delta|\sigma\sqrt{n}}\varphi(\frac{\mu}{\sigma\delta}z)h(r) \quad \text{as} \quad a \to \infty,$$

where $\delta = \Delta'(v) \neq 0$ *and* $q_a(n, r) = \frac{\mathrm{d}}{\mathrm{dr}}P(\tau_a = n, R_a \leq r)$.

Theorem 2.2. *Let* $E[\Delta(\varepsilon_1)^+] < \infty$, *then under the conditions of Theorem 2.1 the following asymptotic relation holds:*

$$P(\tau_a = n) \sim \frac{\mu}{|\delta|\sigma\sqrt{n}}\varphi\left(\frac{\mu}{\sigma\delta}z\right) \quad \text{as} \quad a \to \infty.$$

Corollary 2.1. *Let $\{\gamma_n,\ n=1,2,\dots\}$ be a sequence of independent identically distributed and positive random variables. Then the conditions of Theorem 2.2 are satisfied for the sequence $\varepsilon_n=\ln\gamma_n$ and $p_n=\gamma_1\cdot\gamma_2\cdot\ldots\cdot\gamma_n$, $t_a=\inf\{n:\ p_n>e^a\}$, and therefore, the following asymptotic relation holds:*

$$P(t_a=n)\sim\frac{\mu_1}{\sigma_1\sqrt{n}}\varphi\left(\frac{\mu_1}{\sigma_1}z\right)\quad as\quad a\to\infty,$$

where $\mu_1=E\ln(\gamma_n)$ and $\sigma_1^2=\mathrm{Var}(\ln\gamma_n)$.

Proof of Theorem 2.1. For the case $\delta=\Delta'(v)>0$, the theorem is proved in [1].

So, we prove the theorem in the case $\delta=\Delta'(v)<0$. Defining $M_n=M_n(a,r)=\{y:\ a<n\Delta(y)\le a+r\}$, and observing that the function Δ in a vicinity of the point $x=v$ has a decreasing inverse Δ^{-1}, we can write

$$\begin{aligned}P(\tau_a=n,R_a\le r)&=P(\tau_a=n,0<T_{\tau_a}-a\le r)=P(\tau_a=n,a<n\Delta(\bar S_{\tau_n})\le a+r)\\&=P(\tau_a=n,\bar S_{\tau_a}\in M_n)=P(\tau_a\ge n,\bar S_{\tau_a}\in M_n)\\&=\int_{M_n}P(\tau_a\ge n,\bar S_{\tau_a}\in M_n|\bar S_n=y)p_{\bar S_n}(y)\mathrm{d}y\\&=\int_{M_n}P(\tau_a\ge n|\bar S_{\tau_a}=y)np_{\bar S_n}(ny)\mathrm{d}y=\int_{\Delta^{-1}(\frac{a+r}{n})}^{\Delta^{-1}(\frac{a}{n})}l_a(n,y)np_{\bar S_n}(ny)\mathrm{d}y,\end{aligned}$$

where $l_a(n,y)=P(\tau_a\ge n|\bar S_n=y)$. Next, we have

$$\begin{aligned}q_a(n,r)&=\frac{\mathrm{d}}{\mathrm{d}r}P(\tau_a=n,R_a\le r)\\&=\frac{\mathrm{d}}{\mathrm{d}r}\int_{\Delta^{-1}(\frac{a+r}{n})}^{\Delta^{-1}(\frac{a}{n})}l_a(n,y)np_{\bar S_n}(ny)\mathrm{d}y=\sum_{y:n\Delta(y)=a+r}\frac{1}{|\Delta'(v)|}l_a(n,y)p_{\bar S_n}(ny).\end{aligned}$$

If $n\Delta(y)=a+r$ is the root of equation, given that Δ is a strictly convex function and $\Delta'(v)<0$, then there is a unique root y_0 with the condition $y_0\to v$ as $a\to\infty$, such that

$$q_a(n,y_0)=-\frac{1}{\Delta'(v)}l_a(n,y_0)p_{\bar S_n}(ny_0). \tag{2.2}$$

According to Taylor formula for function Δ in a vicinity of point $x=v$, we have

$$n\Delta(y_0)=n\Delta(v)+n\Delta'(v)(y_0-v)+(y_0-v)\mathrm{O}(n),$$

$$a+r=\mu\left(\frac{a}{\mu}+Z_a\sqrt{\frac{a}{\mu}}\right)+(y_0-v)(n\Delta'(v)+no(1)),$$

implying that $r=\mu Z_a\sqrt{\frac{a}{\mu}}+(y_0-v)n(\Delta'(v)+o(1))$. Thus, we have

$$y_0=v-\frac{\mu z}{\Delta'(v)}\cdot\frac{1}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)\quad\text{as}\quad a\to\infty. \tag{2.3}$$

According to the Theorem 2.7 and Corollary 5.1 from [1], we have

$$l_a(n, y_0) \to \mu h(r) \quad \text{as} \quad a \to \infty. \tag{2.4}$$

Therefore, from (2.2) – (2.4) we obtain

$$q_a(n, y_0) \sim -\frac{1}{\Delta'(v)} \cdot \mu h(r) \left(\frac{1}{\sigma\sqrt{n}}\right) \varphi\left[\frac{ny_0 - nv}{\sigma\sqrt{n}}\right]$$

$$q_a(n, y_0) \sim -\frac{1}{\Delta'(v)} \cdot \mu h(r) \left(\frac{1}{\sigma\sqrt{n}}\right) \varphi\left[-\frac{\frac{\mu z}{\Delta'(v)}\frac{n}{\sqrt{n}}}{\sigma\sqrt{n}}\right]$$

$$\sim -\frac{1}{\Delta'(v)} \cdot \mu h(r) \left(\frac{1}{\sigma\sqrt{n}}\right) \varphi\left(-\frac{\Delta(v)}{\sigma\Delta'(v)}z\right) \sim -\frac{\Delta(v)}{\sigma\Delta'(v)\sqrt{n}}\varphi\left(\frac{\Delta(v)}{\sigma\Delta'(v)}z\right) h(r).$$

Thus, we have proved that for $\delta = \Delta'(v) \neq 0$ the following asymptotic relation holds:

$$q_a(n, y) \sim \frac{\Delta(v)}{\sigma|\Delta'(v)|\sqrt{n}}\varphi\left(\frac{\Delta(v)}{\sigma\Delta'(v)}z\right) h(r).$$

Theorem 2.1 is proved.

Proof of Theorem 2.2. We have

$$P(\tau_a = n) \sim \frac{\mu}{|\delta|\sigma\sqrt{n}}\varphi\left(\frac{\mu}{\delta\sigma}z\right) \quad \text{as} \quad a \to +\infty.$$

For each $c > 0$, we can write

$$\sqrt{n}P(\tau_a = n) = \sqrt{n}\int_0^\infty q_a(n, r)\mathrm{d}r = \sqrt{n}\int_0^c q_a(n, r)\mathrm{d}r)$$
$$+ \sqrt{n}\int_c^\infty q_a(n, r)\mathrm{d}r = q_{a,1}(n, c) + q_{a,2}(n, c).$$

From Theorem 2.1, we have

$$q_{a,1}(n, c) = \sqrt{n}\int_0^c q_a(n, r)\mathrm{d}r = \sqrt{n}\int_0^c \frac{\mu}{|\delta|\sigma\sqrt{n}}\varphi\left(\frac{\mu}{\delta\sigma}z\right) h(r)\mathrm{d}r$$
$$= \frac{\mu}{|\delta|\sigma}\varphi\left(\frac{\mu}{\delta\sigma}z\right)\int_0^c h(r)\mathrm{d}r = \frac{\mu}{|\delta|\sigma}\varphi\left(\frac{\mu}{\delta\sigma}z\right) H(c),$$

where $H(c) \to 1$ as $c \to \infty$. If $c \to \infty$ and $a \to \infty$, then we have $q_{a,1}(n, c) = \frac{\mu}{|\delta|\sigma}\varphi\left(\frac{\mu}{\delta\sigma}z\right)$. Thus, to complete the proof, it is enough to show that $q_{a,2}(n, c) \to 0$ as $c \to \infty$.

Since Δ is a convex function, we can write

$$T_n = n\Delta\left(\frac{S_n}{n}\right) = n\Delta\left(\frac{S_{n-1}}{n} + \frac{\varepsilon_n}{n}\right) = n\Delta\left(\frac{n-1}{n}\overline{S}_{n-1} + \frac{1}{n}\varepsilon_n\right)$$
$$< n\left[\frac{n-1}{n}\Delta(\overline{S}_{n-1}) + \frac{1}{n}\Delta(\varepsilon_n)\right] = T_{n-1} + \Delta(\varepsilon_n).$$

and

$$q_{a,2}(n,c) = \sqrt{n}\int_c^\infty q_a(n,r)\mathrm{d}r = \sqrt{n}\int_c^\infty \frac{d}{dr}P(\tau_a = n,\ R_a \le r)\mathrm{d}r$$
$$= \sqrt{n}\int_c^\infty \frac{d}{dr}[P(\tau_a = n) - P(\tau_a = n,\ R_a > r)]\mathrm{d}r = \sqrt{n}P(\tau_a = n,\ R_a > c)$$
$$= \sqrt{n}P(\tau_a = n,\ T_n - a > c) = \sqrt{n}P(\tau_a = n,\ T_n > a + c)$$
$$\le \sqrt{n}P(T_{n-1} \le a, a + c < T_n < T_{n-1} + \Delta(\varepsilon_n)) = \sqrt{n}P(a + c - \Delta(\varepsilon_n) < T_{n-1} \le a)$$
$$= \sqrt{n}\int_c^\infty P\left[a + c - \Delta(\varepsilon_n) < T_{n-1} \le a \mid \Delta(\varepsilon_n) = s\right]\mathrm{d}Q(s)$$
$$= \sqrt{n}\int_c^\infty P(a + c - s < T_{n-1} \le a)\mathrm{d}Q(s),$$

where $Q(s) = P(\Delta(\varepsilon_n) < s)$.

Based on the definition of e_n and WLLN: $\frac{e_n}{\sqrt{n}} \xrightarrow{P} 0$, we obtain

$$\lim_{n\to\infty} P\left[\frac{T_n - n\mu}{\delta\sigma\sqrt{n}} \le x\right] = \lim_{n\to\infty} P\left[\frac{e_n + z_n - n\mu}{\delta\sigma\sqrt{n}} \le x\right]$$
$$(2.5)\qquad = \lim_{n\to\infty} P\left[\frac{z_n - n\mu}{\delta\sigma\sqrt{n}} \le x\right] = \lim_{n\to\infty} P\left[\frac{S_n - nv}{\sigma\sqrt{n}} \le x\right] = \phi(x),$$

where ϕ is the standard normal distribution function. According to (2.5), we have

$$(2.6)\qquad P(a + c - s < T_n \le a) - P(a + c - s < S_n \le a) \to 0 \quad \text{as} \quad n \to \infty.$$

According to the local limit theorem and (2.1), there is $K > 0$ such that

$$(2.7)\qquad P(a + c - s < S_n \le a) \le K(s - c).$$

From (2.6) and (2.7), we conclude that there is a constant $M > 0$ such that

$$q_{a,2}(n,c) \le \sqrt{n}M\int_c^\infty (s - c)\mathrm{d}Q(s) \le \sqrt{n}M\int_c^\infty s\mathrm{d}Q(s).$$

Since by assumption $E[\Delta(\varepsilon_1)^+] < \infty$, the last term of the above relation tends to 0 as $c \to \infty$, and the result follows. Theorem 2.2 is proved.

Proof of Corollary 2.1. Given that the sequence $\{\varepsilon_n = \ln\gamma_n\}$ satisfies the condition of Theorem 2.2, with $S_n = \ln p_n = \ln\gamma_1 + \cdots + \ln\gamma_n$ and $\Delta(x) = x$, we have

$$t_a = \inf\{n : S_n > a\} = \inf\{n : \ln\gamma_1 + \cdots + \ln\gamma_n > a\}$$
$$= \inf\{n : \ln\gamma_1\cdots\gamma_n > a\} = \inf\{n : \gamma_1\cdots\gamma_n > e^a\} = \inf\{n : p_n > e^a\},$$

and the result easily follows from Theorem 2.2.

Corollary 2.1 is proved.

Example. Let $\{\varepsilon_n,\ n = 1, 2, \ldots\}$ be a sequence of independent identically distributed random variables having exponential distribution with parameter $\lambda = 1$. In this case, the sum $S_n = \sum_{k=1}^{n} \varepsilon_k$ has a Gamma distribution with parameters $(n, 1)$. Then for $\Delta(x) = \frac{1}{x}$, $x > 0$, we have

$$\tau_a = \inf\{n : \frac{n^2}{S_n} > a\}, \qquad \mu = \Delta(v) = 1, \qquad v = E\varepsilon_n = 1,$$
$$\sigma^2 = \mathrm{Var}\,\varepsilon_1 = 1, \qquad \delta = \Delta'(v) = -1.$$

So, by Theorem 2.2, we obtain

$$P(\tau_a = n) \sim \frac{1}{\sqrt{n}}\varphi(z) \quad \text{as} \quad a \to \infty.$$

Список литературы

[1] M. Woodroofe, Nonlinear Renewal Theory Sequential Analysis, SIAM, Philadelphia, (1982).

[2] S. Lally, "Limit theorems for first passage times in linear and nonlinear renewal theory", Adv. Appl. Probab., 16, no. 4, 766 – 803 (1984).

[3] M. Woodroofe, R. Keener, "Asymptotic expansions in boundary crossing problems", Ann. Probab., 15, 102 – 114 (1987).

[4] A. Gut, Stopped Random Walks, Limit Theorems and Applications, Springer, New York (1988).

[5] D. Siegmund, Sequential Analysis, Tests and Confidence Intervals, Springer, New York (1985).

Поступила 6 апреля 2016