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SOME PERTURBATION OF ℓ_p-LOCALIZED FRAMES

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Abstract. In this paper, we give some sufficient conditions under which perturbations preserve ℓ_p-localized frames. Using an arbitrary given sequence, we provide a simple way for constructing ℓ_p-localized sequences.

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1. Introduction

The concept of frames in a Hilbert space was originally introduced by Duffin and Schaeffer [6] in the context of non-harmonic Fourier series. Frames are redundant sets of vectors in a Hilbert space, which yield one natural representation of each vector in the space, but may have infinitely many different representations for any given vector. This redundancy makes frames is useful in applications. For instance, in signal processing, this concept has become very useful in analyzing the completeness and stability of linear discrete signal representations. Since the last decade, various generalizations of the frames have been proposed such as frame of subspaces, pseudo-frames, oblique frames, continuous frames, fusion frames and g-frames.

Given a separable Hilbert space $\mathcal{H}$ with inner product $< .,. >$, a sequence $\{f_k\}_{k=1}^{\infty}$ is called a frame for $\mathcal{H}$ if there exist constants $A > 0$, $B < \infty$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} | < f, f_k > |^2 \leq B\|f\|^2.$$

The constant A is called a lower frame bound and B is called an upper frame bound. If only an upper bound B exists, then $\{f_k\}_{k=1}^{\infty}$ is called a B-Bessel sequence or simply Bessel when the constant is implicit. If $A = B$, then the sequence $\{f_k\}_{k=1}^{\infty}$ is called a tight frame, and if $A = B = 1$, it is called a Parseval frame. A sequence $\{f_k\}_{k=1}^{\infty}$ in Hilbert space $\mathcal{H}$ is called a frame sequence in $\mathcal{H}$ if it is a frame for Hilbert space $\overline{span}\{f_k\}_{k=1}^{\infty}$.

A bounded linear operator T defined by

$$T : \ell_2(\mathbb{N}) \to \mathcal{H}, \quad T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$$

is called the pre frame operator of $\{f_k\}_{k=1}^{\infty}$ (see [4, Theorem 3.1.3]). Also, a linear operator S defined by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{k=1}^{\infty} < f, f_k > f_k$$

is called the frame operator of $\{f_k\}_{k=1}^{\infty}$. The frame operator S is bounded, invertible, self-adjoint, and positive (see [4, Lemma 5.1.6]). It is easy to show that $S = TT^*$, where T^* is the adjoint operator of T (see [4, page 100]). Two frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are said to be dual frames for $\mathcal{H}$ if

$$f = \sum_{k=1}^{\infty} < f, f_k > g_k = \sum_{k=1}^{\infty} < f, g_k > f_k, \quad \text{for any} \quad f \in \mathcal{H}.$$

Observe that the frame $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a dual frame of the frame $\{f_k\}_{k=1}^{\infty}$, and is called the canonical dual frame of $\{f_k\}_{k=1}^{\infty}$. For more information concerning frames we refer to [4, 7, 9].

The fundamental and useful property of frames is their overcompleteness. Balan et al. [1] – [3] used the ℓ_p-localization of a sequence $\mathcal{F} = \{f_i\}_{i \in I}$ (I is an index set), and a sequence $\mathcal{E} = \{e_j\}_{j \in G}$ (G is a discrete abelian group), in a separable Hilbert space $\mathcal{H}$ with respect to the associated map $a : I \to G$ for excess, overcompleteness and decay of the coefficients of the expansion of elements of $\mathcal{F}$ in terms of the elements of $\mathcal{E}$. They constructed an infinite subset of a frame which can be removed yet still leave a frame. They also proved that any sufficiently localized frame can be written as a finite union of Riesz sequences.

Perturbation theory is a very important tool in several areas of mathematics. In applications where bases appear, a famuos classical perturbation result is given by Paley and Wiener [8]. The Paley-Wiener theorem state that a sequence that is sufficiently near to a basis in a Hilbert space automatically forms a basis. A version of Paley-Wiener Theorem for frames can be found in Christensen [5].

In this paper, we concentrate on perturbation theory for ℓ_p-localized frames. We show that some perturbations of the localization map $a : I \to G$ preserve the ℓ_p-localization property (Theorems 2.2 and 2.3). Also, using the convolution on ℓ_1, biorthogonal sequences and orthogonal projections we obtain new ℓ_p-localized sequences (Theorems 2.4 - 2.9). Finally, using an arbitrary given sequence, we provide a simple way for constructing ℓ_p-localized sequences (Theorem 2.10).

2. Main Results

Throughout the paper, I will stand for a countable index set, G will denote an additive discrete group of the form

$$G = \prod_{i=1}^{d} a_i\mathbb{Z} \times \prod_{j=1}^{e} \mathbb{Z}_{n_j},$$

which is equipped with a metric defined as follows. For $m_j \in \mathbb{Z}_{n_j}$, we set $\delta(m_j) = 0$ if $m_j = 0$, and $\delta(m_j) = 1$, otherwise. Then given $g = (a_1n_1, ..., a_dn_d, m_1, ..., m_e) \in G$, we set

$$|g| = sup\{|a_1n_1|, ..., |a_dn_d|, \delta(m_1), ..., \delta(m_e)\}.$$

The metric is then defined by $d(f, g) = |f - g|$. Also, let $a : I \to G$ be a map and let

$$\ell_p(G) := \left\{ r = (r_j)_{j\in G} : \sum_{j\in G} |r_j|^p < \infty \right\}.$$

Balan et al. [2] defined the ℓ_p-localization as follows:

Definition 2.1. Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ be sequences in $\mathcal{H}$, and let $a : I \to G$ be an associated map. a) $\mathcal{F}$ is ℓ_p-localized with respect to the sequence $\mathcal{E}$ and the map a, or simply that $(\mathcal{F}, a, \mathcal{E})$ is ℓ_p-localized, if

$$\sum_{j\in G} \sup_{i\in I} |\langle f_i, e_{j+a(i)}\rangle|^p < \infty.$$

Equivalently, there exists an element $r \in \ell_p(G)$ such that

$$|\langle f_i, e_j\rangle| \le r_{a(i)-j}, \forall i \in I, \forall j \in G.$$

b) $(\mathcal{F}, a)$ is ℓ_p-self-localized, if there exists an element $r \in \ell_p(G)$ such that

$$|\langle f_i, f_j\rangle| \le r_{a(i)-a(j)}, \forall i, j \in I.$$

Also, if $\mathcal{F} = \{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with a canonical dual frame $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$, then c) $(\mathcal{F}, a)$ is ℓ_p-localized with respect to its canonical dual frame $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$, if there exists an element $r \in \ell_p(G)$ such that

$$\left|\left\langle f_i, \tilde{f}_i\right\rangle\right| \le r_{a(i)-a(j)}, \quad i, j \in I.$$

The next theorem states that the ℓ_2-localization property is stable under perturbation of the localization map $a : I \to G$.

Theorem 2.2. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a sequence in $\mathcal{H}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ be a Bessel sequence in $\mathcal{H}$, and let $a : I \to G$ be an associated map. If $(\mathcal{F}, a, \mathcal{E})$ is ℓ_2-localized and $b : I \to G$ is a map such that $a = b$ except on a finite subset F of I, then $(\mathcal{F}, b, \mathcal{E})$ is ℓ_2-localized.*

Proof. We have $\sum_{j\in G}\sup_{i\in I}|\langle f_i,e_{j+a(i)}\rangle|^2<\infty$, because $(\mathcal{F},a,\mathcal{E})$ is ℓ_2-localized. Let $i_0\in I=F\cup(I\setminus F)$. If $i_0\in F$, then for all $j\in G$ we have

$$|\langle f_{i_0},e_{j+b(i_0)}\rangle|^2\le\sup_{i\in F}|\langle f_i,e_{j+b(i)}\rangle|^2\le\sup_{i\in F}|\langle f_i,e_{j+b(i)}\rangle|^2+\sup_{i\in(I\setminus F)}|\langle f_i,e_{j+b(i)}\rangle|^2.$$

Also, if $i_0\in(I\setminus F)$, then for all $j\in G$ we have

$$\begin{aligned}|\langle f_{i_0},e_{j+b(i_0)}\rangle|^2&\le\sup_{i\in(I\setminus F)}|\langle f_i,e_{j+b(i)}\rangle|^2\\&\le\sup_{i\in F}|\langle f_i,e_{j+b(i)}\rangle|^2+\sup_{i\in(I\setminus F)}|\langle f_i,e_{j+b(i)}\rangle|^2.\end{aligned}$$

Thus, for all $j\in G$ we have $\sup_{i\in I}|\langle f_i,e_{j+b(i)}\rangle|^2\le\sup_{i\in F}|\langle f_i,e_{j+b(i)}\rangle|^2+\sup_{i\in(I\setminus F)}|\langle f_i,e_{j+b(i)}\rangle|^2$, because i_o is an arbitrary element of I.

Now for all $i\in I$ we have $G+b(i)=G$, because G is a group. Also, $a(i)=b(i)$ for all $i\in I\setminus F$. Thus, we can write

$$\begin{aligned}\sum_{j\in G}\sup_{i\in I}|\langle f_i,e_{j+b(i)}\rangle|^2&\le\sum_{j\in G}\sup_{i\in F}|\langle f_i,e_{j+b(i)}\rangle|^2+\sum_{j\in G}\sup_{i\in I\setminus F}|\langle f_i,e_{j+b(i)}\rangle|^2\\&=\sum_{j\in G}\sup_{i\in F}|\langle f_i,e_{j+b(i)}\rangle|^2+\sum_{j\in G}\sup_{i\in I\setminus F}|\langle f_i,e_{j+a(i)}\rangle|^2\\&\le\sum_{j\in G}\sum_{i\in F}|\langle f_i,e_{j+b(i)}\rangle|^2+\sum_{j\in G}\sup_{i\in I\setminus F}|\langle f_i,e_{j+a(i)}\rangle|^2\\&=\sum_{i\in F}\sum_{j\in G}|\langle f_i,e_{j+b(i)}\rangle|^2+\sum_{j\in G}\sup_{i\in I\setminus F}|\langle f_i,e_{j+a(i)}\rangle|^2\\&=\sum_{i\in F}\sum_{j\in G+b(i)}|\langle f_i,e_j\rangle|^2+\sum_{j\in G}\sup_{i\in I\setminus F}|\langle f_i,e_{j+a(i)}\rangle|^2\\&=\sum_{i\in F}\sum_{j\in G}|\langle f_i,e_j\rangle|^2+\sum_{j\in G}\sup_{i\in I\setminus F}|\langle f_i,e_{j+a(i)}\rangle|^2\\&\le\sum_{i\in F}\sum_{j\in G}|\langle f_i,e_j\rangle|^2+\sum_{j\in G}\sup_{i\in I}|\langle f_i,e_{j+a(i)}\rangle|^2\\&\le B\sum_{i\in F}\|f_i\|^2+\sum_{j\in G}\sup_{i\in I}|\langle f_i,e_{j+a(i)}\rangle|^2<\infty,\end{aligned}$$

where B is the Bessel bound of $\mathcal{E}$. Thus, $(\mathcal{F},b,\mathcal{E})$ is ℓ_2-localized. □

For two localization maps a and b, the ℓ_p-localization with respect to a and the ℓ_p-localization with respect to b are equivalent if $a-b$ is constant.

Theorem 2.3. *Let $\mathcal{F}=\{f_i\}_{i\in I}$ and $\mathcal{E}=\{e_j\}_{j\in G}$ be sequences in $\mathcal{H}$, and let $a:I\to G$ be an associated map. Suppose that $(\mathcal{F},a,\mathcal{E})$ is ℓ_p-localized. If $b:I\to G$ is a map such that $b-a$ is a constant function, then $(\mathcal{F},b,\mathcal{E})$ is ℓ_p-localized.*

Proof. If $(\mathcal{F},a,\mathcal{E})$ is ℓ_p-localized, then there exists $r\in\ell_p(G)$ such that

$$|\langle f_i,e_j\rangle|\le r_{a(i)-j},\quad i\in I,\quad j\in G.$$

Also, if $b : I \to G$ is a map such that $b - a$ is constant function, then there exists $\lambda \in G$ such that $a(i) = b(i) - \lambda$ for all $i \in I$. Observe that the sequence $s = \{s(j)\}_{j\in j}$ defined by $s(j) := r_{j-\lambda}$ belongs to $\ell_p(G)$, because

$$\sum_{j\in G} |s_j|^p = \sum_{j\in G} |r_{j-\lambda}|^p = \sum_{j\in\lambda+G} |r_j|^p = \sum_{j\in G} |r_{j-\lambda}|^p < \infty.$$

Finally, for all $i \in I$ and all $j \in G$ we have

$$|\langle f_i, e_j\rangle| \leq r_{a(i)-j} = r_{b(i)-j-\lambda} = s_{b(i)-j},$$

and hence $(\mathcal{F}, b, \mathcal{E})$ is ℓ_p-localized. □

Using the convolution on ℓ_1, in the next theorem we obtain new ℓ_1-localized sequences.

Theorem 2.4. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ be sequences in $\mathcal{H}$, and let $a : I \to G$ be an associated map. If $(\mathcal{F}, a, \mathcal{E})$ is ℓ_1-localized and $\mathcal{E}' = \{e'_j\}_{j\in G}$, where $e'_j = \sum_{k\in G} s_k e_{j+k}$ for some $s = \{s_k\}_{k\in G} \in \ell_1(G)$, then $(\mathcal{F}, a, \mathcal{E}')$ is ℓ_1-localized.*

Proof. If $(\mathcal{F}, a, \mathcal{E})$ is ℓ_1-localized, then there exists $r \in \ell_1(G)$ such that

$$|\langle f_i, e_j\rangle| \leq r_{a(i)-j}, \forall i \in I, \forall j \in G.$$

Now, for all $i \in I$ and $j \in G$, we have

$$|\langle f_i, e'_j\rangle| = \left|\left\langle f_i, \sum_{k\in G} s_k e_{j+k}\right\rangle\right| \leq \sum_{k\in G} |s_k| |\langle f_i, e_{j+k}\rangle| \leq \sum_{k\in G} |s_k| r_{a(i)-j-k} = (|s| * r)_{a(i)-j},$$

where $|s| * r$ is the convolution of two sequence $|s|$ and r. Thus, $(\mathcal{F}, a, \mathcal{E}')$ is ℓ_1-localized. □

In the next theorem, we obtain a new ℓ_p-localized sequence for a given ℓ_p-localized sequence.

Theorem 2.5. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ be a sequence in $\mathcal{H}$, and let $a : I \to G$ be an associated map. If $(\mathcal{F}, a, \mathcal{E})$ is ℓ_p-localized and $\mathcal{E}' = \{e'_j\}_{j\in G}$, where $e'_j = e_j + \sum_{k\in I} s_{a(k)-j} f_k$ for some $s = \{s_k\}_{k\in G} \in \ell_p(G)$, then $(\mathcal{F}, a, \mathcal{E}')$ is ℓ_p-localized.*

Proof. If $(\mathcal{F}, a, \mathcal{E})$ is ℓ_p-localized, then there exists $r \in \ell_p(G)$ such that

$$|\langle f_i, e_j\rangle| \leq r_{a(i)-j}, \forall i \in I, \forall j \in G.$$

Now, for all $i \in I$ and $j \in G$, we have

$$\begin{aligned}|\langle f_i, e'_j\rangle| &= \left|\left\langle f_i, e_j + \sum_{k\in I} s_{a(k)-j} f_k\right\rangle\right| \leq |\langle f_i, e_j\rangle| + \left|\sum_{k\in I} s_{a(k)-j} \langle f_i, f_k\rangle\right| \\ &= |\langle f_i, e_j\rangle| + |s|_{a(i)-j} \|f_i\|^2 \leq (r + |s|)_{a(i)-j}.\end{aligned}$$

Thus, $(\mathcal{F}, a, \mathcal{E}')$ is ℓ_p-localized. □

The next theorem shows that if p and q are conjugate exponents, then using Holder inequality, from a given sequence that is ℓ_p-localized and ℓ_q-localized, we can obtain a new ℓ_2-localized sequence.

Theorem 2.6. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ be sequences in $\mathcal{H}$, $a : I \to G$ be an associated map, and let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The following assertions hold.*

a) *If $(\mathcal{F}, a, \mathcal{E})$ is ℓ_p-localized and ℓ_q-localized, then $(\mathcal{F}, a, \mathcal{E})$ is ℓ_2-localized.*

b) *If $(\mathcal{F}, a)$ is ℓ_p-self localized and ℓ_q-self localized, then $(\mathcal{F}, a)$ is ℓ_2-self localized.*

c) *If $\mathcal{F} = \{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ and $(\mathcal{F}, a)$ is ℓ_p-localized and ℓ_q-localized with respect to its canonical dual frame $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$, then $(\mathcal{F}, a)$ is ℓ_2-localized with respect to its canonical dual frame $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$.*

Proof. If $(\mathcal{F}, a, \mathcal{E})$ is ℓ_p-localized and ℓ_q-localized, then there exist $r \in \ell_p(G)$ and $s \in \ell_q(G)$ such that

$$|\langle f_i, e_j\rangle| \leq r_{a(i)-j}, \text{ and } |\langle f_i, e_j\rangle| \leq s_{a(i)-j}, \forall i \in I, \forall j \in G.$$

Thus, for all $i \in I$ and $j \in G$ we have

$$|\langle f_i, e_j\rangle|^2 = |\langle f_i, e_j\rangle||\langle f_i, e_j\rangle| \leq r_{a(i)-j}s_{a(i)-j} = (rs)_{a(i)-j}.$$

By Holder inequality we have $rs \in \ell_1(G)$, and hence $(\mathcal{F}, a, \mathcal{E})$ is ℓ_2-localized. The proof of parts b) and c) is similar to that of part a), and so is omitted. □

Remark. Using the generalized Holder inequality and arguments similar to those applied in the proof of Theorem 2.6, it can easily be shown that if $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ are sequences in $\mathcal{H}$, $a : I \to G$ is an associated map, and $(\mathcal{F}, a, \mathcal{E})$ is ℓ_{P_t}-localized for $t = 1, 2, ..., n$ such that $\sum_{t=1}^{n} \frac{1}{p_t} = \frac{1}{r} \leq 1$, then $(\mathcal{F}, a, \mathcal{E})$ is ℓ_{nr}-localized.

The next theorem shows that biorthogonal sequences of $\mathcal{E} = \{e_j\}_{j\in G}$ give a new ℓ_p-localized sequence with respect to $\mathcal{E}$.

Theorem 2.7. *Let $\mathcal{F} = \{f_i\}_{i\in G}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ be sequences in $\mathcal{H}$, and let $\mathcal{E}' = \{e'_j\}_{j\in G}$ be a biorthogonal sequence of $\mathcal{E}$. If $(\mathcal{F}, id, \mathcal{E})$ is ℓ_p-localized, then $(\mathcal{F} + \mathcal{E}', id, \mathcal{E})$ is also ℓ_p-localized.*

Proof. If $(\mathcal{F}, id, \mathcal{E})$ is ℓ_p-localized, then there exist $r \in \ell_p(G)$ such that

$$|\langle f_i, e_j\rangle| \leq r_{i-j}, \forall i \in I, \forall j \in G.$$

Now, for all $i \in I$ and $j \in G$, we have

$$|\langle f_i + e'_i, e_j\rangle| \leq |\langle f_i, e_j\rangle| + |\langle e'_i, e_j\rangle| \leq r_{i-j} + s_{i-j} = (r+s)_{i-j},$$

where $s = \{s(m)\}_{m\in G} \in \ell_p(G)$ is defined by $s(m) = 0$ for $m \neq 0$ and $s(0) = 1$. Thus, $(\mathcal{F} + \mathcal{E}', id, \mathcal{E})$ is ℓ_p-localized. $\square$

Using the convolution on ℓ_1, in the next theorem we obtain new ℓ_1-localized sequences.

Theorem 2.8. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ be sequences in $\mathcal{H}$, $a : I \to G$ be an associated map, and let $\mathcal{E}' = \{e'_j\}_{j\in G}$, where $e'_j = \sum_{k=1}^{n} e_{j+j_k}$ for some $j_1, j_2, ..., j_n \in G$. If $(\mathcal{F}, a, \mathcal{E})$ is ℓ_1-localized, then $(\mathcal{F}, a, \mathcal{E}')$ is also ℓ_1-localized.*

Proof. If $(\mathcal{F}, a, \mathcal{E})$ is ℓ_1-localized, then there exist $r \in \ell_1(G)$ such that

$$|\langle f_i, e_j\rangle| \leq r_{a(i)-j}, \forall i \in I, \forall j \in G.$$

Now, for all $i \in I$ and $j \in G$, we have

$$|\langle f_i, e'_j\rangle| = \left|\left\langle f_i, \sum_{k=1}^{n} e_{j+j_k}\right\rangle\right| \leq \sum_{k=1}^{n} |\langle f_i, e_{j+j_k}\rangle| \leq \sum_{k=1}^{n} r_{a(i)-j-j_k} = (r * s)_{a(i)-j},$$

where $s = \{s(m)\}_{m\in G} \in \ell_1(G)$ is defined by $s(m) = 1$ for $m \in \{j_1, j_2, ..., j_n\}$ and $s(m) = 0$, otherwise. Thus, $(\mathcal{F}, a, \mathcal{E}')$ is ℓ_1-localized. $\square$

The next theorem gives a new ℓ_p-localized sequence by acting a familly of orthogonal projections on a given ℓ_p-self localized sequence.

Theorem 2.9. *Let $\mathcal{F} = \{f_i\}_{i\in G}$ be a bounded sequence in $\mathcal{H}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ such that $e_j = f_j - p_j f_j$, where p_j is the orthogonal projection onto $M_j := \overline{span}\{f_i\}_{i\neq j}$. If $(\mathcal{F}, id)$ is ℓ_p-self localized, then $(\mathcal{F}, id, \mathcal{E})$ is also ℓ_p-localized.*

Proof. If $(\mathcal{F}, id)$ is ℓ_p-self localized, then there exist $r \in \ell_p(G)$ such that

$$|\langle f_i, f_j\rangle| \leq r_{i-j}, \forall i, j \in G.$$

Also, there exist $M > 0$ such that $\|f_i\| \leq M$, for all $i \in G$. Observe that the sequence $s = \{s(m)\}_{m\in G}$, defined by $s(m) = 2r(m)$ for $m \neq 0$ and $s(0) = 2M^2$, belongs to $\ell_p(G)$. Hence, for all $i \neq j \in G$, we have

$$|\langle f_i, e_j\rangle| = |\langle f_i, f_j - p_j f_j\rangle| \leq |\langle f_i, f_j\rangle| + |\langle f_i, p_j f_j\rangle| = 2|\langle f_i, f_j\rangle| \leq s_{i-j}.$$

Also, for all $i \in G$ we have

$$|\langle f_i, e_i\rangle = |\langle f_i, f_i - p_i f_i\rangle| \leq |\langle f_i, f_i\rangle| + |\langle f_i, p_i f_i\rangle| \leq 2\|f_i\|^2 \leq 2M^2 = s_{i-i}.$$

Thus, $(\mathcal{F}, id, \mathcal{E})$ is ℓ_p-localized. $\square$

A simple way to construct ℓ_p-localized sequences is given in the next theorem.

Theorem 2.10. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_j\}_{j\in \mathbb{Z}}$ be sequences in $\mathcal{H}$, $a : I \to \mathbb{Z}$ be an arbitrary map, $\mathcal{F}' = \{f'_i\}_{i\in I}$, where $f'_i = \frac{2^{-|a(i)|} f_i}{1+\|f_i\|}$ and $\mathcal{E}' = \{e'_j\}_{j\in \mathbb{Z}}$, where $e'_j = \frac{2^{-|a(j)|} e_j}{1+\|e_j\|}$. Then the following asertions hold:*

a) $(\mathcal{F}', a, \mathcal{E}')$ *is* ℓ_p*-localized.*

b) $(\mathcal{F}', a)$ *is* ℓ_p*-self localized.*

c) $(\mathcal{E}', id)$ *is* ℓ_p*-self localized.*

Proof. For all $i \in I$ and $j \in \mathbb{Z}$ we have

$$|\langle f_i', e_j' \rangle| = \left| \left\langle \frac{2^{-|a(i)|} f_i}{1 + \|f_i\|}, \frac{2^{-|a(j)|} e_j}{1 + \|e_j\|} \right\rangle \right| \le 2^{-(|a(i)|+|j|)} \le 2^{-|a(i)-j|} = r_{a(i)-j},$$

where $r = \{r(m)\}_{m \in \mathbb{Z}} \in \ell_p(\mathbb{Z})$ is defined by $r(m) = 2^{-|m|}$. Thus, $(\mathcal{F}', a, \mathcal{E}')$ is ℓ_p-localized, and assertion a) is proved. The proof of parts b) and c) is similar to that of part a), and so is omitted. □

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