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# HANKEL AND BEREZIN TYPE OPERATORS ON WEIGHTED BESOV SPACES OF HOLOMORPHIC FUNCTIONS ON POLYDISKS

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Abstract. Let $S$ be the space of functions of regular variation and let $\omega = (\omega_1, \dots, \omega_n)$, $\omega_j \in S$. The weighted Besov space of holomorphic functions on polydisks, denoted by $B_p(\omega)$ $(0 < p < +\infty)$, is defined to be the class of all holomorphic functions $f$ defined on the polydisk $U^n$ such that $\|f\|^p_{B_p(\omega)} = \int_{U^n} |Df(z)|^p \prod_{j=1}^n \frac{\omega_j(1-|z_j|)dm_{2n}(z)}{(1-|z_j|^2)^{2-p}} < +\infty$, where $dm_{2n}(z)$ is the $2n$-dimensional Lebesgue measure on $U^n$ and $D$ stands for a special fractional derivative of $f$. We prove some theorems concerning fractional boundedness of the generalized little Hankel and Berezin type operators on the spaces $B_p(\omega)$ and $L_p(\omega)$ (the weighted $L_p$-space).

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## 1. Introduction and auxiliary constructions

Numerous authors have contributed to the theory of holomorphic Besov spaces in the unit disk in $\mathbb{C}$ and in the unit ball in $\mathbb{C}^n$ (see, e.g., J. Arazy et al. [1], K. Stroethoff [17], O. Blasco [3], A. Karapetyants and F. Kodzoeva [10], K. Zhu [19], and references therein). The study of holomorphic Besov spaces on the polydisk is of special interest. Since the polydisk is a product of $n$ disks, one would expect that the natural extensions of results from one-dimensional case would be valid here, but it turns out that, in general, this is not true. Thus, the results for polydisk generally are different from that of for one-dimensional disk and for $n$-dimensional ball. For example, let us recall the classical theorem by Privalov stating that if $f \in \text{Lip}\,\alpha$, then $Kf \in \text{Lip}\,\alpha$, where $Kf$ is a Cauchy type integral. It is known that the analogue of this theorem for multidimensional Lipschitz classes is not true (see [9]), even though its analogue for a sphere is valid (see [14]). In many cases, especially when the classes are defined by means of derivatives, the generalization of functional spaces to the polydisk is different from those on a unit ball. For generalization of

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holomorphic Besov spaces to the polydisk we refer to [8]. Let

$$U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n, \ |z_j| < 1, \ 1 \le j \le n\}$$

be the unit polydisk in the $n$-dimensional complex plane $\mathbb{C}^n$, and let

$$T^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n, \ |z_i| = 1, \ 1 \le i \le n\}$$

be its torus. We denote by $H(U^n)$ the set of holomorphic functions on $U^n$, by $L^\infty(U^n)$ the set of bounded measurable functions on $U^n$, and by $H^\infty(U^n)$ the subspace of $L^\infty(U^n)$ consisting of holomorphic functions.

Let $S$ be the class of all non-negative measurable functions $\omega$ on $(0,1)$, for which there exist positive numbers $M_\omega$, $q_\omega$, $m_\omega$, $(m_\omega, q_\omega \in (0,1))$, such that

$$m_\omega \le \frac{\omega(\lambda r)}{\omega(r)} \le M_\omega$$

for all $r \in (0,1)$ and $\lambda \in [q_\omega, 1]$. Some properties of functions from $S$ can be found in [15]. We set

$$-\alpha_\omega = \frac{\log m_\omega}{\log q_\omega^{-1}}; \qquad \beta_\omega = \frac{\log M_\omega}{\log q_\omega^{-1}},$$

and assume that $0 < \beta_\omega < 1$. For example, $\omega \in S$ if $\omega(t) = t^\alpha$ with $-1 < \alpha < \infty$.

In what follows, for convenience of notation, for $\zeta = (\zeta_1, ..., \zeta_n)$ and $z = (z_1, ..., z_n)$, we set

$$\omega(1-|z|) = \prod_{j=1}^n \omega_j(1-|z_j|), \ 1-|z| = \prod_{j=1}^n (1-|z_j|), \ 1-\overline{\zeta}z = \prod_{j=1}^n (1-\overline{\zeta}_j z_j).$$

Further, for $m = (m_1, ..., m_n)$, we set

$$(m+1) = (m_1+1)...(m_n+1), \ (m+1)! = (m_1+1)!...(m_n+1)!,$$

$$(1-|z|)^m = \prod_{j=1}^n (1-|z_j|)^{m_j}.$$

Throughout the paper, we assume that $\omega_j \in S$, $1 \le j \le n$. Using the results of [15] one can prove that

$$\omega_j(t) = \exp\left\{\eta_j(t) + \int_t^1 \frac{\varepsilon_j(u)}{u} du\right\},$$

where $\eta(u)$, $\varepsilon(u)$ are bounded measurable functions and $-\alpha_{\omega_j} \le \varepsilon_j(u) \le \beta_{\omega_j}$ $(1 \le j \le n)$. Without loss of generality, we can assume that $\eta(u) = 0$. Then $t^{\alpha_{\omega_j}} \le \omega_j(t) \le t^{-\beta_{\omega_j}}$ is always true. Now define the notion of fractional differential.

**Definition 1.1.** *For a holomorphic function* $f(z) = \sum_{(k)=(0)}^{(\infty)} a_k z^k$, $z \in U^n$, *and for* $\beta = (\beta_1, ..., \beta_n)$, $\beta_j > -1$, $(1 \le j \le n)$, *we define the fractional differential* $D^\beta f$ *as follows:*

$$D^\beta f(z) = \sum_{(k)=(0)}^{(\infty)} \prod_{j=1}^n \frac{\Gamma(\beta_j + 1 + k_j)}{\Gamma(\beta_j+1)\Gamma(k_j+1)} a_k z^k, \ k = (k_1, ..., k_n), \quad z \in U^n,$$

*where* $\Gamma(\cdot)$ *is the Gamma function and* $\sum_{(k)=(0)}^{(\infty)} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty}$.

If $\beta = (1, \ldots, 1)$, then we put $D^\beta f(z) \equiv Df(z)$, and hence

$$Df(z_1, \ldots, z_n) = \frac{\partial^n (f(z_1, \ldots, z_n) z_1 \cdots z_n)}{\partial z_1 \ldots \partial z_n}.$$

If $n = 1$ then $Df$ is the usual derivative of function $zf(z)$.

Next, we define the weighted $L_p(\omega)$ spaces of holomorphic functions.

**Definition 1.2.** *Let* $0 < p < +\infty$ *and* $\beta_{\omega_j} < -1$ $(1 \leq j \leq n)$. *We denote by* $L_p(\omega)$ *the set of all measurable functions on* $U^n$, *for which*

$$\|f\|_{L_p(\omega)}^p := \int_{U^n} |f(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^2} dm_{2n}(z) < +\infty.$$

Note that $L_p(\omega)$ is the $L_p$-space with respect to measure $\omega(1-|z|)(1-|z|^2)^{-2} dm_{2n}(z)$. Using the conditions imposed on $\omega$ ($\omega_j \in S$), we conclude that this measure is bounded. Now we define the weighted holomorphic Besov spaces on the polydisk.

**Definition 1.3.** *Let* $0 < p < +\infty$ *and* $f \in H(U^n)$. *A function* $f$ *is said to be in Besov space* $B_p(\omega)$ *if*

$$\|f\|_{B_p(\omega)}^p := \int_{U^n} |Df(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} dm_{2n}(z) < +\infty.$$

From the definition of differential $Df$ it follows that $\|\cdot\|_{B_p(\omega)}$ is indeed a norm. Notice that it is not necessary to add $|f(0)|$). This follows from the fact that here $Df = 0$ implies $f = 0$ for holomorphic $f$. As in the one-dimensional case, $B_p(\omega)$ is a Banach space with respect to the norm $\|\cdot\|_{B_p(\omega)}$. For properties of weighted holomorphic Besov spaces we refer to [8].

Toeplitz operators on various spaces have been studied in a number of papers (see [5, 6, 18, 11], and references therein). Notice that some problems concerning Toeplitz operators can be solved by means of Hankel operators and vice versa. In the classical Hardy theory of holomorphic functions on the unit disk there is only one type of Hankel operator. In the $B_p(\omega)$ theory we have two: the little and big Hankel operators. The analogue of Hankel operators of the Hardy theory here are the little Hankel operators, which were studied by many authors (see [13, 2, 8]).

Now we define the little Hankel operators. Denote by $\overline{B}_p(\omega)$ the space of conjugate holomorphic functions on $B_p(\omega)$. For an integrable function $f$ on $U^n$, we define the generalized little Hankel operator with symbol $g \in L^\infty(U^n)$ by

$$h_g^\alpha(f)(z) = \overline{P}_\alpha(fg)(z) = \int_{U^n} \frac{(1-|\zeta|^2)^\alpha}{(1-\zeta\overline{z})^{\alpha+2}} f(\zeta) g(\zeta) dm_{2n}(\zeta),$$

$\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > -1$, $1 \leq j \leq n$.

Observe that in the special case $n = 1$, $\alpha = 0$ we have the classical little Hankel operator (see [20]). In Section 2 we study the boundedness of little Hankel operator on $B_p(\omega)$. For the cases $0 < p < 1$ and $p = 1$ we have the following results.

**Theorem 1.1.** *Let $0 < p < 1$, $f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$) and $g \in L^\infty(U^n)$. Then $h_g^\alpha(f) \in \overline{B}_p(\omega)$ if and only if $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \le j \le n$.*

**Theorem 1.2.** *Let $f \in B_1(\omega)$ and $g \in L^\infty(U^n)$. Then $h_g^\alpha(f) \in \overline{B}_1(\omega)$ if and only if $\alpha_j > \alpha_{\omega_j} - 2$, $1 \le j \le n$.*

The result in the case $p > 1$ is different from those for the cases $0 < p < 1$ and $p = 1$. Specifically, in this case we have the following assertion.

**Theorem 1.3.** *Let $1 < p < +\infty$, $f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$) and $g \in L^\infty(U^n)$. If $\alpha_j > \alpha_{\omega_j}$, $1 \le j \le n$, then $h_g^\alpha(f) \in \overline{B}_p(\omega)$.*

The Berezin transform, which is an analogue of Poisson transform in the spaces $A^p(\alpha)$ (respectively, in $B_p(\omega)$), plays an important role especially in the study of Hankel and Toeplitz operators. In particular, some properties of these operators (for example, compactness, boundedness, etc.) can be proved by means of the Berezin transform (see [17, 12, 20]). On the other hand, the Berezin-type operators are of independent interest.

In Section 3, it will be shown that some properties of Berezin-type operators of the one-dimensional classical case remain valid in more general situations.

For an integrable function $f$ on $U^n$ and for $g \in L^\infty(U^n)$ we define the Berezin-type operator as follows:

$$B_g^\alpha f(z) = \frac{(\alpha+1)}{\pi^n}(1-|z|^2)^{\alpha+2}\int_{U^n}\frac{(1-|\zeta|^2)^\alpha}{|1-z\overline{\zeta}|^{4+2\alpha}}f(\zeta)g(\zeta)dm_{2n}(\zeta).$$

In the special case $\alpha = 0$, $g \equiv 1$, the operator $B_g^\alpha$ will be called Berezin transform.

We have the following statements.

**Theorem 1.4.** *Let $0 < p < 1$, $f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$) and $g \in L^\infty(U^n)$, and let $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \le j \le n$. Then $B_g^\alpha(f) \in L^p(\omega)$.*

**Theorem 1.5.** *Let $1 < p < +\infty$, $f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$) and $g \in L^\infty(U^n)$, and let $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \le j \le n$. Then $B_g^\alpha(f) \in L_p(\omega)$.*

**Theorem 1.6.** *Let $f \in B_1(\omega)$ (or $f \in \overline{B}_1(\omega)$) and $g \in L^\infty(U^n)$. Then $B_g^\alpha(f) \in L_1(\omega)$ if and only if $\alpha_j > \alpha_{\omega_j}$, $1 \le j \le n$.*

Note that, in general, the operators $h_g^\alpha$ and $B_g^\alpha$ are not bounded.

To prove the main results, we need more notation and some auxiliary results. Observe first that the partition of the polydisk into dyadic quadrangles plays an important role (see [4, 16]). Define

$$\Delta_{k_j,l_j} = \{z_j \in U : 1 - \frac{1}{2^{k_j}} \le |z_j| < 1 - \frac{1}{2^{k_j+1}},\ \frac{\pi l_j}{2^{k_j}} \le \arg z_j < \frac{\pi(l_j+1)}{2^{k_j}}\},$$
$$\Delta^*_{k_j,l_j} = 4/3\Delta_{k_j,l_j},$$

where $k = (k_1, \dots, k_n)$ $(k_j \ge 0)$, $l_j$ are some integers such that $-2^{k_j} \le l_j \le 2^{k_j+1} - 1$ $(1 \le j \le n)$ and $2^k = (2^{k_1}, \dots, 2^{k_n})$. Then $\Delta_{k,l} = \Delta_{k_1,l_1} \times \dots \times \Delta_{k_n,l_n}$ and $\Delta^*_{k,l}$ can be defined similarly. The system $\{\Delta_{k,l}\}$ is called the system of dyadic quadrangles.

**Proposition 1.1.** *Let $\zeta_{k_j,l_j}$ be the center of $\Delta_{k_j,l_j}$, $1 \le j \le n$. Then*

$$1 - |\zeta_{k_j,l_j}| \asymp 1 - |\zeta_j| \quad \zeta_j \in \Delta_{k_j,l_j} \quad \text{and} \quad (1 - |\zeta_{k_j,l_j}|)^2 \asymp |\Delta_{k_j,l_j}| \quad 1 \le j \le n.$$

Note that the partition of the polydisk into dyadic quadrangles is important for obtaining some integral estimates particularly in the case $0 < p \le 1$ (see [16]). Besides, the system $\{\Delta_{kl}\}$, as well as the system $\{\Delta^*_{kl}\}$, are coverings of $U^n$, and one can observe that the interiors of $\Delta_{kl}$ for distinct indices are disjoint, which is no longer true for $\Delta^*_{kl}$. On the other hand, $\{\Delta^*_{k,l}\}$ is a finite covering in the sense that any quadrangle $\{\Delta^*_{kl}\}$ has nonempty intersection only with a finite number of quadrangles from $\{\Delta_{kl}\}$, and this number is independent of $k$ and $l$. Also, note that such partitions for the spaces $A^p_\alpha$ were used for the first time by F. A. Shamoyan [16] in the study of weighted classes of functions in the polydisk and unit ball in $\mathbb{C}^n$. The following two lemmas will be used in the proofs of main results of the paper.

**Lemma 1.1.** *Let $m = (m_1, \dots, m_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, $\beta_j \ge 0$, $1 \le j \le n$. If $f \in B_p(\omega)$, then*

$$|f(z)| \le C \int_{U^n} \frac{(1 - |\zeta|^2)^m}{|1 - \overline{\zeta} z|^{m+1}} |Df(\zeta)| dm_{2n}(\zeta), \tag{1.1}$$

*where $m_j \ge \alpha_{\omega_j} - 1$ $(1 \le j \le n)$.*

The proof follows from [8, Lemma 2.5].

**Lemma 1.2.** *Let $n = 1$. Assume that $a + 1 - \beta_\omega > 0$, $b > 1$ and $b - a - 2 > \alpha_\omega$. Then*

$$\int_U \frac{(1 - |\zeta|^2)^a \omega(1 - |\zeta|^2)}{|1 - z\overline{\zeta}|^b} dm_2(\zeta) \le \frac{\omega(1 - |z|^2)}{(1 - |z|^2)^{b-a-2}}. \tag{1.2}$$

The proof can be found in [6, Lemma 2].

## 2. Little Hankel operators on $B_p(\omega)$

In this section we study the little Hankel operators $h_g^\alpha$ on $B_p(\omega)$ $(0 < p < +\infty)$. We denote the restriction of $\|\cdot\|_{L^p(\omega)}$ to $\overline{B}_p(\omega)$ by $\|\cdot\|_{\overline{B}_p(\omega)}$, and first consider the case $0 < p < 1$.

**Theorem 2.1.** *Let $0 < p < 1$, $f \in B_p(\omega)$ (or $f \in \overline{B}^p(\omega)$) and $g \in L^\infty(U^n)$. Then $h_g^\alpha(f) \in \overline{B}_p(\omega)$ if and only if $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \le j \le n$.*

*Proof.* Let $0 < p < 1$, $f \in B^p(\omega)$ (or $f \in \overline{B}^p(\omega)$), $g \in L^\infty(\omega)$ and $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \le j \le n$. We show that $h_g^\alpha(f) \in \overline{B}^p(\omega)$. Denote $I := \|h_g^\alpha f\|_{\overline{B}^p(\omega)}$. Using the partition of the polydisk, Lemma 3 of [16] and Proposition 1.1, we can write

$$I = \int_{U^n} \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} \left( \int_{U^n} \frac{(1-|\zeta|^2)^\alpha}{|1-\overline{z}\zeta|^{\alpha+3}} |f(\zeta)||g(\zeta)| dm_{2n}(\zeta) \right)^p dm_{2n}(z) \le$$
$$C(g) \int_{U^n} \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} \sum_{k,l} \left( \int_{\Delta_{k,l}} \frac{(1-|\zeta|)^\alpha}{|1-\overline{z}\zeta|^{\alpha+2}} |f(\zeta)| dm_{2n}(\zeta) \right)^p dm_{2n}(z) \le$$
$$C(g) \int_{U^n} \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} \sum_{k,l} \max_{\zeta\in\overline{\Delta}_{k,l}} |f(\zeta)|^p |\Delta_{k,l}|^p \frac{(1-|\zeta_{k,l}|)^{\alpha p}}{|1-\overline{z}\zeta_{k,l}|^{(\alpha+3)p}} dm_{2n}(z) =$$
$$C(g) \sum_{k,l} \max_{\zeta\in\overline{\Delta}_{k,l}} |f(\zeta)|^p |\Delta_{k,l}|^p (1-|\zeta_{k,l}|)^{\alpha p} \int_{U^n} \frac{\omega(1-|z|)(1-|z|^2)^{p-2}}{|1-\overline{z}\zeta_{k,l}|^{(\alpha+3)p}} dm_{2n}(z),$$

where $\zeta_{k,l}$ is the center of $\Delta_{k,l}$ and $C(g) := C(\alpha, p, \omega)\|g\|_\infty$.

Recalling that the system $\{\Delta_{k,l}^*\}$ forms a finite covering of $U^n$, by (1.2) and Lemma 4 of [16], we obtain

$$I \le C(g) \sum_{k,l} \max_{\zeta\in\overline{\Delta}_{k,l}} |f(\zeta)|^p (1-|\zeta_{k,l}|)^{-2+2} \omega(1-|\zeta_{k,l}) \le$$
$$C(g) \sum_k \sum_l \int_{\Delta_{k,l}^*} |f(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^2}) dm_{2n}(\zeta) \le$$
$$C(g) \int_{U^n} |f(z)|^p \frac{\omega(1-|z|)}{(1-|z|^2)^2}) dm_{2n}(\zeta).$$

Next, using (1.1) and Lemma 1.2, we get

$$I \le C(g)\int_{U^n}\frac{\omega(1-|z|)}{(1-|z|^2)^2}\left(\int_{U^n}\frac{(1-|t|^2)^m}{|1-\bar{t}\zeta|^{m+1}}|Df(t)|dm_{2n}(t)\right)^p dm_{2n}(\zeta)\le$$

$$C(g)\int_{U^n}\frac{\omega(1-|z|)}{(1-|z|^2)^2}\sum_{k,l}\left(\int_{\Delta_{k,l}}\frac{(1-|t|^2)^m}{|1-\bar{t}\zeta|^{m+1}}|Df(t)|dm_{2n}(t)\right)^p dm_{2n}(\zeta)\le$$

$$C(g)\int_{U^n}\frac{\omega(1-|z|)}{(1-|z|^2)^2}\sum_{k,l}\max_{t\in\overline{\Delta}_{k,l}}|Df(t)|^p|\Delta_{k,l}|^p\frac{(1-|t_{k,l}|^2)^{mp}}{|1-\bar{t}\zeta|^{(m+1)p}}dm_{2n}(\zeta)\le$$

$$C(g)\sum_{k,l}\max_{t\in\overline{\Delta}_{k,l}}|Df(t)|^p|\Delta_{k,l}|^p\frac{\omega(1-|t_{k,l}|)(1-|t_{k,l}|^2)^{mp}}{(1-|t_{k,l}2)^{(m+1)p-2+2}}=$$

$$C(g)\sum_{k,l}\max_{t\in\overline{\Delta}_{k,l}}|Df(t)|^p(1-|t_{k,l}|^2)^{p-2+2}.$$

An application of Lemma 4 of [16] yields

$$I\le C(g))\sum_k\sum_l\int_{\Delta^*_{k,l}}\frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}}|Df(z)|dm_{2n}(z)\le$$

$$\int_{U^n}\frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}}|Df(z)|dm_{2n}(z)=C(\alpha,p,\omega)\|g\|_\infty\|f\|^p_{B_p(\omega)},$$

showing that $h^\alpha_g(f)\in\overline{B}_p(\omega)$.

Conversely, let $h^\alpha_g(f)\in\overline{B}_p(\omega)$ for all $g\in L^\infty(U^n)$. For $r=(r_1,...,r_n)$, $r_j\in(0,1)$, and $k=(k_1,...,k_n)$ we take the function

$$f_r(z)=C_r(1-rz)^{-k},\ k_j>(\alpha_{\omega_j}+2)/p,\ 1\le j\le n, \tag{2.1}$$

where $C_r=(1-r)^k\omega^{-1/p}(1-r)$, and observe that $\|f_r\|_{B_p(\omega)}\sim const.$

Consider the following domains

$$\widetilde{U}_j=\{z_j\in U, |\arg z_j|<(1-r_j)/2;\ (4r_j-1)/3<|z_j|<(1+2r_j)/3\}$$

and $\widetilde{U}^n=\widetilde{U}_1\times\ldots\times\widetilde{U}_n$. Taking the function $g_r(\zeta)$ as $g_r(\zeta)=\exp^{-\arg f_r(\zeta)}$ and a polydisk $V^n$ centered at $(r_1,...,r_n)$ with radius of $(1-r_1)...(1-r_n)$ such that $\overline{V}^n\subset\widetilde{U}^n$ (here $\overline{V}^n$ is the closure of $V^n$), we get

$$\|h^\alpha_{g_r}f_r\|_{\overline{B}_p(\omega)}\ge\int_{U^n}\frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}}\left(\int_{V^n}\frac{(1-|\zeta|)^\alpha}{|1-\bar{z}\zeta|^{\alpha+3}}|f_r(\zeta)|dm_{2n}(\zeta)\right)^p dm_{2n}(z).$$

Let

$$\max_{\zeta\in\overline{V}^n}|1-\bar{z}\zeta|=|1-\bar{z}\widetilde{\zeta}|,$$

then we have

$$\|h^\alpha_{g_r}f_r\|_{\overline{B}_p(\omega)}\ge C_1(\alpha,p,\omega)\frac{(1-r)^{\alpha p}}{\omega(1-r)}\int_{U^n}\frac{\omega(1-|z|)}{(1-|z|)^{2-p}}\left(\int_{V^n}\frac{dm_{2n}(\zeta)}{|1-\bar{z}\zeta|^{\alpha+3}}\right)^p dm_{2n}(z)$$

$$\ge C_1(\alpha,p,\omega)\frac{(1-r)^{(\alpha+2)p}}{\omega(1-r)}\int_{U^n}\frac{\omega(1-|z|)dm_{2n}(z)}{|1-\bar{z}\widetilde{\zeta}|^{(\alpha+3)p}(1-|z|)^{2-p}}.$$

Assuming the opposite that $(\alpha_j+2)p\le\alpha_{\omega_j}$ for some $j$, for the corresponding integral with $\omega_j(t)=t^{\alpha_{\omega_j}}$, we get

$$\int_{U^n}\frac{\omega(1-|z|)dm_{2n}(z)}{|1-\overline{z}\widetilde{\zeta}|^{(\alpha+3)p}(1-|z|)^{2-p}}\sim const,\quad \text{if}\quad (\alpha_j+2)p<\alpha_{\omega_j}$$

and

$$\int_{U^n}\frac{\omega(1-|z|)dm_{2n}(z)}{|1-\overline{z}\widetilde{\zeta}|^{(\alpha+3)p}(1-|z|)^{2-p}})\sim\log\frac{1}{1-|\widetilde{\zeta_j}|},\quad \text{if}\quad (\alpha_j+2)p=\alpha_{\omega_j}+2.$$

Consequently,

$$\frac{(1-r_j)^{(\alpha_j+2)p}}{\omega_j(1-r_j)}\to\infty,\quad \frac{(1-r_j)^{(\alpha_j+2)p}}{\omega_j(1-r_j)}\log\frac{1}{1-r_j}\to\infty\quad\text{as}\quad r_j\to 1-0,$$

and the result follows. □

**Corollary 2.1.** *Let $0<p<1$, $\alpha_j>\alpha_{\omega_j}/p-2$, $1\le j\le n$ and $g\in L^\infty(U^n)$. Then the operator $h_g^\alpha$ is bounded on $B_p(\omega)$, (and on $\overline{B}_p(\omega)$). Moreover, we have $\|h_g^\alpha(f)\|\le C\|f\|\cdot\|g\|$.*

In the case $p=1$ we have the following result.

**Theorem 2.2.** *Let $f\in B_1(\omega)$ and $g\in L^\infty(U^n)$. Then $h_g^\alpha(f)\in\overline{B}_1(\omega)$ if and only if $\alpha_j>\alpha_{\omega_j}-2$, $1\le j\le n$.*

*Proof.* Let $f\in B_1(\omega)$, $g\in L^\infty(U^n)$ and $\widetilde{C}:=C(\alpha,\omega)\|g\|_\infty$. Then by (1.1) and (1.2) we have

$$\|h_g^\alpha(f)\|_{\overline{B}_1(\omega)}\le\|g\|_\infty\int_{U^n}(1-|\zeta|^2)^\alpha|f(\zeta)|\int_{U^n}\frac{\omega(1-|z|)dm_{2n}(z)}{|1-\zeta\overline{z}|^{\alpha+3}(1-|z|)}dm_{2n}(\zeta)\le$$
$$\widetilde{C}\int_{U^n}|f(\zeta)|\frac{\omega(1-|\zeta|^2)}{(1-|\zeta|)^2}dm_{2n}(\zeta)\le\widetilde{C}\int_{U^n}\frac{\omega(1-|\zeta|)}{(1-|\zeta|)^2}\int_{U^n}\frac{(1-|t|^2)^m}{|1-\overline{t}\zeta|^{m+1}}|Df(t)|\times$$
$$dm_{2n}(t)dm_{2n}(\zeta)=\widetilde{C}\int_{U^n}(1-|t|^2)^m|Df(t)|\int_{U^n}\frac{\omega(1-|\zeta|)dm_{2n}(\zeta)dm_{2n}(t)}{|1-\overline{t}\zeta|^{m+1}(1-|\zeta|^2)^2}.$$

Using (1.1) again, we get

$$\|h_g^\alpha(f)\|_{\overline{B}_1(\omega)}\le\widetilde{C}\int_{U^n}\frac{\omega(1-|t|)}{(1-|t|)}|Df(t)|dm_{2n}(t)=\widetilde{C}\|f\|_{\overline{B}_1(\omega)},$$

showing $h_g^\alpha f\in\overline{B}_1(\omega)$.

The proof of necessity of the condition $\alpha_j>\alpha_{\omega_j}$, $1\le j\le n$ is similar to that of in Theorem 2.1, and so, we omit the details. □

**Corollary 2.2.** *Let $\alpha_j>\alpha_{\omega_j}$, $1\le j\le n$ and $g\in L^\infty(U^n)$. Then the operator $h_g^\alpha$ is bounded on $B_1(\omega)$ and $\|h_g^\alpha(f)\|\le C\|f\|\cdot\|g\|$.*

For the case $1<p<+\infty$ we have the following result.

**Theorem 2.3.** *Let $1 < p < +\infty$, $f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$) and $g \in L^\infty(U^n)$. If $\alpha_j > \alpha_{\omega_j}$, $1 \le j \le n$, then $h_g^\alpha(f) \in \overline{B}_p(\omega)$.*

*Proof.* By Hölder inequality and (1.2), we get

$$|Dh_g^\alpha(f)(z)| \le \int_{U^n} \frac{(1-|\xi|^2)^\alpha}{|1-\xi\bar{z}|^{\alpha+3}} |f(\xi)| \cdot |g(\xi)|\, dm_{2n}(\xi) \le$$
$$\|g\|_\infty \int_{U^n} \frac{(1-|xi|^2)^\alpha |f(\xi)|}{|1-\xi\bar{z}|^{\alpha+3}}\, dm_{2n}(\xi) \le \|g\|_\infty \times$$
$$\left(\int_{U^n} \frac{(1-|\xi|^2)^\alpha |f(\xi)|^p}{|1-\xi\bar{z}|^{\alpha+3}}\, dm_{2n}(\xi)\right)^{1/p} \cdot \left(\int_{U^n} \frac{(1-|\xi|^2)^\alpha\, dm_{2n}(\xi)}{|1-\xi\bar{z}|^{\alpha+3}}\right)^{1/q} \le$$
$$\frac{C(\alpha,q)\|g\|_\infty}{(1-|z|)^{1/q}} \left(\int_{U^n} \frac{(1-|\xi|^2)^\alpha |f(\xi)|^p}{|1-\xi\bar{z}|^{\alpha+3}}\, dm_{2n}(\xi)\right)^{1/p}.$$

Setting $C = C(\alpha,q)\|g\|_\infty$ and using (1.1), we can write

$$\|h_g^{(\alpha)}(f)\|_{B_p(\omega)} = \int_{U_n} \frac{\omega(1-|z|)}{(1-|z|^2)^{2-p}} |Dh_g^\alpha(f)(z)|^p\, dm_{2n}(z) \le$$
$$C\int_{U^n} \frac{\omega(1-|z|)}{|1-|z|^2)^{2-p+p/q}} \int_{U^n} \frac{(1-|\xi|^2)^\alpha |f(\xi)|^p}{|1-\xi\bar{z}|^{\alpha+3}}\, dm_{2n}(\xi) dm_{2n}(z) \le$$
$$C\int_{U^n} |f(\xi)|^p (1-|\xi|^2)^\alpha \int_{U^n} \frac{\omega(1-|z|) dm_{2n}(z) dm_{2n}(\xi)}{|1-\xi\bar{z}|^{\alpha+3}(|1-|z|^2)^{2-p+p/q}} \le$$
$$C_1 \int_{U^n} (1-|\xi|^2)^\alpha |f(\xi)|^p \frac{\omega(1-|\xi|)(1-|\xi|^2)^{p-2-p/q}}{(1-|\xi|^2)^{\alpha+1}}\, dm_{2n}(\xi) =$$
$$\int_{U^n} (1-|xi|^2)^{p-2-p/q-1} \omega(1-|\xi|) |f(\xi)|^p\, dm_{2n}(\xi).$$

On the other hand, by (1.1) we get

$$|f(\xi)|^p \le \left(\int_{U^n} \frac{(1-|t|^2)^m}{|1-\bar{t}\xi|^{m+1}} |Df(t)|\, dm_{2n}(t)\right)^p \le$$
$$\left(\int_{U^n} \frac{(1-|t|^2)^{m-\delta}(1-|t|^2)^\delta}{|1-\bar{t}\xi|^{m+1}} |Df(t)|\, dm_{2n}(t)\right)^p \le$$
$$\int_{U^n} \frac{(1-|t|^2)^{m-\delta}}{|1-\bar{t}\xi|^{m+1}} (1-|t|^2)^{\delta p} |Df(t)|^p\, dm_{2n}(t) \cdot \frac{C(m,\delta,q)}{(1-|\xi|^2)^{(\delta-1)p/q}},$$

for some $\delta > 1$. Therefore, we have

$$\|h_g^{\alpha)}(f)\|B_p(\omega) \le C_1 \int U^n (1-|t|^2)^{m-\delta+\delta p} |Df(t)|^p \int_{U^n} \frac{(1-|\xi|^2)^{p-3-\delta p/q}}{|1-\bar{t}\xi|^{m+1}} \times$$
$$\omega(1-|\xi|) dm_{2n}(\xi) \int_{U^n} (1-|t|^2)^{m-\delta+\delta p} |Df(t)|^p \le$$
$$C_2 \int_{U^n} \frac{\omega(1-|\xi|)(1-|\xi|)^{(1-\delta)p/q-2}}{|1-\bar{t}\xi|^{m+1}} dm_{2n}(\xi) d_{2n}(t) \le$$
$$\int_{U^n} (1-|t|^2)^{m-\delta+\delta p} \frac{|Df(t)|^p \omega(1-|t|) dm_{2n}(t)}{(1-|t|^2)^{m-1+2-(1-\delta)p/q}} =$$
$$\int_{U^n} (1-|t|^2)^{p-2} |Df(t)|^p \omega(1-|t|)\, dm_{2n}(t) = \|f\|_{B_p}(\omega).$$

Thus, $\|h_g^\alpha(f)\|_{B_p(\omega)} \leq C_3\|f\|_{B_p(\omega)}\|g\|_\infty$, where $C_3 = C_2 \cdot C^p(m,\delta,q)$, and the result follows. □

**Corollary 2.3.** *Let $\alpha_j > \alpha_{\omega_j}$, $1 \leq j \leq n$ and $g \in L^\infty(U^n)$. Then the operator $h_g^\alpha$ is bounded on $B_p(\omega)$ and $\|h_g^\alpha(f)\|_{B_p(\omega)} \leq C_3\|f\|_{B_p(\omega)} \cdot \|g\|_\infty$.*

## 3. Berezin-type operators on $B_p(\omega)$

In this section we establish the boundedness of Berezin-type operators $B_g^\alpha$ on weighted Besov spaces $B_p(\omega)$. We first consider the case $0 < p < 1$.

**Theorem 3.1.** *Let $0 < p < 1$, $f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$) and $g \in L^\infty(U^n)$, and let $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \leq j \leq n$. Then $B_g^\alpha(f) \in L^p(\omega)$.*

*Proof.* Let $f \in B_p(\omega)$ or $f \in \overline{B}_p(\omega)$. We show that $B_\alpha f \in L_p(\omega)$. To this end, we estimate the corresponding integral:

$$I := \int_{U^n} \frac{\omega(1-|z|)}{(1-|z|^2)^2}\left((1-|z|^2)^{\alpha+2}\int_{U^n}\frac{(1-|\zeta|^2)^\alpha|f(\zeta)||g(\zeta)|}{|1-z\overline{\zeta}|^{4+2\alpha}}dm_{2n}(\zeta)\right)^p dm_{2n}(z).$$

Using the partition of the polydisk, we can write

$$\begin{aligned} I &\leq \|g\|_\infty \int_{U^n}(1-|z|^2)^{(\alpha+2)p-2}\omega(1-|z|) \\ &\times \sum_{k,l}\left(\int_{\Delta_{k,l}}\frac{(1-|\zeta|)^\alpha}{|1-\overline{\zeta}z|^{4+2\alpha}}|f(\zeta)|dm_{2n}(\zeta)\right)^p dm_{2n}(z) \leq C(\alpha,\omega,p)\|g\|_\infty \\ &\times \int_{U^n}(1-|z|^2)^{(\alpha+2)p-2}\omega(1-|z|)\sum_{k,l}\max_{\zeta\in\overline{\Delta}_{k,l}}|f(\zeta)|^p|\Delta_{k,l}|^p\frac{(1-|\zeta_{k,l}|)^{\alpha p}dm_{2n}(z)}{|1-\overline{\zeta}_{k,l}z|^{(4+2\alpha)p}} \\ &= C(\alpha,\omega,p)\|g\|_\infty\sum_{k,l}\max_{\zeta\in\overline{\Delta}_{k,l}}|f(\zeta)|^p|\Delta_{k,l}|^p \\ &\times \int_{U^n}(1-|z|^2)^{(\alpha+2)p-2}\omega(1-|z|)\frac{(1-|\zeta_{k,l}|)^{\alpha p}dm_{2n}(z)}{|1-\overline{\zeta}_{k,l}z|^{(4+2\alpha)p}}. \end{aligned}$$

Taking into account that $p(4+2\alpha_j) > (\alpha_j+2)p + \alpha_{\omega_j}$ $(1 \leq j \leq n)$, we can use (1.1) and Lemma 4 of [16], to obtain

$$\begin{aligned} I &\leq C(\alpha,p,\omega)\|g\|_\infty\sum_{k,l}\max_{\zeta\in\overline{\Delta}_{k,l}}|f(\zeta)|^p(1-|\zeta_{k,l}|)^{2-2}\omega(1-|\zeta_{k,l}) \\ &\leq C(\omega,\alpha,p)\|g\|_\infty\int_{U^n}|f(z)|^p\frac{\omega(1-|\zeta|)}{(1-|z|^2)}dm_{2n}(\zeta). \end{aligned}$$

Now we estimate the last integral. Using Lemma 1.1 we obtain

$$I \leq C(\omega,\alpha,p)\|g\|_\infty\int_{U^n}\frac{\omega(1-|\zeta|)}{(1-\zeta|^2)}\left(\int_{U^n}\frac{(1-|t|^2)^m}{|1-\overline{t}\zeta|^{m+1}}|Df(t)|dm_{2n}(t)\right)^p dm_{2n}(\zeta).$$

Then, in view of the following inequality

$$\left(\int_{U^n} \frac{(1-|t|^2)^m}{|1-\bar{t}\zeta|^{m+1}}|Df(t)|dm_{2n}(t)\right)^p$$
$$\leq \sum_{k,l}\left(\int_{\Delta_{k,l}} \frac{(1-|t|^2)^m}{|1-\bar{t}\zeta|^{m+1}}|Df(t)|dm_{2n}(t)\right)^p$$
$$\leq \sum_{k,l} \max_{t\in\overline{\Delta}_{k,l}} |Df(t)|^p|\Delta_{k,l}|^p \frac{(1-|t_{k,l}|^2)^{mp}}{|1-\overline{t_{k,l}}\zeta|^{(m+1)p}}$$

and Lemma 4 of [16], we conclude that

$$I \leq C(\omega,\alpha,p)\|g\|_\infty \sum_{k,l} \max_{t\in\overline{\Delta}_{k,l}} |Df(t)|^p|\Delta_{k,l}|^p(1-|t_{k,l}|^2)^{mp} \times$$
$$\int_{U^n} \frac{\omega(1-|z|)(1-|\zeta|)^{-2}}{|1-\overline{t_{k,l}}\zeta|^{(m+1)p}}dm_{2n}(z) \leq$$
$$C(\omega,\alpha,p)\|g\|_\infty \sum_{k,l} \max_{t\in\overline{\Delta}_{k,l}} |Df(t)|^p|\Delta_{k,l}|^p(1-|t_{k,l}|^2)^{mp} \frac{\omega(1-|t_{k,l}|)}{(1-|t_{k,l}|^2)^{(m+1)p}} =$$
$$C(\omega,\alpha,p)\|g\|_\infty \sum_{k,l} \max_{t\in\overline{\Delta}_{k,l}} |Df(t)|^p\omega(1-|t_{k,l}|)(1-|t_{k,l}|^2)^p \leq$$
$$\int_{U^n} |Df(t)|^p \frac{\omega(1-|t|)}{(1-|t|^2)^{2-p}}dm_{2n}(t).$$

Thus, we have

$$I \leq C(\omega,\alpha,p)\|g\|_\infty\|f\|_{B_p(\omega)},$$

and the result follows. □

**Remark 3.1.** The condition $\alpha_j + 2 > \alpha_{\omega_j}/p, (1 \leq j \leq n)$ in Theorem 3.1 is also necessary. Moreover, if the operator $B_g^\alpha$ is bounded on $L^p(\omega)$, then $\alpha_j + 2 > (\alpha_{\omega_j}+2)/p, (1 \leq j \leq n)$.

The proof is similar to the corresponding part of Theorem 2.1, and we omit it.

**Corollary 3.1.** *Let $0 < p < 1$, $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \leq j \leq n$ and $g \in L^\infty(U^n)$. Then the operator $B_g^\alpha$ is bounded on $B_p(\omega)$ (and on $\overline{B}_p(\omega)$).*

**Theorem 3.2.** *Let $1 < p < +\infty$, $f \in B_p(\omega)$ (or $f \in \overline{B}_p(\omega)$) and $g \in L^\infty(U^n)$, and let $\alpha_j > \alpha_{\omega_j}/p - 2$, $1 \leq j \leq n$. Then $B_g^\alpha(f) \in L_p(\omega)$.*

*Proof.* Let $f \in B_p(\omega)$ or $f \in \overline{B}_p(\omega)$. Our aim is to show that $B_\alpha f \in L^p(\omega)$. We have

$$|B_g^\alpha(f)(z)|^p \le (1-|z|^2)^{(\alpha+2)p} \frac{C(\alpha,\pi,p)}{(1-|z|^2)^{(\alpha+2)p/q}}$$
$$\le \int_{U^n} \frac{(1-|\xi|^2)^\alpha |f(\xi)|^p |g(\xi)|^p}{|1-z\bar{\xi}|^{2\alpha+4}} dm_{2n}(\xi) \le C(\alpha,\pi,p)(1-|z|^2)^{\alpha+2}$$
$$\times \int_{U^n} \frac{(1-|\xi|^2)^\alpha |f(\xi)|^p}{|1-\bar{\xi}z|^{2\alpha+4}} dm_{2n}(\xi) \le C(\alpha,\pi,p)(1-|z|^2)^{\alpha+2} \cdot \|g\|_\infty$$
$$\times \int_{U^n} \frac{(1-|\xi|^2)^\alpha}{|1-z\bar{\xi}|^{2\alpha+4}} \int_{U^n} \frac{(1-|t|^2)^{m-\delta}(1-|t|^2)^{\delta p}|Df(t)|^p}{|1-\bar{t}\xi|^{m+1}(1-|\xi|^2)^{(\delta-1)p/q}} dm_{2n}(t) dm_{2n}(\xi)$$
$$= C(\alpha,\pi,p)(1-|z|^2)^{\alpha+2}\|g\|_\infty \int_{U^n} (1-|t|^2)^{m-\delta+\delta p}|Df(t)|^p$$
$$\times \int_{U^n} \frac{(1-|\xi|^2)^{\alpha-(\delta-1)p/q}}{|1-\bar{\xi}z|^{2\alpha+4}|1-\bar{t}\xi|^{m+1}} dm_{2n}(\xi) dm_{2n}(t).$$

Therefore

$$\|B_g^\alpha(f)\|_{L_p(\omega)} = \int_{U^n} (1-|t|^2)^{m-\delta+\delta p}|Df(t)|^p \int_{U^n} \frac{(1-|\xi|^2)^{\alpha-(\delta-1)p/q}}{|1-\bar{t}\xi|^{m+1}}$$
$$\times \int_{U^n} \frac{\omega(1-|z|)(1-|z|^2)^\alpha}{|1-\bar{\xi}z|^{2\alpha+4}} dm_{2n}(z) dm_{2n}(\xi) dm_{2n}(t) \le \int_{U^n} (1-|t|^2)^{m-\delta+\delta p} \times$$
$$|Df(t)|^p \int_{U^n} \frac{(1-|xi|^2)^{\alpha-(\delta-1)p/q}\omega(1-|\xi|)}{|1-\bar{t}\xi|^{m+1}(1-|\xi|)^{\alpha+2}} dm_{2n}(\xi) dm_{2n}(t)$$
$$= \int U^n (1-|t|^2)^{m-\delta+\delta p}|Df(t)|^p \int_{U^n} \frac{\omega(1-|\xi|)}{|1-\bar{t}\xi|^{m+1}} (1-|xi|^2)^{-2-(\delta-1)p/q} dm_{2n}(\xi)$$
$$\int_{U^n} (1-|t|^2)^{m-\delta+\delta p} \frac{\omega(1-|t|)|Df(t)|^p}{(1-|t|^2)^{m-1+2(\delta-1)p/q}} dm_{2n}(t)$$
$$= \int_{U^n} \frac{\omega(1-|t|)|Df(t)|^p}{(1-|t|^2)^{2-p}} dm_{2n}(t) = \|f\|_{B_p(\omega)}\|g\|_\infty C(\alpha,\pi,p),$$

and the result follows. □

For the case $p = 1$ we have the following result.

**Theorem 3.3.** *Let $f \in B_1(\omega)$ (or $f \in \overline{B}_1(\omega)$) and $g \in L^\infty(U^n)$. Then $B_g^\alpha(f) \in L_1(\omega)$ if and only if $\alpha_j > \alpha_{\omega_j}$, $1 \le j \le n$.*

*Proof.* Let $f \in B_1(\omega)$ or $f \in \overline{B}_1(\omega)$. Our aim is to show that $B_g^\alpha f \in L_1(\omega)$. We have

$$|B_g^\alpha(f)| \le (1-|z|^2)^{\alpha+2} \int_{U^n}\int_{U^n} \frac{(1-|xi|^2)^\alpha |f(\xi)| \cdot |g(\xi)|\, dm_{2n}(\xi)}{|1-\bar{xi}|z|^{4+2\alpha}}$$
$$\le \|g\|_\infty (1-|z|^2)^{\alpha+2} \int_{U^n} \frac{(1-|\xi|^2)^\alpha}{|1-\bar{\xi}z|^{4+2\alpha}} \int_{U^n} \frac{(1-|t|^2)^m |Df(t)|}{|1-\bar{t}\xi|^{m+1}} dm_{2n}(t) dm_{2n}(\xi).$$

Then, using (1.2), we get

$$\|B_g^\alpha\|_{L_1(\omega)} \leq \int_{U^n}\int_{U^n}\frac{(1-|\xi|^2)^\alpha}{|1-\bar{\xi}z|^{2\alpha+4}}\int_{U^n}\frac{(1-|t|^2)^m|Df(t)|\omega(1-|z|)}{|1-t\xi|^{m+1}(1-|z|^2)^{2-\alpha-2}}\,dm_{2n}(z)dm_{2n}(t)dm_{2n}(\xi)$$

$$= \int_{U^n}(1-|t|^2)^m|Df(t)|\int_{U^n}\frac{(1-|\xi|^2)^\alpha}{|1-\bar{t}\xi|^{m+1}}\int_{U^n}\frac{\omega(1-|z|)\,dm_{2n}(z)dm_{2n}(\xi)dm_{2n}(t)}{|1-\bar{\xi}z|^{4+2\alpha}(1-|z|^2)^{-\alpha}}$$

$$\leq \int_{U^n}(1-|t|^2)^m|Df(t)|\frac{(1-|t|^2)^\alpha\omega(1-|t|)}{(1-|t|^2)^{m-1+2+\alpha}}\,dm_{2n}(t)$$

$$= \int_{U^n}\frac{|Df(t)|\omega(1-|t|)\,dm_{2n}(t)}{(1-|t|)} = \|f\|_{B_1(\omega)}\cdot\|g\|_\infty,$$

showing that $B_g^\alpha f \in L_1(\omega)$.

To prove the necessity of the condition $\alpha_j > \alpha_{\omega_j}$, $1 \leq j \leq n$, we proceed as in the proof of Theorem 2.1. We again use the described technique of selection of $f_r$ by (2.1) for $p = 1$ and $V^n$, and take $f_r(\zeta) \equiv |f_r(\zeta)|$, to obtain

$$\|B_\alpha(f_r)\|_{L^1(\omega)} \geq \int_{U^n}\omega(1-|z|)(1-|z|^2)^{\alpha+2}\int_{V^n}\frac{(1-|\zeta|)^\alpha}{|1-\bar{\zeta}z|^{2\alpha+4}}|f_r(\zeta)|dm_{2n}(\zeta)dm_{2n}(z)$$

$$\geq C_1(\alpha,\omega)\frac{(1-r)^\alpha}{\omega(1-r)(1-r)^2}\int_{U^n}\frac{\omega(1-|z|)(1-|z|^2)^{\alpha+2}}{|1-rz|^{2\alpha+4}}dm_{2n}(z).$$

As in the case of little Hankel operators, assuming the opposite that $\alpha_j \leq \alpha_{\omega_j}$ for some $j$, we get a contradiction. □

**Corollary 3.2.** *Let $\alpha_j > \alpha_{\omega_j}$, $1 \leq j \leq n$ and $g \in L^\infty(U^n)$. Then the operator $B_g^\alpha$ is bounded on $L^1(\omega)$.*

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