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ALMOST EVERYWHERE STRONG SUMMABILITY OF FEJÉR MEANS OF RECTANGULAR PARTIAL SUMS OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

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Abstract. In this paper we prove a BMO-estimate for rectangular partial sums of two-dimensional Walsh-Fourier series, and using this result we establish almost everywhere exponential summability of rectangular partial sums of double Walsh-Fourier series.

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1. Introduction

We denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} = [0, 1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \le p \le 2^n$.

Let $r_0(x)$ be the function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2) \\ -1, & \text{if } x \in [1/2, 1), \end{cases} \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by $r_n(x) = r_0(2^n x)$, $n \ge 1$. Let $w_0, w_1, \ldots$ denote the Walsh functions, that is, $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s$, then $w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x)$. The Walsh-Dirichlet kernel is defined by

$$D_0(x) = 0, \; D_n(x) = \sum_{k=0}^{n-1} w_k(x), \quad n \ge 1.$$

Given $x \in \mathbb{I}$, the expansion

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \tag{1.1}$$

where each $x_k = 0$ or 1, will be called a dyadic expansion of x. If $x \in \mathbb{I} \backslash \mathbb{Q}$, then (1.1) is uniquely determined. For the dyadic expansion $x \in \mathbb{Q}$ we choose the one

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for which $\lim_{k\to\infty} x_k = 0$. The dyadic addition of numbers $x, y \in \mathbb{I}$ in terms of their dyadic expansions

$$x \dotplus y := \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

Denote $I_N := [0, 2^{-N})$ and $I_N(x) := x \dotplus I_N$. We consider the double system $\{w_n(x) \times w_m(y) : n, m \in \mathbb{N}\}$ on the unit square $\mathbb{I}^2 = [0,1) \times [0,1)$. Throughout the paper the notation $a \lesssim b$ will stand for $a \leq c \cdot b$, where c is an absolute constant.

The norm (or pre-norm) of the space $L_p(\mathbb{I}^2)$ is defined by

$$\|f\|_p = \left(\int_{\mathbb{I}^2} |f|^p \right)^{1/p} \qquad (0 < p < +\infty).$$

If $f \in L_1(\mathbb{I}^2)$, then

$$\hat{f}(n, m) = \int_{\mathbb{I}^2} f(x_1, x_2) \, w_n(x_1) w_m(x_2) dx_1 dx_2$$

is the (n, m)-th Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}(x_1, x_2; f) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m, n) \, w_m(x_1) w_n(x_2).$$

Denote

$$S_n^{(1)}(x_1, x_2; f) := \sum_{l=0}^{n-1} \widehat{f}(l, x_2) \, w_l(x_1), \quad S_m^{(2)}(x_1, x_2; f) := \sum_{r=0}^{m-1} \widehat{f}(x_1, r) \, w_r(x_2),$$

where

$$\widehat{f}(l, x_2) = \int_{\mathbb{I}} f(x_1, x_2) \, w_l(x_1) \, dx_1, \quad \widehat{f}(x_1, r) = \int_{\mathbb{I}} f(x_1, x_2) \, w_r(x_2) \, dx_2.$$

Recall the definition of the space $BMO[\mathbb{I}^2]$. Let $f \in L_1(\mathbb{I}^2)$. We say that f has a bounded mean oscillation ($f \in BMO[\mathbb{I}^2]$) if

$$\|f\|_{BMO} := \sup_Q \left(\frac{1}{|Q|} \int_Q |f - f_Q|^2 \right)^{1/2} < \infty,$$

where $f_Q = \frac{1}{|Q|} \int_Q f$ and the supremum is taken over all dyadic squares $Q \subset \mathbb{I}^2$.

For an arbitrary sequence of numbers $\xi := \{\xi_{n_1 n_2} : n_1, n_2 = 0, 1, 2, \ldots\}$, and $\delta_k^n := \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ we define

$$BMO[\xi] := \sup_{0 \leq n_1, n_2 < \infty} \left\| \sum_{k_1=0}^{2^{n_1}-1} \sum_{k_2=0}^{2^{n_2}-1} \xi_{k_1 k_2} \mathbb{I}_{\delta_{k_1}^{n_1}}(t_1) \, \mathbb{I}_{\delta_{k_2}^{n_2}}(t_2) \right\|_{BMO},$$

where $\mathbb{I}_E$ is the characteristic function of a set $E \subset \mathbb{I}^2$.

We denote by $L(\log L)^{\alpha}(\mathbb{I}^2)$ the class of measurable functions f satisfying

$$\int\limits_{\mathbb{I}^2} |f|\left(\log^+|f|\right)^{\alpha} < \infty,$$

where $\log^+ u := \mathbb{I}_{(1,\infty)}\log u$. Denote by $S_n^T(x,f)$ the partial sums of the trigonometric Fourier series of f, and let

$$\sigma_n^T(x,f) = \frac{1}{n+1}\sum_{k=0}^{n} S_k^T(x,f)$$

be the $(C,1)$ means. Fejér [1] proved that $\sigma_n^T(f)$ converges to f uniformly for any 2π-periodic continuous function. In [15], Lebesgue established almost everywhere convergence of $(C,1)$ means for $f \in L_1(\mathbb{T}), \mathbb{T} := [-\pi,\pi)$. The strong summability problem, that is, convergence of the strong means

$$\frac{1}{n+1}\sum_{k=0}^{n}\left|S_k^T(x,f) - f(x)\right|^p, \quad x \in \mathbb{T}, \quad p > 0, \tag{1.2}$$

was first considered by Hardy and Littlewood in [13]. They showed that for any $f \in L_r(\mathbb{T})$ $(1 < r < \infty)$ the strong means tend to 0 a.e. as $n \to \infty$. The Fourier series of a function $f \in L_1(\mathbb{T})$ is said to be (H,p)-summable at $x \in T$, if the strong means (1.2) converge to 0 as $n \to \infty$. The (H,p)-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [16] for $p = 2$, and later by Zygmund [29] for general case $1 \le p < \infty$. In [18], Oskolkov obtained the following result: let $f \in L_1(\mathbb{T})$ and let Φ be a continuous positive convex function on $[0,+\infty)$ with $\Phi(0) = 0$ and $\ln\Phi(t) = O(t/\ln\ln t)$, $t \to \infty$, then for almost all x

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\Phi\left(\left|S_k^T(x,f) - f(x)\right|\right) = 0. \tag{1.3}$$

It was noted in [18] that Totik announced a conjecture that (1.3) holds almost everywhere for any $f \in L_1(\mathbb{T})$, provided that $\ln\Phi(t) = O(t)$, $t \to \infty$. In [19] Rodin proved the following statement.

Theorem R. *Let $f \in L_1(\mathbb{T})$. Then for any $A > 0$*

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left(\exp\left(A\left|S_k^T(x,f) - f(x)\right|\right) - 1\right) = 0$$

for a.e. $x \in \mathbb{T}$.

G. Karagulyan [15] proved that the following is true.

Theorem K. *Suppose that a continuous increasing function $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$, satisfies the condition*

$$\limsup_{t\to+\infty}\frac{\log\Phi(t)}{t} = \infty.$$

Then there exists a function $f \in L_1(\mathbb{T})$ for which

$$\limsup_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(\left|S_k^T(x,f)\right|\right) = \infty$$

for any $x \in \mathbb{T}$.

For Walsh system Rodin [22] proved that the following is true.

Theorem R2 (Rodin). *If $\Phi(t) : [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$, is an increasing continuous function satisfying*

$$\limsup_{t\to\infty} \frac{\log \Phi(t)}{t} < \infty, \tag{1.4}$$

then the partial sums of Walsh-Fourier series of any function $f \in L_1(\mathbb{I})$ satisfy the condition

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi\left(\left|S_k(x;f) - f(x)\right|\right) = 0$$

almost everywhere on $\mathbb{I}$.

In [3] it was established that, as in the trigonometric case [15], the condition (1.4) is sharp for a.e. Φ-summability of Walsh-Fourier series. More precisely, in [3] was proved the following statement.

Theorem GGK. *If an increasing function $\Phi(t) : [0,\infty) \to [0,\infty)$ satisfies the condition*

$$\limsup_{t\to\infty} \frac{\log \Phi(t)}{t} = \infty,$$

then there exists a function $f \in L_1(\mathbb{I})$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi\left(\left|S_k(x;f)\right|\right) = \infty \quad \textit{for any} \quad x \in \mathbb{I}.$$

The two-dimensional Fejér summability of $f \in L\log^+ L(\mathbb{I}^2)$ was proved by Zygmund [30] for trigonometric Fourier series and by Myricz et al. [18] (see also Weisz [26]) for Walsh-Fourier series. The two-dimensional strong summability, that is,

$$\frac{1}{2^{n_1+n_2}} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \left|S_{k_1k_2}(x_1,x_2;f) - f(x_1,x_2)\right|^p \to 0 \quad \text{a.e. as } n\to\infty$$

was established by Gogoladze [10] for trigonometric Fourier series and for $f \in L\log^+ L(\mathbb{T}^2)$. The same result for multi-dimensional Walsh-Fourier series is due to Rodin [21] (see also Weisz [27]). These results show that in the case of two-dimensional functions both the $(C;1,1)$ summability and the $(C;1,1)$ strong summability have the same maximal convergence space $L\log^+ L$.

In [9], a BMO-estimate for quadratic partial sums of two-dimensional trigonometric Fourier series was proved, from which almost everywhere exponential summability of quadratic partial sums of double Fourier series was derived.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems. The problem of summability of multiple Fourier series have been investigated by Gogoladze [11, 12], Wang [26], Zhag [29], Glukhov [4], Goginava [5, 6], Goginava and Gogoladze [7, 8] and Gat et al. [2]. In this paper we study the problem of BMO-estimation for rectangular partial sums of two-dimensional Walsh-Fourier series.

The main results of the present paper are the following two theorems.

Theorem 1.1. *If $f \in L(\log L)^2(\mathbb{I}^2)$, then*

$$\left|\left\{(x_1,x_2)\in\mathbb{I}^2 : BMO[S_{n_1n_2}(x_1,x_2;f)] > \lambda\right\}\right| \lesssim \frac{1}{\lambda}\left(1+\int\limits_{\mathbb{I}^2}|f|(\log|f|)^2\right).$$

The next theorem shows that the rectangular sums of two-dimensional Walsh-Fourier series of a function $f \in L(\log L)^2(\mathbb{I}^2)$ are almost everywhere exponentially summable to the function f.

Theorem 1.2. *Suppose that $f \in L(\log L)^2(\mathbb{I}^2)$. Then for any $A > 0$*

$$\lim_{m_1,m_2\to\infty}\frac{1}{2^{m_1+m_2}}\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\left(\exp\left(A\left|S_{n_1n_2}(x_1,x_2;f)-f(x_1,x_2)\right|\right)-1\right)=0$$

for a.e. $(x_1,x_2)\in\mathbb{I}^2$.

2. Auxiliary Results

In this section we recall some known results and prove two lemmas needed in the proofs of main results. Consider the following operator, introduced by Schipp [23]:

$$V_n(x;f) := \left(\sum_{l=0}^{2^n-1}\left(\int\limits_{l2^{-n}}^{(l+1)2^{-n}}\sum_{j=0}^{n-1}2^{j-1}\mathbb{I}_{I_j}(t)\,S_{2^n}(x\dotplus t\dotplus e_j,f)\,dt\right)^2\right)^{1/2},$$

and denote $V(f) := \sup\limits_n V_n(f)$. The following theorems were proved by Schipp [23].

Theorem Sch1. [23] *Let $f \in L_1(\mathbb{I})$. Then*

$$\mu\{|Vf|>\lambda\}\lesssim\frac{\|f\|_1}{\lambda}\qquad\|Vf\|\lesssim 1+\int\limits_{\mathbb{I}}|f|\log^+|f|\quad\left(f\in L\log^+L(\mathbb{I})\right).$$

Theorem Sch2. [23] *The following estimate holds:*

$$\left\{\frac{1}{2^n}\sum_{m=0}^{2^n-1}|S_m(x;f)|^2\right\}^{1/2}\lesssim V_n(x;|f|).$$

We set

$$V_{m_1m_2}(x_1,x_2;f)$$
$$:=\left(\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}\left(\int_{l_12^{-m_1}}^{(l_1+1)2^{-m_1}}\int_{l_22^{-m_2}}^{(l_2+1)2^{-m_2}}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1)\sum_{j_2=0}^{m_2-1}2^{j_2-1}\mathbb{I}_{I_{j_2}}(t_2)\right.\right.$$
$$\left.\left.\times S_{2^{m_1}2^{m_2}}(x_1\dotplus t_1\dotplus e_{j_1},x_2\dotplus t_2\dotplus e_{j_2};f)\,dt_1dt_2\right)^2\right)^{1/2}.$$

For a two-dimensional integrable function f we introduce the following functions:

$$:\ V_n^{(1)}(x_1,x_2;f)=\left(\sum_{l=0}^{2^n-1}\left(\int_{l2^{-n}}^{(l+1)2^{-n}}\sum_{j=0}^{n-1}2^{j-1}\mathbb{I}_{I_j}(t)\,S_{2^n}^{(1)}(x_1\dotplus t\dotplus e_j,x_2;f)^2\,dt\right)^2\right)^{1/2}$$

$$:\ V_n^{(2)}(x_1,x_2;f)=\left(\sum_{l=0}^{2^n-1}\left(\int_{l2^{-n}}^{(l+1)2^{-n}}\sum_{j=0}^{n-1}2^{j-1}\mathbb{I}_{I_j}(t)\,S_{2^n}^{(2)}(x_1,x_2\dotplus t\dotplus e_j;f)\,dt\right)^2\right)^{1/2},$$

and

$$V^{(s)}(x_1,x_2;f):=\sup_n\left|V_n^{(s)}(x_1,x_2;f)\right|,\quad s=1,2.$$

Lemma 2.1. *The following estimate holds:*

$$\left\{\frac{1}{2^{m_1+m_2}}\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}|f*(D_{n_1}\otimes D_{n_2})|^2\right\}^{1/2}\lesssim V_{m_1m_2}(x_1,x_2;|f|).$$

Proof. Let

$$\varepsilon_{ji}=\begin{cases}-1, & \text{if } j=0,1,...,i-1\\ 1, & \text{if } j=i.\end{cases}$$

In [23], Schipp proved that

$$\begin{aligned}D_m(t) &= \sum_{k=0}^{n-1}\mathbb{I}_{I_k\setminus I_{k+1}}(t)\sum_{j=0}^{k}\varepsilon_{kj}2^{j-1}w_m(t\dotplus e_j)\\ &\quad-\frac{1}{2}w_m(t)+(m+1/2)\mathbb{I}_{I_n}(t),\quad m<2^n.\end{aligned}$$

Then we can write

$$H_{m_1m_2}(x_1,x_2;f)\tag{2.1}$$
$$:=\left\{\frac{1}{2^{m_1+m_2}}\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}|S_{2^{m_1}2^{m_2}}(f)*(D_{n_1}\otimes D_{n_2})|^2\right\}^{1/2}$$

$$\leq \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}\left(x_1 \dotplus t_1, x_2 \dotplus t_2; f\right) \sum_{k_1=0}^{m_1-1} \mathbb{I}_{I_{k_1} \setminus I_{k_1+1}}\left(t_1\right) \right.\right.$$
$$\times \sum_{j_1=0}^{k_1} \varepsilon_{k_1 j_1} 2^{j_1-1} w_{n_1}\left(t_1 \dotplus e_{j_1}\right) \sum_{k_2=0}^{m_2-1} \mathbb{I}_{I_{k_2} \setminus I_{k_2+1}}\left(t_2\right)$$
$$\left.\left. \times \sum_{j_2=0}^{k_2} \varepsilon_{k_2 j_2} 2^{j_2-1} w_{n_2}\left(t_2 \dotplus e_{j_2}\right) dt_1 dt_2 \right|^2 \right\}^{1/2}$$
$$+ \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}\left(x_1 \dotplus t_1, x_2 \dotplus t_2; f\right) \sum_{k_1=0}^{m_1-1} \mathbb{I}_{I_{k_1} \setminus I_{k_1+1}}\left(t_1\right) \right.\right.$$
$$\left.\left. \times \sum_{j_1=0}^{k_1} \varepsilon_{k_1 j_1} 2^{j_1-1} w_{n_1}\left(t_1 \dotplus e_{j_1}\right) \frac{w_{n_2}\left(t_2\right)}{2} dt_1 dt_2 \right|^2 \right\}^{1/2}$$
$$+ \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}\left(x_1 \dotplus t_1, x_2 \dotplus t_2; f\right) \sum_{k_1=0}^{m_1-1} \mathbb{I}_{I_{k_1} \setminus I_{k_1+1}}\left(t_1\right) \right.\right.$$
$$\left.\left. \times \sum_{j_1=0}^{k_1} \varepsilon_{k_1 j_1} 2^{j_1-1} w_{n_1}\left(t_1 \dotplus e_{j_1}\right) \left(n_2 + 1/2\right) \mathbb{I}_{I_{m_2}}\left(t_2\right) dt_1 dt_2 \right|^2 \right\}^{1/2}$$
$$+ \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}\left(x_1 \dotplus t_1, x_2 \dotplus t_2; f\right) \sum_{k_2=0}^{m_2-1} \mathbb{I}_{I_{k_2} \setminus I_{k_2+1}}\left(t_2\right) \right.\right.$$
$$\left.\left. \times \sum_{j_2=0}^{k_2} \varepsilon_{k_2 j_2} 2^{j_2-1} w_{n_2}\left(t_2 \dotplus e_{j_2}\right) \frac{w_{n_1}\left(t_1\right)}{2} dt_1 dt_2 \right|^2 \right\}^{1/2}$$
$$+ \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}\left(x_1 \dotplus t_1, x_2 \dotplus t_2; f\right) \right.\right.$$
$$\left.\left. \times \frac{w_{n_1}\left(t_1\right)}{2} \frac{w_{n_2}\left(t_2\right)}{2} dt_1 dt_2 \right|^2 \right\}^{1/2}$$
$$+ \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}\left(x_1 \dotplus t_1, x_2 \dotplus t_2; f\right) \right.\right.$$
$$\left.\left. \times \frac{w_{n_1}\left(t_1\right)}{2} \left(n_2 + 1/2\right) \mathbb{I}_{I_{m_2}}\left(t_2\right) dt_1 dt_2 \right|^2 \right\}^{1/2}$$
$$+ \left\{ \frac{1}{2^{m_1+m_2}} \sum_{n_1=0}^{2^{m_1}-1} \sum_{n_2=0}^{2^{m_2}-1} \left| \int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}\left(x_1 \dotplus t_1, x_2 \dotplus t_2; f\right) \sum_{k_2=0}^{m_2-1} \mathbb{I}_{I_{k_2} \setminus I_{k_2+1}}\left(t_2\right) \right.\right.$$

$$\times \sum_{j_2=0}^{k_2} \varepsilon_{k_2 j_2} 2^{j_2-1} w_{n_2}(t_2 \dotplus e_{j_2})(n_1+1/2)\mathbb{I}_{I_{m_1}}(t_1)\,dt_1dt_2\Bigg|^2\Bigg\}^{1/2}$$

$$+\left\{\frac{1}{2^{m_1+m_2}}\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\Bigg|\int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}(x_1 \dotplus t_1, x_2 \dotplus t_2; f)\right.$$

$$\left.\times\frac{w_{n_2}(t_2)}{2}(n_1+1/2)\mathbb{I}_{I_{m_1}}(t_1)\,dt_1dt_2\Bigg|^2\right\}^{1/2}$$

$$+\left\{\frac{1}{2^{m_1+m_2}}\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\Bigg|\int_{\mathbb{I}^2} S_{2^{m_1}2^{m_2}}(x_1 \dotplus t_1, x_2 \dotplus t_2; f)\right.$$

$$\left.\times(n_1+1/2)\mathbb{I}_{I_{m_1}}(t_1)(n_2+1/2)\mathbb{I}_{I_{m_2}}(t_2)\,dt_1dt_2\Bigg|^2\right\}^{1/2} := \sum_{i=1}^{9} R_i.$$

There is a suitable vector

$$\left\{\beta^{(1)}_{n_1n_2}(x_1,x_2) : 0 \le n_1 < 2^{m_1}, 0 \le n_2 < 2^{m_2}\right\}$$

such that

$$\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\left|\beta^{(1)}_{n_1n_2}(x_1,x_2)\right|^2 = 1$$

and

$$2^{(m_1+m_2)/2}R_1 = \int_{\mathbb{I}^2}\sum_{j_1=0}^{m_1-1} 2^{j_1-1}\left(\sum_{k_1=j_1}^{m_1-1}\varepsilon_{k_1j_1}\mathbb{I}_{I_{k_1}\setminus I_{k_1+1}}(t_1 \dotplus e_{j_1})\right) \tag{2.2}$$

$$\times\sum_{j_2=0}^{m_2-1} 2^{j_1-1}\left(\sum_{k_2=j_2}^{m_2-1}\varepsilon_{k_2j_2}\mathbb{I}_{I_{k_2}\setminus I_{k_2+1}}(t_1 \dotplus e_{j_2})\right)$$

$$\times S_{2^{m_1}2^{m_2}}(x_1 \dotplus t_1 \dotplus e_{j_1}, x_2 \dotplus t_2 \dotplus e_{j_2}; f)$$

$$\times\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\beta^{(1)}_{n_1n_2}(x_1,x_2)\,w_{n_1}(t_1)\,w_{n_2}(t_2)\,dt_1dt_2$$

$$\le \int_{\mathbb{I}^2}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1)\sum_{j_2=0}^{m_2-1}2^{j_2-1}\mathbb{I}_{I_{j_2}}(t_2)$$

$$\times S_{2^{m_1}2^{m_2}}(x_1 + t_1 + e_{j_1}, x_2 + t_2 + e_{j_2}; |f|)$$

$$\times\left|\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\beta^{(1)}_{n_1n_2}(x_1,x_2)\,w_{n_1}(t_1)\,w_{n_2}(t_2)\right|dt_1dt_2.$$

Analogously, we can prove that

$$2^{(m_1+m_2)/2}R_2 \lesssim \int_{\mathbb{I}^2}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1) \tag{2.3}$$

$$\times S_{2^{m_1}2^{m_2}}(x_1 \dotplus t_1 \dotplus e_{j_1}, x_2 \dotplus t_2 \dotplus e_0; |f|)$$

$$\times\left|\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\beta^{(2)}_{n_1n_2}(x_1,x_2)\,w_{n_1}(t_1)\,w_{n_2}(e_0)\,w_{n_2}(t_2)\right|dt_1dt_2,$$

(2.4)

$$2^{(m_1+m_2)/2}R_4\lesssim\int_{\mathbb{I}^2}\sum_{j_2=0}^{m_2-1}2^{j_2-1}\mathbb{I}_{I_{j_2}}(t_2)\,S_{2^{m_1}2^{m_2}}(x_1+t_1+e_0,x_2+t_2+e_{j_2};|f|)$$

$$\times\left|\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\beta^{(4)}_{n_1n_2}(x_1,x_2)\,w_{n_1}(t_1)\,w_{n_1}(e_0)\,w_{n_2}(t_2)\right|dt_1dt_2.$$

(2.5) $$2^{(m_1+m_2)/2}R_5\lesssim\int_{\mathbb{I}^2}S_{2^{m_1}2^{m_2}}(x_1+t_1+e_0,x_2+t_2+e_0;|f|)$$

$$\times\left|\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\beta^{(5)}_{n_1n_2}(x_1,x_2)\,w_{n_1}(t_1)\,w_{n_1}(e_0)\,w_{n_2}(t_2)\,w_{n_2}(e_0)\right|dt_1dt_2,$$

where

$$\sum_{n_1=0}^{2^{m_1}-1}\left|\beta^{(s)}_{n_1}(x_1,x_2)\right|^2=1\ \ (x_1,x_2)\in\mathbb{I}^2,\ s=2,4,5.$$

Now, we estimate R_3. There is a suitable vector $\left\{\beta^{(3)}_{n_1}(x_1,x_2):0\le n_1<2^{m_1}\right\}$ such that

$$\sum_{n_1=0}^{2^{m_1}-1}\left|\beta^{(3)}_{n_1}(x_1,x_2)\right|^2=1,\ \ (x_1,x_2)\in\mathbb{I}^2$$

and

(2.6) $$2^{(m_1+m_2)/2}R_3$$

$$\le c2^{3m_2/2}\int_{I\times I_{m_2}}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1)\,S_{2^{m_1}2^{m_2}}(x_1+t_1+e_{j_1},x_2+t_2;|f|)$$

$$\times\left|\sum_{n_1=0}^{2^{m_1}-1}\beta^{(3)}_{n_1}(x_1,x_2)\,w_{n_1}(t_1)\right|dt_1dt_2.$$

Analogously, we can prove that

(2.7) $$2^{(m_1+m_2)/2}R_6\lesssim2^{(3/2)m_2}\int_{\mathbb{I}\times I_{m_2}}S_{2^{m_1}2^{m_2}}(x_1+t_1+e_0,x_2+t_2;|f|)$$

$$\times\left|\sum_{n_1=0}^{2^{m_1}-1}\beta^{(6)}_{n_1}(x_1,x_2)\,w_{n_1}(e_0)\,w_{n_1}(t_1)\right|dt_1dt_2,$$

(2.8) $$2^{(m_1+m_2)/2}R_7\lesssim2^{(3/2)m_1}\int_{I_{m_1}\times\mathbb{I}}\sum_{j_2=0}^{m_2-1}2^{j_2-1}\mathbb{I}_{I_{j_2}}(t_2)$$

$$\times S_{2^{m_1}2^{m_2}}(x_1+t_1,x_2+t_2+e_{j_2};|f|)$$

$$\times\left|\sum_{n_1=0}^{2^{m_1}-1}\beta_{n_1}^{(7)}(x_1,x_2)\,w_{n_2}(t_2)\right|dt_1dt_2,$$

$$(2.9)\qquad 2^{(m_1+m_2)/2}R_8\lesssim 2^{(3/2)m_1}\int\limits_{I_{m_1}\times\mathbb{I}} S_{2^{m_1}2^{m_2}}(x_1\dotplus t_1,x_2\dotplus t_2\dotplus e_0;|f|)$$

$$\times\left|\sum_{n_2=0}^{2^{m_2}-1}\beta_{n_2}^{(8)}(x_1,x_2)\,w_{n_2}(e_0)\,w_{n_2}(t_2)\right|dt_1dt_2,$$

$$(2.10)\qquad 2^{(m_1+m_2)/2}R_9\lesssim 2^{(3/2)(m_1+m_2)}\int\limits_{I_{m_1}\times I_{m_2}} S_{2^{m_1}2^{m_2}}(x_1\dotplus t_1,x_2\dotplus t_2;|f|),$$

where

$$\sum_{n_1=0}^{2^{m_1}-1}\left|\beta_{n_1}^{(s)}(x_1,x_2)\right|^2=1,\ \sum_{n_2=0}^{2^{m_2}-1}\left|\beta_{n_2}^{(8)}(x_1,x_2)\right|^2=1\ \ (x_1,x_2)\in\mathbb{I}^2.\ \ (x_1,x_2)\in\mathbb{I}^2,\ \ s=6,7$$

Next, we set

$$P_{m_1m_2}^{(1)}(x_1,x_2):=\sum_{n_1=0}^{2^{m_1}-1}\sum_{n_2=0}^{2^{m_2}-1}\beta_{n_1n_2}^{(1)}(x_1,x_2)\,w_{n_1}(t_1)\,w_{n_2}(t_2),$$

and use (2.2) to obtain

$$2^{(m_1+m_2)/2}R_1$$

$$=\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}\left|P_{m_1m_2}^{(1)}\left(\frac{l_1}{2^{m_1}},\frac{l_2}{2^{m_2}}\right)\right|\int\limits_{l_12^{-m_1}}^{(l_1+1)2^{-m_1}}\int\limits_{l_22^{-m_2}}^{(l_2+1)2^{-m_2}}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1)$$

$$\times\sum_{j_2=0}^{m_2-1}2^{j_2-1}\mathbb{I}_{I_{j_2}}(t_2)\,S_{2^{m_1}2^{m_2}}(x_1\dotplus t_1\dotplus e_{j_1},x_2\dotplus t_2\dotplus e_{j_2};|f|)\,dt_1dt_2$$

$$\le\left(\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}\left|P_{m_1m_2}^{(1)}\left(\frac{l_1}{2^{m_1}},\frac{l_2}{2^{m_2}}\right)\right|^2\right)^{1/2}$$

$$\times\left(\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}\left(\int\limits_{l_12^{-m_1}}^{(l_1+1)2^{-m_1}}\int\limits_{l_22^{-m_2}}^{(l_2+1)2^{-m_2}}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1)\sum_{j_2=0}^{m_2-1}2^{j_2-1}\mathbb{I}_{I_{j_2}}(t_2)\right.\right.$$

$$\left.\left.\times S_{2^{m_1}2^{m_2}}(x_1\dotplus t_1\dotplus e_{j_1},x_2\dotplus t_2\dotplus e_{j_2};|f|)\,dt_1dt_2\right)^2\right)^{1/2}.$$

Taking into account that

$$\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}\left|P_{m_1m_2}^{(1)}\left(\frac{l_1}{2^{m_1}},\frac{l_2}{2^{m_2}}\right)\right|^2$$

$$=2^{m_1+m_2}\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}\int\limits_{l_12^{-m_1}}^{(l_1+1)2^{-m_1}}\int\limits_{l_22^{-m_2}}^{(l_2+1)2^{-m_2}}\left|P_{m_1m_2}^{(1)}(t_1,t_2)\right|^2dt_1dt_2$$

$$2^{m_1+m_2}\int_{\mathbb{I}^2}\left|P^{(1)}_{m_1m_2}(t_1,t_2)\right|^2 dt_1dt_2\le 2^{m_1+m_2},$$

we obtain

(2.11) $$R_1\lesssim V_{m_1m_2}(x_1,x_2;|f|).$$

Analogously, using (2.3)-(2.10) we can prove that for $s=2,...,9$

(2.12) $$R_s\lesssim V_{m_1m_2}(x_1,x_2;|f|).$$

Combining (2.1), (2.11) and (2.12) we conclude the proof of lemma 2.1. □

Lemma 2.2. *The following estimate holds:*

$$V_{m_1m_2}(x_1,x_2;|f|)\lesssim V^{(1)}\left(x_1,x_2;V^{(2)}(|f|)\right).$$

Proof. There is a suitable vector$\{a_{l_1l_2}(x_1,x_2):0\le l_1<2^{m_1},0\le l_2<2^{m_2}\}$ such that

$$\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}|a_{l_1l_2}(x_1,x_2)|^2=1$$

and

$$V_{m_1m_2}(x_1,x_2;|f|)=\sum_{l_1=0}^{2^{m_1}-1}\int_{l_12^{-m_1}}^{(l_1+1)2^{-m_1}}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1)$$
$$\times\sum_{l_2=0}^{2^{m_2}-1}a_{l_1l_2}(x_1,x_2)\left(\int_{l_22^{-m_2}}^{(l_2+1)2^{-m_2}}\sum_{j_2=0}^{m_2-1}2^{j_2-1}\mathbb{I}_{I_{j_2}}(t_2)\right.$$
$$\left.\times S_{2^{m_1}2^{m_2}}(x_1+t_1+e_{j_1},x_2+t_2+e_{j_2};|f|)\,dt_2\right)dt_1$$
$$\le\sum_{l_1=0}^{2^{m_1}-1}\int_{l_12^{-m_1}}^{(l_1+1)2^{-m_1}}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1)\left(\sum_{l_2=0}^{2^{m_2}-1}|a_{l_1l_2}(x_1,x_2)|^2\right)^{1/2}\sum_{l_2=0}^{2^{m_2}-1}$$
$$\left(\int_{l_22^{-m_2}}^{(l_2+1)2^{-m_2}}\sum_{j_2=0}^{m_2-1}2^{j_2-1}\mathbb{I}_{I_{j_2}}(t_2)\right.$$
$$\left.\times S^{(2)}_{2^{m_2}}(x_1+t_1+e_{j_1},x_2+t_2+e_{j_2};|f|)\,dt_2\right)^2\Bigg)^{1/2}dt_1$$
$$\le\sum_{l_1=0}^{2^{m_1}-1}\left(\sum_{l_2=0}^{2^{m_2}-1}|a_{l_1l_2}(x_1,x_2)|^2\right)^{1/2}\int_{l_12^{-m_1}}^{(l_1+1)2^{-m_1}}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1)$$
$$\times V^{(2)}(x_1+t_1+e_{j_1},x_2;|f|)\,dt_1$$
$$\le\left(\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}|a_{l_1l_2}(x_1,x_2)|^2\right)^{1/2}\left(\sum_{l_1=0}^{2^{m_1}-1}\left(\int_{l_12^{-m_1}}^{(l_1+1)2^{-m_1}}\sum_{j_1=0}^{m_1-1}2^{j_1-1}\mathbb{I}_{I_{j_1}}(t_1)\right.\right.$$

$$\times V^{(2)}\left(x_1 \dotplus t_1 \dotplus e_{j_1}, x_2; |f|\right) dt_1\Big)^2\Bigg)^{1/2} \lesssim V^{(1)}\left(x_1, x_2; V^{(2)}\right).$$

Lemma 2.2 is proved. □

3. Proof of Main Results

Proof of Theorem 1.1. Set

$$f_{n_1,n_2}(x_1,x_2,t_1,t_2) := \sum_{k_1=0}^{2^{n_1}-1}\sum_{k_2=0}^{2^{n_2}-1} S_{k_1,k_2}(x_1,x_2;f)\, \mathbb{I}_{\delta_{k_1}^{n_1}}(t_1)\, \mathbb{I}_{\delta_{k_2}^{n_2}}(t_2),$$

$$J_1 := \left[j_1 2^{-m}, (j_1+1)2^{-m}\right), J_2 := \left[j_2 2^{-m}, (j_2+1)2^{-m}\right).$$

Then we can write ($n_1 \le n_2$)

$$\|f_{n_1,n_2}(x_1,x_2,\cdot,\cdot)\|_{BMO} \tag{3.1}$$

$$= \sup_m \sup_{0\le j_1,j_2<2^m} \left(\frac{1}{|J_1\times J_2|}\int_{J_1\times J_2} \left| f_{n_1,n_2}(x_1,x_2,t_1,t_2) - \frac{1}{|J_1\times J_2|}\int_{J_1\times J_2} f_{n_1,n_2}(x_1,x_2,u_1,u_2)\, du_1 du_2\right|^2 dt_1 dt_2\right)^{1/2}$$

$$\le \left(\sup_{m\le n_1}\sup_{0\le j_1,j_2<2^m} + \sup_{n_1<m\le n_2}\sup_{0\le j_1,j_2<2^m} + \sup_{m>n_2}\sup_{0\le j_1,j_2<2^m}\right)$$

$$\left(\frac{1}{|J_1\times J_2|}\int_{J_1\times J_2} \left| f_{n_1,n_2}(x_1,x_2,t_1,t_2) - \frac{1}{|J_1\times J_2|}\int_{J_1\times J_2} f_{n_1,n_2}(x_1,x_2,u_1,u_2)\, du_1 du_2\right|^2 dt_1 dt_2\right)^{1/2}$$

$$:= P_1(n_1,n_2) + P_2(n_1,n_2) + P_3(n_1,n_2).$$

Let $n_1 \le n_2 < m$. Since $f_{n_1,n_2}(x_1,x_2,t_1,t_2)$ is constant on $\left[\frac{j_1}{2^m}, \frac{j_1+1}{2^m}\right) \times \left[\frac{j_2}{2^m}, \frac{j_2+1}{2^m}\right)$, for fixed $(x_1,x_2) \in \mathbb{I}^2$ we conclude that

$$P_3(n_1,n_2) = 0. \tag{3.2}$$

Let $m \le n_1$. Then for P_1 we can write

$$P_1(n_1,n_2) = \sup_{m\le n_1}\sup_{0\le j_1,j_2<2^m}\left(2^{2m}\int_{J_1\times J_2}\left|\sum_{k_1=j_1 2^{n_1-m}}^{(j_1+1)2^{n_1-m}-1}\sum_{k_2=j_2 2^{n_2-m}}^{(j_2+1)2^{n_2-m}-1} S_{k_1,k_2}(x_1,x_2;f)\,\mathbb{I}_{\delta_{k_1}^{n_1}}(t_1)\,\mathbb{I}_{\delta_{k_2}^{n_2}}(t_2) - 2^{2m}\right.\right.$$

$$\times \int_{J_1\times J_2}\sum_{k_1=j_1 2^{n_1-m}}^{(j_1+1)2^{n_1-m}-1}\sum_{k_2=j_2 2^{n_2-m}}^{(j_2+1)2^{n_2-m}-1} S_{k_1,k_2}(x_1,x_2;f)$$

$$\times \mathbb{I}_{\delta_{k_1}^{n_1}}(u_1)\,\mathbb{I}_{\delta_{k_2}^{n_2}}(u_2)\,du_1du_2\Big|^2 dt_1dt_2\Bigg)^{1/2}$$

$$= \sup_{m\le n_1}\sup_{0\le j_1,j_2<2^m}\Bigg(2^{m-n_1}2^{m-n_2}\sum_{k_1=j_12^{n_1-m}}^{(j_1+1)2^{n_1-m}-1}\sum_{k_2=j_22^{n_2-m}}^{(j_2+1)2^{n_2-m}-1}$$

$$\Big|S_{k_1,k_2}(x_1,x_2;f)-2^{m-n_1}2^{m-n_2}$$

$$\times\sum_{k_1=j_12^{n_1-m}}^{(j_1+1)2^{n_1-m}-1}\sum_{k_2=j_22^{n_2-m}}^{(j_2+1)2^{n_2-m}-1}S_{k_1,k_2}(x_1,x_2;f)\Big|^2\Bigg)^{1/2}$$

$$= \sup_{m_1\le n_1,m_2\le n_2}\sup_{0\le j_1<2^{m_1},0\le j_2<2^{m_2}}\Bigg(2^{-m_1}2^{-m_2}\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}$$

$$\Big|S_{l_1+j_12^{m_1},l_2+j_22^{m_2}}(x_1,x_2;f)-2^{-m_1}2^{-m_2}$$

$$\times\sum_{q_1=0}^{2^{m_1}-1}\sum_{q_2=0}^{2^{m_2}-1}S_{q_1+j_12^{m_1},q_2+j_22^{m_2}}(x_1,x_2;f)\Big|^2\Bigg)^{1/2}.$$

Since

$$\begin{aligned}
&S_{l_1+j_12^{m_1},l_2+j_22^{m_2}}(x_1,x_2;f)\\
=\;&S_{j_12^{m_1},j_22^{m_2}}(x_1,x_2;f)+S_{l_1,j_22^{m_2}}(x_1,x_2;fw_{j_12^{m_1}})\,w_{j_12^{m_1}}(x_1)\\
&+S_{j_12^{m_1},l_2}(x_1,x_2;fw_{j_22^{m_2}})\,w_{j_22^{m_2}}(x_2)\\
&+S_{l_1,l_2}(x_1,x_2;fw_{j_12^{m_1}}\otimes w_{j_22^{m_2}})\,w_{j_12^{m_1}}(x_1)\,w_{j_22^{m_2}}(x_2),
\end{aligned}$$

we can write

$$(3.3)\qquad P_1(n_1,n_2)\le \sup_{m_1\le n_1,m_2\le n_2}\sup_{0\le j_1<2^{m_1},0\le j_2<2^{m_2}}\Bigg(2^{-m_1-m_2}\sum_{l_1=0}^{2^{m_1}-1}\sum_{l_2=0}^{2^{m_2}-1}$$

$$S_{l_1,l_2}(x_1,x_2;fw_{j_12^{m_1}}\otimes w_{j_22^{m_2}})-2^{-m_1-m_2}$$

$$-\times\sum_{q_1=0}^{2^{m_1}-1}\sum_{q_2=0}^{2^{m_2}-1}S_{q_1,q_2}(x_1,x_2;fw_{j_12^{m_1}}\otimes w_{j_22^{m_2}})\Big|^2\Bigg)^{1/2}$$

$$+\sup_{m_1\le n_1,m_2\le n_2}\sup_{0\le j_1<2^{m_1},0\le j_2<2^{m_2}}\Bigg(2^{-m_1}\sum_{l_1=0}^{2^{m_1}-1}\Big|S_{l_1,j_22^{m_2}}(x_1,x_2;fw_{j_12^{m_1}})$$

$$-2^{-m_1}\sum_{q_1=0}^{2^{m_1}-1}S_{q_1,j_22^{m_2}}(x_1,x_2;fw_{j_12^{m_1}})\Big|^2\Bigg)^{1/2}$$

$$+\sup_{m_1\le n_1,m_2\le n_2}\sup_{0\le j_1<2^{m_1},0\le j_2<2^{m_2}}\Bigg(2^{-m_2}\sum_{l_2=0}^{2^{m_2}-1}\Big|S_{j_12^{m_1},l_2}(x_1,x_2;fw_{j_22^{m_2}})$$

$$-2^{-m_1}\sum_{q_2=0}^{2^{m_2}-1}S_{j_12^{m_1},q_2}(x_1,x_2;fw_{j_22^{m_2}})\Big|^2\Bigg)^{1/2}$$

$$:= P_{11}(n_1,n_2) + P_{12}(n_1,n_2) + P_{13}(n_1,n_2).$$

Next, from Lemmas 2.1 and 2.2 we obtain

$$P_{11}(n_1,n_2) \lesssim \sup_{m_1\le n_1, m_2\le n_2} \sup_{0\le j_1<2^{m_1}, 0\le j_2<2^{m_2}} V_{m_1,m_2}\left(x_1,x_2; \left|f w_{j_1 2^{m_1}} \otimes w_{j_2 2^{m_2}}\right|\right)$$

$$\lesssim V^{(1)}\left(x_1,x_2; V^{(2)}(|f|)\right).$$

Consequently, by Theorem Sch1 we can write

$$\left|\left\{(x_1,x_2)\in \mathbb{I}^2 : \sup_{n_1,n_2} P_{11}(n_1,n_2) > \lambda\right\}\right| \tag{3.4}$$

$$\lesssim \frac{1}{\lambda}\int_{\mathbb{I}}\left(\int_{\mathbb{I}} V^{(2)}(x_1,x_2;|f|)\,dx_1\right)dx_2$$

$$\lesssim \frac{1}{\lambda}\int_{\mathbb{I}}\left(\int_{\mathbb{I}} |f(x_1,x_2)|\log^+|f(x_1,x_2)|\,dx_2 + 1\right)dx_1$$

$$\lesssim \frac{1}{\lambda}\left(\int_{\mathbb{I}^2} |f|\log^+|f| + 1\right).$$

Since

$$S_{l_1, j_2 2^m}(x_1,x_2; f w_{j_1 2^m}) = S^{(1)}_{l_1}\left(x_1,x_2; S_{j_2 2^m}(f)\, w_{j_1 2^m}\right),$$

we can apply Lemma 2 to obtain $P_{12}(n_1,n_2)$

$$\lesssim \sup_{m_1\le n_1, m_2\le n_2} \sup_{0\le j_1<2^{m_1}, 0\le j_2<2^{m_2}} \left(2^{-m_1}\sum_{l_1=0}^{2^{m_1}-1}\left|S^{(1)}_{l_1}\left(x_1,x_2; S_{j_2 2^{m_2}}(f)\, w_{j_1 2^{m_1}}\right)\right|^2\right)^{1/2}$$

$$\le \sup_{m_1\le n_1, m_2\le n_2} \sup_{0\le j_1<2^{m_1}, 0\le j_2<2^{m_2}} V^{(1)}\left(x_1,x_2; \left|S^{(2)}_{j_2 2^{m_2}}(f)\, w_{j_1 2^{m_1}}\right|\right)$$

$$\le V^{(1)}\left(x_1,x_2; S^{(2)}_*(f)\right),$$

where $S^{(2)}_*(f) := \sup_n \left|S^{(2)}_n(f)\right|$. If $f \in L\left(\log^+ L\right)^2(\mathbb{I}^2)$, then $f(x_1,\cdot) \in L\left(\log^+ L\right)^2(\mathbb{I})$ for a.e. $x_1 \in \mathbb{I}$, and from the well-known theorem (see [25]), we have $S^{(2)}_*(x_1,\cdot,f) \in L_1(\mathbb{I})$ for a.e. $x_1 \in \mathbb{I}$. Moreover, we have

$$\int_{\mathbb{I}} S^{(2)}_*(x_1,x_2;f)\,dx_2 \lesssim \int_{\mathbb{I}} |f(x_1,x_2)|\left(\log^+|f(x_1,x_2)|\right)^2 dx_2 + 1$$

for a. e. $x_1 \in \mathbb{I}$. Hence,

$$\left|\left\{(x_1,x_2)\in \mathbb{I}^2 : \sup_{n_1,n_2} P_{12}(n_1,n_2) > \lambda\right\}\right| \tag{3.5}$$

$$\lesssim \left|\left\{(x_1,x_2)\in \mathbb{I}^2 : V^{(1)}\left(x_1,x_2; S^{(2)}_*(f)\right) > \lambda\right\}\right|$$

$$\lesssim \frac{1}{\lambda}\int_{\mathbb{I}}\left(\int_{\mathbb{I}} S^{(2)}_*(x_1,x_2;f)\,dx_1\right)dx_2$$

$$\lesssim \frac{1}{\lambda} \int_{\mathbb{I}} \left(\int_{\mathbb{I}} |f(x_1,x_2)| \left(\log^+ |f(x_1,x_2)|\right)^2 dx_2 + 1 \right) dx_1$$

$$\lesssim \frac{1}{\lambda} \left(\int_{\mathbb{I}^2} |f| \left(\log^+ |f|\right)^2 + 1 \right).$$

Analogously, we can prove that

$$\left| \left\{ (x_1,x_2) \in \mathbb{I}^2 : \sup_{n_1,n_2} P_{13}(n_1,n_2) > \lambda \right\} \right| \lesssim \frac{1}{\lambda} \left(\int_{\mathbb{I}^2} |f| \left(\log^+ |f|\right)^2 + 1 \right). \tag{3.6}$$

Combining (3.3)- (3.6) we get

$$\left| \left\{ (x_1,x_2) \in \mathbb{I}^2 : \sup_{n_1,n_2} P_1(n_1,n_2) > \lambda \right\} \right| \lesssim \frac{1}{\lambda} \left(\int_{\mathbb{I}^2} |f| \left(\log^+ |f|\right)^2 + 1 \right). \tag{3.7}$$

Analogously, we can prove that

$$\left| \left\{ (x_1,x_2) \in \mathbb{I}^2 : \sup_{n_1,n_2} P_2(n_1,n_2) > \lambda \right\} \right| \lesssim \frac{1}{\lambda} \left(\int_{\mathbb{I}^2} |f| \left(\log^+ |f|\right)^2 + 1 \right). \tag{3.8}$$

Combining (3.1), (3.2), (3.7) and (3.8) we complete the proof of the theorem. □

Proof of Theorem 1.2. The result can easily be deduced from Theorem 1.1 and John-Nirenberg theorem (see [9]). So, we omit the details. □

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