Известия НАН Армении, Математика, том 53, н. 1, 2018, стр. 68 – 73.

ON AN EQUIVALENCY OF RARE DIFFERENTIATION BASES OF RECTANGLES

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Abstract. The paper considers differentiation properties of density bases formed of bounded open sets. We prove that two quasi-equivalent subbases of some density basis differentiate the same class of non-negative functions. Applications for bases formed of rectangles are discussed.

MSC2010 numbers: 42B08, 42B25.

Keywords: dyadic rectangles; differentiation basis; rare basis.

1. Introduction

Let $\mathcal{R}^n$ be the family of open rectangles $\prod_{i=1}^n (a_i, b_i)$ in $\mathbb{R}^n$ and $\mathcal{R}^n_{\text{dyadic}} \subset \mathcal{R}^n$ be the family of dyadic rectangles of the form

$$\prod_{k=1}^{n} \left(\frac{j_i - 1}{2^{m_i}}, \frac{j_i}{2^{m_i}}\right), \quad j_i, m_i \in \mathbf{Z}, \quad i = 1, 2, \ldots, n. \tag{1.1}$$

For a given set $E \subset \mathbb{R}^n$ we denote

$$\operatorname{diam}(E) = \sup_{x,y \in E} \|x - y\|.$$

Definition 1.1. *A family $\mathcal{M}$ of bounded open sets from $\mathbb{R}^n$ is said to be a differentiation basis (or simply basis), if for any point $x \in \mathbb{R}^n$ there exists a sequence of sets $E_k \in \mathcal{M}$ such that $x \in E_k$, $k = 1, 2, \ldots$ and $\operatorname{diam}(E_k) \to 0$ as $k \to \infty$.*

Let $\mathcal{M}$ be a differentiation basis and $L_{\text{loc}}(\mathbb{R}^n)$ be the space of locally integrable functions:

$$L_{\text{loc}}(\mathbb{R}^n) = \{f : f \in L(K) \text{ for any compact } K \subset \mathbb{R}^n\}.$$

For any function $f \in L_{\text{loc}}(\mathbb{R}^n)$ we define

$$\delta_{\mathcal{M}}(x, f) = \limsup_{\operatorname{diam}(E) \to 0,\, x \in E \in \mathcal{M}} \left| \frac{1}{|E|} \int_E f(t)dt - f(x) \right|.$$

The integral of a function $f \in L_{\text{loc}}(\mathbb{R}^n)$ is said to be differentiable at a point $x \in \mathbb{R}^n$ with respect to the basis $\mathcal{M}$, if $\delta_{\mathcal{M}}(x, f) = 0$. The integral of a function is said to be differentiable with respect to the basis $\mathcal{M}$ if it is differentiable at almost every point. Consider the following classes of functions

$$\mathcal{F}(\mathcal{M}) = \{f \in L_{\text{loc}}(\mathbb{R}^n) : \delta_{\mathcal{M}}(x, f) = 0 \text{ almost everywhere }\},$$

$$\mathcal{F}^+(\mathcal{M}) = \{f \in L_{\text{loc}}(\mathbb{R}^n) : f(x) \geq 0,\ \delta_{\mathcal{M}}(x, f) = 0 \text{ almost everywhere }\}.$$

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function. Denote by $\Phi(L)(\mathbb{R}^n)$ the class of measurable functions f defined on $\mathbb{R}^n$ such that $\Phi(|f|) \in L^1(\mathbb{R}^n)$. If Φ satisfies the Δ_2-condition $\Phi(2x) \leq k\Phi(x)$, then $\Phi(L)$ turns to be an Orlicz space with the norm

$$\|f\|_\Phi = \inf\left\{c > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f|}{c}\right) \leq 1\right\}.$$

The following classical theorems determine the optimal Orlicz space, which functions have a.e. differentiable integrals with respect to the entire family of rectangles $\mathcal{R}^n$ is the space

$$L(1 + \log^+ L)^{n-1}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n),$$

corresponding to the case $\Phi(t) = t(1 + \log^+ t)^{n-1}$ ([1]).

Theorem A. (see [2]).

$$L(1 + \log^+ L)^{n-1}(\mathbb{R}^n) \subset \mathcal{F}(\mathcal{R}^n).$$

Theorem B. (see [6]). *If the function* Φ *satisfies*

$$\Phi(t) = o(t \log^{n-1} t) \text{ as } t \to \infty,$$

then $\Phi(L)(\mathbb{R}^n) \not\subset \mathcal{F}(\mathcal{R}^n)$. *Moreover, there exists a positive function* $f \in \Phi(L)(\mathbb{R}^n)$ *such that* $\delta_{\mathcal{R}^n}(x, f) = \infty$ *everywhere.*

Such theorems are valid also for the basis $\mathcal{R}^n_{\text{dyadic}}$. The first one trivially follows from embedding $L(1 + \log^+ L)^{n-1}(\mathbb{R}^n) \subset \mathcal{F}(\mathcal{R}^n) \subset \mathcal{F}(\mathcal{R}^n_{\text{dyadic}})$. The second can be deduced from the following

Theorem C. (see [10] (also [11, 12])) $\mathcal{F}^+(\mathcal{R}^n_{\text{dyadic}}) = \mathcal{F}^+(\mathcal{R}^n)$.

Let $\Delta = \{\nu_k : k = 1, 2, \ldots\}$ be an increasing sequence of positive integers. This sequence generates the rare basis $\mathcal{R}^n_{\text{dyadic}}(\Delta)$ of dyadic rectangles of the form (1.1) with $m_i \in \Delta$, $i = 1, 2, \ldots, n$. This kind of bases first considered in the papers [8], [9], [7], [4]. A. Stokolos [8] proved that the analogous of Saks theorem holds for any basis $\mathcal{R}^n_{\text{dyadic}}(\Delta)$ with an arbitrary Δ sequence. That means $L(1 + \log^+ L)^{n-1}(\mathbb{R}^n)$ is again the largest Orlicz space containing in $\mathcal{F}(\mathcal{R}^n_{\text{dyadic}}(\Delta))$. The necessary and sufficient condition for the equivalency of rare dyadic basis $\mathcal{R}^2_{\text{dyadic}}(\Delta)$ and complete dyadic basis $\mathcal{R}^2_{\text{dyadic}}$ is established in [4]. G. Oniani and T. Zerekidze [5] characterised translation invariant as well as net type bases formed of rectangles that are equivalent to the basis of all rectangles in the class of all non-negative functions. G. A. Karagulyan [3] proved some theorems, establishing an equivalency of some convergence conditions for multiple martingale sequences, those in particular imply some results of the papers [8], [9], [7].

Definition 1.2. *A basis $\mathcal{M}$ is said to be density basis if $\mathcal{M}$ differentiates the integral of any characteristic function $\mathbf{I}_E$ of measurable set E:*

$$\delta_{\mathcal{M}}(x, \mathbf{I}_E) = 0 \text{ at almost every } x \in \mathbf{R}^n.$$

We will say that the basis $\mathcal{M}$ differentiates a class of functions $\mathcal{F}$, if basis $\mathcal{M}$ differentiates the integrals of all functions of $\mathcal{F}$.

Theorem D. (see [1], III, Theorem 1.4) *If $\mathcal{M}$ is a density basis, then it differentiates L^∞.*

Note that any subbasis $\mathcal{M}'$ of a density basis $\mathcal{M}$ is also density basis, since in this case $\delta_{\mathcal{M}'}(x, f) \le \delta_{\mathcal{M}}(x, f)$ for any $x \in \mathbf{R}^n$ and $f \in L_{loc}(\mathbf{R}^n)$.

Definition 1.3. *Let $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{M}$ be subbases. We will say that basis $\mathcal{M}_2$ is quasi-coverable by basis $\mathcal{M}_1$ (with respect to basis $\mathcal{M}$) if for any $R \in \mathcal{M}_2$ there exist $R_k \in \mathcal{M}_1$, $k = 1, 2, \dots, p$ and $R' \in \mathcal{M}$ such that*

$$R \subseteq \bar{R} \subseteq R', \quad \bar{R} = \bigcup_{k=1}^{p} R_k \tag{1.2}$$

$$\operatorname{diam}(R') \le c \cdot \operatorname{diam}(R), \quad |R'| \le c|R_k|, \quad k = 1, 2, \dots, p, \tag{1.3}$$

$$\sum_{k=1}^{p} |R_k| \le c|\bar{R}|, \quad |\bar{R}| \le c|R|, \tag{1.4}$$

where constant $c \ge 1$ depends only on the bases $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}$. We will say two bases are quasi-equivalent if they are quasi-coverable with respect to each other.

In this paper we prove that quasi-equivalent subbases $\mathcal{M}_1, \mathcal{M}_2$ of density basis $\mathcal{M}$ differentiate the same class of non-negative functions, namely $\mathcal{F}^+(\mathcal{M}_1) = \mathcal{F}^+(\mathcal{M}_2)$.

2. Main theorem

Theorem 2.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be subbases of density basis $\mathcal{M}$ in $\mathbf{R}^n$. If the bases $\mathcal{M}_1$ and $\mathcal{M}_2$ are quasi-equivalent with respect to $\mathcal{M}$ then $\mathcal{F}^+(\mathcal{M}_1) = \mathcal{F}^+(\mathcal{M}_2)$.*

Proof of Theorem. First, let us suppose that $\mathcal{F}^+(\mathcal{M}_1) \setminus \mathcal{F}^+(\mathcal{M}_2) \neq \varnothing$. That means there exists a non-negative function $f \in L_{loc}(\mathbf{R}^n)$ such that

$$\delta_{\mathcal{M}_1}(x, f) = 0, \quad \text{a.e.}, \tag{2.1}$$

$$\delta_{\mathcal{M}_2}(x, f) > 0, \quad x \in E_1, \tag{2.2}$$

where $|E_1| > 0$. From (2.2) it follows that there exist such positive numbers α and γ that the set

$$E_2 = \{x \in \mathbf{R}^n : \delta_{\mathcal{M}_2}(x, f) > \alpha, 0 \le f(x) \le \gamma\}$$

has positive measure. Set $f = f_\gamma + f^\gamma$, where

$$f_\gamma(x) = \begin{cases} f(x), & \text{if } 0 \le f(x) \le \gamma, \\ 0, & \text{if } f(x) > \gamma. \end{cases}$$

Since $\mathcal{M}_2 \subseteq \mathcal{M}$ is density basis, then it differentiates L^∞ and therefore differentiates $f_\gamma \in L^\infty$, namely we have $\delta_{\mathcal{M}_2}(x, f_\gamma) = 0$ almost everywhere. Denote by E_3 the subset of E_2 where $\delta_{\mathcal{M}_2}(x, f_\gamma) = 0$. Clearly $|E_3| = |E_2| > 0$. From this we can deduce that if $x \in E_3 \subset E_2$ then $\delta_{\mathcal{M}_2}(x, f) = \delta_{\mathcal{M}_2}(x, f^\gamma)$ and $f^\gamma(x) = 0$, since $0 \le f(x) \le \gamma$. Furthermore, using (2.1), we get set $E_4 \subset E_3$ of positive measure such that for any $x \in E_4$

$$\delta_{\mathcal{M}_2}(x, f^\gamma) > \alpha, \quad f^\gamma(x) = 0, \quad \delta_{\mathcal{M}_1}(x, f^\gamma) = 0, \tag{2.3}$$

According to (2.3) for any $x \in E_4$ one can choose a number $\delta(x) > 0$ such that the conditions $x \in R \in \mathcal{M}_1$, $\operatorname{diam}(R) < \delta(x)$, imply

$$\frac{1}{|R|}\int_R f^\gamma(u)\,du < \eta, \tag{2.4}$$

where $\eta > 0$ will be conveniently chosen later. For some $\delta > 0$ the set $G = \{x \in E_4 : \delta(x) \ge \delta\}$ has positive measure. Thus, we have

$$\delta_{\mathcal{M}_2}(x, f^\gamma) > \alpha,\ f^\gamma(x) = 0, \quad \text{if } x \in G, \tag{2.5}$$

$$\frac{1}{|R|}\int_R f^\gamma(u)\,du < \eta, \text{ if } R \cap G \ne \varnothing, R \in \mathcal{M}_1, \operatorname{diam}(R) < \delta. \tag{2.6}$$

Since $\mathcal{M}$ differentiates $\mathbf{I}_G$, hence we may fix $x_0 \in G$ with

$$\lim_{\operatorname{diam}(R)\to 0,\ x_0 \in R \in \mathcal{M}} \frac{|R \cap G|}{|R|} = 1,$$

which means that for any $\varepsilon > 0$ there exists $\sigma(\varepsilon)$ such that $\operatorname{diam}(R) < \sigma(\varepsilon)$ and $x_0 \in R \in \mathcal{M}$ imply $|R \cap G| > (1 - \varepsilon)|R|$. Using this relation and (2.5), we can fix R such that

$$x_0 \in R \in \mathcal{M}_2, \quad \operatorname{diam}(R) < \frac{1}{c}\min\left(\sigma\left(\frac{1}{c}\right), \delta\right), \tag{2.7}$$

$$\frac{1}{|R|}\int_R f^\gamma(u)\,du > \alpha, \tag{2.8}$$

As we have that basis $\mathcal{M}_2$ is quasi-coverable with $\mathcal{M}_1$, then for $R \in \mathcal{M}_2$ we can fix $R' \in \mathcal{M}$ and $R_k \in \mathcal{M}_1$, $k = 1, 2, \ldots, p$ such that (1.2),(1.3) and (1.4) hold. From this and (2.7) we get

$$x_0 \in R' \in \mathcal{M}, \quad \operatorname{diam}(R') < \sigma\left(\frac{1}{c}\right),$$

which implies

$$|R' \cap G| > \left(1 - \frac{1}{c}\right)|R'|. \tag{2.9}$$

which together with (1.3) gives that $R_k \cap G \neq \varnothing$, $k = 1, 2, \ldots, p$. Now, since each R_k contains some point from G, we can use (2.6) and come to contradiction against (2.8). Namely, combining (2.6), (2.7) and (1.3) we have $R_k \cap G \neq \varnothing$, $R_k \in \mathcal{M}_1$, $\operatorname{diam}(R_k) < \delta$ and therefore

$$\frac{1}{|R_k|}\int_{R_k} f^\gamma(u)\,du < \eta, \quad k = 1, 2, \ldots, m$$

which together with (1.4) implies

$$\int_{\tilde{R}} f^\gamma(u)\,du \le \int_{\tilde{R}} \sum_{k=1}^{p} \mathrm{I}_{R_k}(u) f^\gamma(u)\,du = \sum_{k=1}^{p} \int_{R_k} f^\gamma(u)\,du < \eta \sum_k |R_k| \le \eta c |\tilde{R}|$$

and

$$\frac{1}{|\tilde{R}|}\int_{\tilde{R}} f^\gamma(u)\,du < \eta c.$$

On the other hand, from non-negativity of function f^γ and from (2.8),(1.4) follows that

$$\frac{1}{|\tilde{R}|}\int_{\tilde{R}} f^\gamma(u)\,du \ge \frac{|R|}{|\tilde{R}|} \cdot \frac{1}{|R|}\int_{R} f^\gamma(u)\,du > \frac{\alpha}{c},$$

which is impossible if choose $\eta < \frac{\alpha}{c^2}$. Thus we have proved that $\mathcal{F}^+(\mathcal{M}_1) \subset \mathcal{F}^+(\mathcal{M}_2)$. In the same way we can prove the inverse inclusion $\mathcal{F}^+(\mathcal{M}_2) \subset \mathcal{F}^+(\mathcal{M}_1)$. □

3. Applications

It is well known that the basis of all open rectangles $\mathcal{R}^n$ differentiates $L^\infty(\mathbf{R}^n)$, i.e. it is a density basis. Therefore we can apply the theorem when $\mathcal{M} = \mathcal{R}^n$ and get criteria for two bases formed of rectangles differentiating the same class of non-negative functions:

Corollary 3.1. *If bases $\mathcal{R}_1$ and $\mathcal{R}_2$ formed of rectangles in $\mathbf{R}^n$ are quasi-equivalent, then $\mathcal{F}^+(\mathcal{R}_1) = \mathcal{F}^+(\mathcal{R}_2)$.*

Let $\Omega = \{\omega_k^i\}_{i,k=1}^{n,\infty}$ be a finite family of sequences with

(3.1) $$\omega_k^i \to 0 \text{ as } k \to \infty \text{ for } i = 1, 2, \ldots, n.$$

Define the basis $\mathcal{R}_\Omega$ as a family of rectangles of the form

$$\prod_{i=1}^{n} \left((m_i - 1)\omega_{k_i}^i, m_i \omega_{k_i}^i\right), \quad m_i \in \mathbf{Z},\ k_i \in \mathbf{N},\ i = 1, 2, \ldots, n.$$

and the basis $\tilde{\mathcal{R}}_\Omega$ as a family of rectangles with side lengths $l_i, i = 1, 2, \ldots, n$ satisfying $c_1 \cdot \omega_{k_i}^i \le l_i \le c_2 \cdot \omega_{k_i}^i$, $k_i \in \mathbf{N}$, $i = 1, 2, \ldots, n$. Then, it can be shown that the bases $\mathcal{R}_\Omega$ and $\tilde{\mathcal{R}}_\Omega$ are quasi-equivalent, therefore

Corollary 3.2. *For any Ω with (3.1) $\mathcal{F}^+(\tilde{\mathcal{R}}_\Omega) = \mathcal{F}^+(\mathcal{R}_\Omega)$.*

Corollary 3.3. *If the family of sequences Ω satisfies*

$$\max_{1\le i\le n}\sup_{k\in\mathbb{N}}\frac{\omega_k^i}{\omega_{k+1}^i}<\infty, \tag{3.2}$$

then

$$\mathcal{F}^+(\mathcal{R}_\Omega)=\mathcal{F}^+(\mathcal{R}^n). \tag{3.3}$$

Proof. Denote by γ the finite quantity of the left hand side of (3.2). Then for coefficients $c_1=1$ and $c_2=\gamma+1$ we have $\mathcal{F}^+(\bar{\mathcal{R}}_\Omega)=\mathcal{F}^+(\mathcal{R}^n)$. Hence from the theorem we deduce (3.3). □

Finally, if we take $\omega_k^i=2^{-\nu_k}$, $k\in\mathbb{N}$, $i=1,2,\ldots,n$, where $\Delta=\{\nu_k: k\ge 1\}$ is an increasing sequence of positive integers, the basis $\mathcal{R}_\Omega$ becomes the basis of all dyadic rectangles $\mathcal{R}^n_{\text{dyadic}}(\Delta)$ corresponding to the sequence Δ.

Corollary 3.4. *If the sequence $\Delta=\{\nu_k: k\ge 1\}$ satisfies*

$$\sup_{k\in\mathbb{N}}(\nu_{k+1}-\nu_k)<\infty$$

then $\mathcal{F}^+(\mathcal{R}^n_{\text{dyadic}}(\Delta))=\mathcal{F}^+(\mathcal{R}^n)$. Particularly, if we take $\Delta=\mathbb{N}$, we get Theorem C.

Acknowledgement. Thanks to the referee for useful remarks.

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Поступила 13 марта 2017