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ON A COMPOSITION PRESERVING INEQUALITIES BETWEEN POLYNOMIALS

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Abstract. The Schur-Szegő composition of two polynomials $f(z) = \sum_{j=0}^{n} A_j z^j$ and $g(z) = \sum_{j=0}^{n} B_j z^j$, both of degree n, is defined by $f * g(z) = \sum_{j=0}^{n} \binom{n}{j}^{-1} A_j B_j z^j$. In this paper, we estimate the minimum and the maximum of the modulus of $f * g(z)$ on $|z| = 1$ and thereby obtain results analogues to Bernstein type inequalities for polynomials.

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1. Introduction and statements of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative, then concerning the estimate of $|P'(z)|$ on the unit disk $|z| \le 1$, we have

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The inequality (1.1) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (see, e.g., [7, 8]), and by applying the maximum modulus principle (see [6, 7]) to the polynomial $Q(z) = z^n \overline{P(1/\bar{z})}$ one concludes that

$$\max_{|z|=1} |P(Rz)| \le R^n \max_{|z|=1} |P(z)|. \tag{1.2}$$

Both the inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then the inequalities (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\max_{|z|=1} |P(Rz)| \le \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|. \tag{1.4}$$

The inequality (1.3) was conjectured by Erdös and later it was verified by Lax [4], whereas the inequality (1.4) was proved by Ankeny and Rivlin [1].

Given two polynomials $f(z) = \sum_{j=0}^{n} A_j z^j$ and $g(z) = \sum_{j=0}^{n} B_j z^j$, both of degree n, their **Schur-Szegö composition** is defined by

$$f * g(z) = \sum_{j=0}^{n} \frac{A_j B_j}{\binom{n}{j}} z^j.$$

For any nth degree polynomial $P(z)$, one can easily see that if $q(z) = \sum_{j=0}^{n} \binom{n}{j} j\, z^j$, $g(z) = \sum_{j=0}^{n} \binom{n}{j} z^j$ and $f(z) = Rz$, then

$$P * q(z) = zP'(z) \quad \text{and} \quad (P \circ f) * g(z) = P(Rz).$$

In view of these observations, it is natural to look for results analogous to the above inequalities for the Schur-Szegö composition of polynomials. Our first result in this direction is the following theorem.

Theorem 1.1. *Let $P(z)$ be a polynomial of degree n and let $h(z) = \sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all its zeros in the disk $|z| \le 1$. Then for every $R > 1$,*

$$\max_{|z|=1} |(P \circ f) * h(z)| \le R^n |l_n| \max_{|z|=1} |P(z)| \tag{1.5}$$

where $f(z) = Rz$. The result is sharp, as is shown by the extremal polynomial $P(z) = az^n$ $a \ne 0$.

A variety of interesting results can be easily deduced from Theorem 1.1 as special cases. Here we mention few of them.

The following Corollary is obtained by letting $R \to 1$ in inequality (1.5).

Corollary 1.1. *Let $P(z)$ be a polynomial of degree n and let $h(z) = \sum_{j=0}^{n} l_j z^j$ be polynomial of degree n having all its zeros in the disk $|z| \le 1$, then*

$$\max_{|z|=1} |P * h(z)| \le |l_n| \max_{|z|=1} |P(z)|. \tag{1.6}$$

The result is best possible and equality in (1.6) holds for $P(z) = az^n$, $a \ne 0$.

Remark 1.1. For $h(z) = \sum_{j=0}^{n} \binom{n}{j} j z^j$, Corollary 1.1 reduces to inequality (1.1). Whereas if we take $h(z) = \sum_{j=0}^{n} \binom{n}{j} z^j$ in Theorem 1.1, we get inequality (1.2).

If in Corollary 1.1 we choose $h(z) = z^n + z^k$, where $k = 0, 1, \cdots, n-1$, then we obtain the following extension of Visser's inequality (see [9]).

Corollary 1.2. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, then*

$$|a_n| + \frac{|a_k|}{\binom{n}{k}} \le \max_{|z|=1} |P(z)|, \quad k = 0, 1, 2, \ldots, n-1.$$

The result is sharp.

By a different method, Corollary 1.2 was recently proved by S. Gulzar [3].

Next, we state an analog of Theorem 1.1 for the minimum modulus of a polynomial $P(z)$ on $|z| = 1$, when there is a restriction on the zeros of $P(z)$. More precisely, we have the following result.

Theorem 1.2. *Let $P(z)$ and $h(z) = \sum_{j=0}^{n} l_j z^j$ be polynomials of degree n having all their zeros in the disk $|z| \le 1$. Then for $R > 1$ the following inequality holds:*

$$\min_{|z|=1} |(P \circ f) * h(z)| \ge R^n |l_n| \min_{|z|=1} |P(z)|, \tag{1.7}$$

where $f(z) = Rz$. The result is best possible and equality in (1.7) holds for $P(z) = az^n$, $a \neq 0$.

Remark 1.2. A result due to Aziz and Dawood [2, Theorem 1] can be obtained from Theorem 1.2 for a suitable choice of $h(z)$.

Next, letting $R \to 1$ in (1.7), we get the following result.

Corollary 1.3. *Let $P(z)$ and $h(z) = \sum_{j=0}^{n} l_j z^j$ be polynomials of degree n having all their zeros in the disk $|z| \le 1$, then*

$$\min_{|z|=1} |P * h(z)| \ge |l_n| \min_{|z|=1} |P(z)|. \tag{1.8}$$

The result is sharp and the extremal polynomial is $P(z) = az^n$, $a \neq 0$.

The next result is obtained by taking $h(z) = z^n + z^k$, where $k = 0, 1, \cdots, n-1$, in Corollary 1.3.

Corollary 1.4. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \le 1$, then*

$$|a_n| - \frac{|a_k|}{\binom{n}{k}} \ge \min_{|z|=1} |P(z)|, \quad 0 \le k \le n-1. \tag{1.9}$$

The result is sharp.

Theorem 1.1 can be sharpened, if we restrict ourselves to the class of polynomials having no zero in $|z| < 1$. In fact, we have the following result.

Theorem 1.3. *Let $P(z)$ be a polynomial of degree n having no zero in $|z| < 1$ and let $h(z) = \sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all its zeros in $|z| \le 1$, then for every $R > 1$,*

$$\max_{|z|=1} |(P \circ f) * h(z)| \le \frac{R^n |l_n| + |l_0|}{2} \max_{|z|=1} |P(z)|, \tag{1.10}$$

where $f(z) = Rz$. The result is sharp and equality in (1.10) holds for polynomial $P(z) = az^n + b$, $|a| = |b| \neq 0$.

The following corollary immediately follows by letting $R \to 1$ in Theorem 1.3.

Corollary 1.5. *Let $P(z)$ be a polynomial of degree n having no zero in $|z| < 1$ and let $h(z) = \sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\max_{|z|=1} |P * h(z)| \leq \frac{|l_n| + |l_0|}{2} \max_{|z|=1} |P(z)|.$$

The result is best possible and the equality holds for polynomial $P(z) = az^n + b$, $|a| = |b| \neq 0$.

Remark 1.3. It is easy to see that for suitable choices of $h(z)$, the inequalities (1.3) and (1.4) become special cases of Theorem 1.3.

The next corollary is obtained by taking $h(z) = z^n + z^k$, $0 \leq k \leq n-1$ in Corollary 1.5.

Corollary 1.6. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in $|z| < 1$, then*

$$|a_n| + \frac{|a_k|}{\binom{n}{k}} \leq \frac{\Lambda}{2} \max_{|z|=1} |P(z)|, \quad 0 \leq k \leq n-1,$$

where

$$\Lambda = \begin{cases} 1, & \text{if } \ 1 \leq k \leq n-1 \\ 2, & \text{if } \ k = 0. \end{cases}$$

The result is sharp.

Applying Corollary 1.6 to polynomial $z^n P(1/z)$, we obtain the next result.

Corollary 1.7. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$|a_0| + \frac{|a_k|}{\binom{n}{k}} \leq \frac{\Lambda}{2} \max_{|z|=1} |P(z)|, \quad 1 \leq k \leq n,$$

where

$$\Lambda = \begin{cases} 1, & \text{if } \ 1 \leq k \leq n-1 \\ 2, & \text{if } \ k = n. \end{cases}$$

A polynomial $P \in \mathcal{P}_n$ is said to be self-inversive if $P(z) \equiv uQ(z)$, where $|u| = 1$ and $Q(z)$ is the conjugate polynomial of $P(z)$, that is, $Q(z) := z^n \overline{P(1/\overline{z})}$.

Finally, we present the following result for self-inversive polynomials.

Theorem 1.4. *Let $P(z)$ be a self-inversive polynomial of degree n and let $h(z) = \sum_{j=0}^{n} l_j z^j$ be a polynomial of degree n having all its zeros in $|z| \le 1$, then for every $R > 1$,*

$$\max_{|z|=1} |(P \circ f) * h(z)| \le \frac{R^n|l_n| + |l_0|}{2} \max_{|z|=1} |P(z)|,$$

where $f(z) = Rz$. The result is sharp, as is shown by the extremal polynomial $P(z) = az^n + b$, $|a| = |b| \ne 0$.

Remark 1.4. A variety of interesting results can easily be deduced from Theorem 1.4 in the same way as we have deduced from Theorems 1.1 and 1.3.

2. Lemmas

For the proofs of the above stated theorems we need a number of lemmas. The first lemma is a consequence of the Schur-Szegö composition theorem [5].

Lemma 2.1. *Let f and g be polynomials of degree n. If all the zeros of f are of modulus at most r and all the zeros of g are of modulus at most s, then all the zeros of $f * g$ are of modulus at most rs.*

Lemma 2.2. *Let $F(z)$ and $h(z)$ be polynomials of degree n having all their zeros in $|z| \le 1$, and let $P(z)$ be a polynomial of degree n such that $|P(z)| \le |F(z)|$ for $|z| = 1$. Then*

$$|(P \circ f) * h(z)| \le |(F \circ f) * h(z)| \quad \textit{for} \quad |z| = 1, \tag{2.1}$$

where $f(z) = Rz$ with $R > 1$.

Proof. If $P(z) = e^{i\alpha} F(z)$, then (2.1) is trivial. Therefore, we suppose that $P(z) \ne e^{i\alpha} F(z)$ for all $\alpha \in \mathbb{R}$. Let $P(z) = \sum_{j=0}^{n} a_j z^j$, $h(z) = \sum_{j=0}^{n} l_j z^j$ and $F(z) = \sum_{j=0}^{n} b_j z^j$. Furthermore, let $P^*(z) := z^n \overline{P(1/\bar{z})}$ and $F^*(z) := z^n \overline{F(1/\bar{z})}$. Since all the zeros of $F^*(z)$ lie in $|z| \ge 1$, $P^*(z)/F^*(z)$ is analytic in $|z| \le 1$ and $|P^*(z)/F^*(z)| \le 1$ for $|z| = 1$. By the maximum modulus principle, $|P^*(z)| \le |F^*(z)|$ for $|z| \le 1$, or equivalently $|P(z)| \le |F(z)|$ for $|z| \ge 1$. Taking $z = Re^{i\theta}$, $R > 1, 0 \le \theta < 2\pi$, we obtain $|P(Rz)| \le |F(Rz)|$ for $|z| = 1$. By Rouche's theorem, all the zeros of polynomial

$$(P \circ f)(z) - \lambda(F \circ f)(z) = P(Rz) - \lambda F(Rz) = \sum_{j=0}^{n} R^j (a_j - \lambda b_j) z^j$$

lie in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and $f(z) = Rz$.

Applying Lemma 2.1 to the polynomial $P\circ f(z)-\lambda F\circ f(z)$, it follows that all the zeros of polynomial

$$\begin{aligned}(P\circ f-\lambda F\circ f)*h(z)&=\sum_{j=0}^{n}\frac{l_jR^j(a_j-\lambda b_j)}{\binom{n}{j}}z^j\\&=\sum_{j=0}^{n}\frac{l_jR^ja_j}{\binom{n}{j}}z^j-\lambda\sum_{j=0}^{n}\frac{l_jR^jb_j}{\binom{n}{j}}z^j\\&=(P\circ f)*h(z)-\lambda(F\circ f)*h(z)\end{aligned}$$

lie in $|z|<1$. This implies

$$|(P\circ f)*h(z)|\le|(F\circ f)*h(z)| \tag{2.2}$$

for $|z|\ge1$ and $R>1$. If the inequality (2.2) is not true, then there exits a point $z=z_0$ with $|z_0|\ge1$ such that

$$|(P\circ f)*h(z_0)|>|(F\circ f)*h(z_0)|.$$

But all the zeros of $F(Rz)$ lie in $|z|\le1/R<1$. Therefore, by Lemma 2.1, we have

$$(F\circ f)*h(z_0)\neq0.$$

We take

$$\lambda=\frac{(P\circ f)*h(z_0)}{(F\circ f)*h(z_0)},$$

so that λ is a well defined real or complex number with $|\lambda|>1$, and with this choice of λ, we obtain $(P\circ f)*h(z)-\lambda(F\circ f)*h(z_0)=0$, where $|z_0|\ge1$, which contradicts the fact that all the zeros of $(P\circ f-\lambda F\circ f)*h(z)$ lie in $|z|<1$, and the result follows. □

The next Lemma immediately follows from Lemma 2.2 by taking $F(z)=z^n\overline{P(1/\bar{z})}$.

Lemma 2.3. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z|<1$ and $Q(z)=z^n\overline{P(1/\bar{z})}$, then*

$$|(P\circ f)*h(z)|\le|(Q\circ f)*h(z)| \quad \text{for} \quad |z|=1, \tag{2.3}$$

where $f(z)=Rz$ with $R>1$.

Lemma 2.4. *Let $P(z)$ be a polynomial of degree n and let $h(z)=\sum_{j=0}^{n}l_jz^j$ be a polynomial of degree n having all its zeros in $|z|\le1$, then for every $R>1$ and $|z|=1$, we have*

$$|(P\circ f)*h(z)|+|(Q\circ f)*h(z)|\le\{|l_n|R^n+|l_0|\}M, \tag{2.4}$$

where $f(z)=Rz$ and $Q(z)=z^n\overline{P(1/\bar{z})}$.

Proof. Let $M = \max_{|z|=1}|P(z)|$. Since $P(z)$ is a polynomial of degree n and $|P(z)| \leq M$ for $|z| = 1$. By Rouche's theorem $P(z) - \lambda M$ does not vanish in $|z| < 1$ for every complex number λ with $|\lambda| \geq 1$. Applying Lemma 2.3 to the polynomial $P(z) - \lambda M$, we obtain

$$(2.5) \qquad |(P \circ f) * h(z) - \lambda l_0 M| \leq \left|(Q \circ f) * h(z) - \bar{\lambda} l_n R^n M z^n\right| \quad \text{for} \quad |z| = 1.$$

Since $|Q(z)| = |P(z)| \leq M$ for $|z| = 1$, by Theorem 1.1, we have for $|z| = 1$

$$(2.6) \qquad |(Q \circ f) * h(z)| \leq |l_n| R^n M.$$

In view of inequality (2.6), we can choose $\arg \lambda$ such that

$$\left|(Q \circ f) * h(z) - \bar{\lambda} l_n R^n M z^n\right| = \left|\bar{\lambda} l_n R^n M z^n\right| - |(Q \circ f) * h(z)| \quad \text{for} \quad |z| = 1.$$

Thus, in view of (2.5) we can conclude that

$$|(P \circ f) * h(z)| + |(Q \circ f) * h(z)| \leq |\lambda| \{ |l_n| R^n + |l_0| \} M \quad \text{for} \quad |z| = 1,$$

which is equivalent to (2.4). □

3. Proofs of Theorems

Proof of Theorem 1.1. The proof follows from Lemma 2.2 by taking $F(z) = Mz^n$, where $M = \max_{|z|=1}|P(z)|$. □

Proof of Theorem 1.2. Let $m = \min_{|z|=1}|P(z)|$. If $P(z)$ has a zero on $|z| = 1$, then the inequality (1.7) is trivial. Therefore, we assume that $P(z)$ has all its zeros in $|z| < 1$, so that $m > 0$. Also, we have

$$|mz^n| \leq |P(z)| \quad \text{for} \quad |z| = 1.$$

Applying Lemma 2.2 to the polynomials mz^n and $P(z)$, we get the inequality (1.7). This completes the proof of Theorem 1.2. □

Proof of Corollary 1.4. Taking $h(z) = z^n + z^k$, where $k = 0, 1, \cdots, n-1$, in Corollary 1.3, we get

$$(3.1) \qquad \min_{|z|=1} \left| a_n z^n + \frac{a_k}{\binom{n}{k}} z^k \right| \geq \min_{|z|=1} |P(z)|, \quad 0 \leq k \leq n-1.$$

If $z_1, z_2, \cdots, z_n$ are the roots of $P(z)$, then $|z_j| \leq 1$, $j = 1, 2, \cdots, n$, and we have by Viète's formula

$$(-1)^{n-k} \frac{a_k}{a_n} = \sum_{1 \leq i_1 < i_2 \ldots < i_{n-k} \leq n} z_{i_1} z_{i_2} \ldots z_{i_{n-k}}.$$

This implies

$$\left|\frac{a_k}{a_n}\right| \leq \sum_{1 \leq i_1 < i_2 \ldots < i_{n-k} \leq n} |z_{i_1} z_{i_2} \ldots z_{i_{n-k}}| \leq \binom{n}{n-k} = \binom{n}{k},$$

or equivalently

$$|a_n| \geq \frac{|a_k|}{\binom{n}{k}}, \tag{3.2}$$

which gives

$$\min_{|z|=1}\left|a_n z^n + \frac{a_k}{\binom{n}{k}} z^k\right| = \left||a_n| - \frac{|a_k|}{\binom{n}{k}}\right| = |a_n| - \frac{|a_k|}{\binom{n}{k}}. \tag{3.3}$$

Combing (3.1) and (3.3), we obtain (1.9). □

Proof of Theorem 1.3. The inequality (2.3) in conjunction with Lemma 2.4 gives for every $R > 1$ and $|z| = 1$

$$2\,|(P \circ f) * h(z)| \leq |(P \circ f) * h(z)| + |(Q \circ f) * h(z)| \leq \{|l_n|R^n + |l_0|\} \max_{|z|=1}|P(z)|.$$

This is equivalent to inequality (1.10) and completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. Since $P(z)$ is a self-inversive polynomial of degree n, then for some $u \in \mathbb{C}$ with $|u| = 1$, we have $P(z) = uQ(z)$ for all $z \in \mathbb{C}$, where $Q(z)$ is the conjugate polynomial of $P(z)$. This gives

$$|(P \circ f) * h(z)| = |(Q \circ f) * h(z)| \quad \text{for} \quad |z| = 1.$$

Using this in place of (2.3), and proceeding similarly as in the proof of Theorem 1.3, we get the desired result. □

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