Известия НАН Армении, Математика, том 53, н. 1, 2018, стр. 3 – 12.

ON GENERALIZED DERIVATIONS AND CENTRALIZERS OF OPERATOR ALGEBRAS WITH INVOLUTION

S. ALI, A. FOŠNER, W. JING

King Abdulaziz University, Jeddah, Saudi Arabia
University of Primorska, Koper, Slovenia
Fayetteville State University, Fayetteville, USA
E-mails: *shakir.ali.mm@amu.ac.in; ajda.fosner@fm-kp.si; wjing@uncfsu.edu*

Abstract. Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H and $\mathcal{A}(H) \subseteq B(H)$ be a standard operator algebra which is closed under the adjoint operation. Let $F : \mathcal{A}(H) \to B(H)$ be a linear mapping satisfying $F(AA^*A) = F(A)A^*A + Ad(A^*)A + AA^*d(A)$ for all $A \in \mathcal{A}(H)$, where the associated linear mapping $d : \mathcal{A}(H) \to B(H)$ satisfies the relation $d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A)$ for all $A \in \mathcal{A}(H)$. Then F is of the form $F(A) = SA - AT$ for all $A \in \mathcal{A}(H)$ and some $S, T \in B(H)$, that is, F is a generalized derivation. We also prove some results concerning centralizers on $\mathcal{A}(H)$ and semisimple H^*-algebras.

MSC2010 numbers: 47B47, 46K15, 16W10.

Keywords: generalized derivation; generalized Jordan derivation; left centralizer; standard operator algebra; H^*-algebra.

1. Introduction

Let $\delta : \mathcal{R} \to \mathcal{R}$ be an additive map on a ring $\mathcal{R}$. Recall that δ is called a generalized Jordan derivation if there exists a Jordan derivation $d : \mathcal{R} \to \mathcal{R}$ such that the equality

$$\delta(a^2) = \delta(a)a + ad(a) \tag{1.1}$$

holds for all $a \in \mathcal{R}$, and δ is said to be a generalized derivation if there is a derivation d on $\mathcal{R}$ satisfying

$$\delta(ab) = \delta(a)b + ad(b) \tag{1.2}$$

for all $a, b \in \mathcal{R}$.

In [7], it was proved that every generalized Jordan derivation on a 2-torsion free prime ring is a generalized derivation. This result was generalized in [14] to generalized Jordan derivations on 2-torsion free semiprime rings.

In particular, if $d = \delta$ in (1.1) and (1.2), then δ is called a Jordan derivation and derivation, respectively. The first result on Jordan derivation is due to Herstein [6] who proved that any Jordan derivation on a 2-torsion free prime ring is a Jordan derivation. Cusack [4] and Brešar [2] showed that this is also true for Jordan derivations on 2-torsion free semiprime rings. If $c \in \mathcal{R}$ is a fixed element and $\delta(a) = [c, a] = ca - ac$ for all $a \in \mathcal{R}$, then it is easy to see that δ is a derivation which is called an inner derivation determined by c. It is also well known that every linear derivation on standard operator algebra is inner (cf. [3]). Some related results on operator algebras can be found in [5], [8], [12], and references therein.

In [13], Vukman proved that if a linear mapping d on a standard operator algebra, which is closed under the adjoint operation, or a semisimple H^*-algebra, satisfying

$$d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A),$$

then d is a derivation.

Motivated by the above result and the concept of generalized Jordan derivations, in this paper, we aim to show that if F is a linear mapping on a standard operator algebra which is closed under the adjoint operation satisfying

$$F(AA^*A) = F(A)A^*A + Ad(A^*)A + AA^*d(A),$$

where the associated linear mapping d satisfies the relation

$$d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A),$$

then F is a generalized derivation. A similar result is also obtained for the case of linear mappings on semisimple H^*-algebras. It should be noted that in order to prove the result on semisimple H^*-algebras, we need to have some results about left centralizers. Recall that a linear map $\phi : \mathcal{A} \to \mathcal{A}$ on an algebra $\mathcal{A}$ is called a left centralizer if $\phi(xy) = \phi(x)y$ for all $x, y \in \mathcal{A}$. The definition of a right centralizer should be self explanatory.

We now list some basic notation, definitions, and results. Throughout the paper, $\mathcal{L}(H)$ and $\mathcal{B}(H)$ will stand for the algebra of all linear operators and the algebra of all bounded linear operators on a complex Hilbert space H, respectively. By $\mathcal{F}(H) \subseteq \mathcal{B}(H)$ we denote the subalgebra of all bounded finite rank operators. We call a subalgebra $\mathcal{A}(H)$ of $\mathcal{B}(H)$ standard if it contains $\mathcal{F}(H)$. Notice that every standard operator algebra is prime. An operator $P \in \mathcal{B}(H)$ is said to be a projection if $P^* = P$ and $P^2 = P$. Each rank one operator can be expressed as $x \otimes y$, where $x \otimes y(u) = \langle u, y \rangle x$ for all $u \in H$.

Let $\mathcal{A}$ be an algebra over the filed $\mathbb{C}$ of complex numbers. An involution in $\mathcal{A}$ is a map $a \mapsto a^*$ of $\mathcal{A}$ into itself such that

(1) $(a^*)^* = a$
(2) $(a + b)^* = a^* + b^*$
(3) $(\lambda a)^* = \bar{\lambda} a^*$
(4) $(ab)^* = b^* a^*$

for any $a, b \in \mathcal{A}$ and $\lambda \in \mathbf{C}$. An algebra over $\mathbf{C}$ endowed with an involution is called an involution algebra or a $*$-algebra. Recall that a semisimple H^*-algebra is a complex semisimple Banach $*$-algebra whose norm is a Hilbert space norm such that $(x, yz^*) = (xz, y) = (z, x^*y)$ is fulfilled for all elements x, y, z. Let $\mathcal{A}$ be a semisimple H^*-algebra and $\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$ be the collection of minimal closed ideals of $\mathcal{A}$ such that $\mathcal{A} = \oplus_{\alpha\in\Gamma}\mathcal{A}_\alpha$. Then any element $x \in \mathcal{A}$ can be expressed as $x = \sum_{\alpha\in\Gamma} x_\alpha$ and $x_\alpha x_\beta = 0$ for $x_\alpha \in \mathcal{A}_\alpha$ and $x_\beta \in \mathcal{A}_\beta$ with $\alpha \neq \beta$. For every x and y in $\mathcal{A}$ with $x = \sum_\alpha x_\alpha$ and $y = \sum_\alpha y_\alpha$, we have $xy = \sum_\alpha x_\alpha y_\alpha$. A self-adjoint idempotent element $e \in \mathcal{A}$ is called a projection. A nonzero projection is said to be minimal if it can't be represented as a sum of two mutually orthogonal nonzero projections in $\mathcal{A}$. For more information about H^*-algebras, we refer the reader to [1] and [11].

2. Main results

Our first theorem is a generalization of Theorem 1 of [13].

Theorem 2.1. *Let H be a complex Hilbert space, and let $\mathcal{A}(H) \subseteq \mathcal{B}(H)$ be a standard operator algebra, which is closed under the adjoint operation. Suppose there exists a linear mapping $F : \mathcal{A}(H) \to \mathcal{B}(H)$ satisfying the relation*

$$F(AA^*A) = F(A)A^*A + Ad(A^*)A + AA^*d(A) \tag{2.1}$$

for all $A \in \mathcal{A}(H)$, where the associated linear mapping $d : \mathcal{A}(H) \to \mathcal{B}(H)$ satisfies the relation

$$d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A) \tag{2.2}$$

for all $A \in \mathcal{A}(H)$. Then $F(A) = SA - AT$ for all $A \in \mathcal{A}(H)$ and some $S, T \in \mathcal{B}(H)$, which means that F is a linear generalized derivation.

It should be mentioned that in the proof below, we borrow some ideas from [10] and [13].

Proof. First we consider the restriction of F to $\mathcal{F}(H)$. Suppose $A \in \mathcal{F}(H)$. Then $A^* \in \mathcal{F}(H)$. Pick a projection $P \in \mathcal{F}(H)$ such that $AP = PA = A$ and $A^*P = PA^* = A^*$. Hence, in view of relation (2.1), we obtain

$$F(P) = F(P)P + Pd(P)P + Pd(P). \tag{2.3}$$

Right multiplication by P to (2.3) yields that $2Pd(P)P = 0$. This implies that

$$Pd(P)P = 0. \tag{2.4}$$

In view of above relation, we find that

$$Pd(P)A = 0,\ Ad(P)P = 0,\ \text{and } Ad(P)A = 0. \tag{2.5}$$

Using (2.4) in (2.3), we get

$$F(P) = F(P)P + Pd(P). \tag{2.6}$$

Replacing A by $A+P$ in (2.1) and using the fact that $A^* = (A+P)^* = A^* + P$, we obtain

$$\begin{aligned} F((A+P)(A^*+P)(A+P)) &= F(A)A^*A + Ad(A^*)A \\ &+AA^*d(A) + F(AA^* + A^*A + A^2) + 2F(A) + F(A^*) + F(P)P + Pd(P). \end{aligned} \tag{2.7}$$

On the other hand, we find that

$$\begin{aligned} F((A+P)(A^*+P)(A+P)) &= F(A)A^*A + F(A)A + F(A)A^* \\ &+F(A)P + F(P)A^*A + F(P)A^* + F(P)A + F(P)P + Ad(A^*)A + Pd(A^*)A \\ &+Ad(P)A + Pd(P)A + Ad(A^*)P + Pd(A^*)P + Ad(P)P + Pd(P)P + AA^*d(A) \\ &+A^*d(A) + Ad(A) + Pd(A) + AA^*d(P) + A^*d(P) + Ad(P) + Pd(P). \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we obtain

$$\begin{aligned} & F(AA^* + A^*A + A^2) + 2F(A) + F(A^*) \\ = \ & F(A)A + F(A)A^* + F(A)P + F(P)A^*A + F(P)A^* + F(P)A + Pd(A^*)A \\ & +Ad(P)A + Pd(P)A + Ad(A^*)P + Pd(A^*)P + Ad(P)P + Pd(P)P \\ & +A^*d(A) + Ad(A) + Pd(A) + AA^*d(P) + A^*d(P) + Ad(P). \end{aligned}$$

An application of (2.5) and (2.6) yields

$$\begin{aligned} F(AA^* + A^*A + A^2) + 2F(A) + F(A^*) &= F(A)A + F(A)A^* \\ &+F(A)P + F(P)A^*A + F(P)A^* + F(P)A + Pd(A^*)A \\ &+Ad(A^*)P + Pd(A^*)P + A^*d(A) + Ad(A) \\ &+Pd(A) + AA^*d(P) + A^*d(P) + Ad(P). \end{aligned} \tag{2.9}$$

Replacing A by $-A$ in (2.9), we get

$$\begin{aligned} F(AA^* + A^*A + A^2) - 2F(A) - F(A^*) &= F(A)A^* + F(A)A \\ &+F(P)A^*A - F(P)A^* - F(P)A - F(A)P + Pd(A^*)A + Ad(A^*)P \\ &-Pd(A^*)P + A^*d(A) + Ad(A) - Pd(A) + AA^*d(P) - A^*d(P) - Ad(P). \end{aligned} \tag{2.10}$$

Adding (2.9) and (2.10), we arrive at

$$F(AA^* + A^*A + A^2) = F(A)A^* + F(A)A + F(P)A^*A \tag{2.11}$$

$$+Pd(A^*)A + Ad(P)P + A^*d(A) + Ad(A) + AA^*d(P).$$

Subtracting (2.10) from (2.9), we obtain

$$2F(A) + F(A^*) = F(P)A^* + F(P)A + F(A)P + Pd(A^*)P \tag{2.12}$$
$$+Pd(A) + A^*d(P) + Ad(P).$$

Next, substituting iA for A into (2.10) and (2.11), we find that

$$F(A^2 - AA^* - A^*A) = F(A)A - F(A)A^* - F(P)A^*A \tag{2.13}$$
$$-Pd(A^*)A + Ad(P)A + Ad(A) - A^*d(A) - AA^*d(P)$$

$$2iF(A) - iF(A^*) = iF(P)A - iF(P)A^* + iF(A)P \tag{2.14}$$
$$-iPd(A^*)P + iPd(A) - iA^*d(P) + iAd(P).$$

This implies that

$$2F(A) - F(A^*) = F(P)A - F(P)A^* + F(A)P \tag{2.15}$$
$$-Pd(A^*)P + Pd(A) - A^*d(P) + Ad(P).$$

Adding (2.12) and (2.15), we arrive at

$$2F(A) = F(A)P + Ad(P) + F(P)A + Pd(A). \tag{2.16}$$

Now adding (2.11) and (2.13), we get

$$F(A^2) = F(A)A + Ad(A) \tag{2.17}$$

for all $A \in \mathcal{A}(H)$.

By Theorem 1 of [13], we see that d is an inner derivation on $\mathcal{A}(H)$. So, there exists an operator $N \in B(H)$ such that

$$d(A) = NA - AN \tag{2.18}$$

for all $A \in \mathcal{F}(H)$. In view of relations (2.16) and (2.17), we conclude that F maps $\mathcal{F}(H)$ into itself. Also, from (2.17), it is clear that F is a generalized Jordan derivation on $\mathcal{F}(H)$.

Note that $\mathcal{F}(H)$ is prime and hence F is a generalized derivation on $\mathcal{F}(H)$ by Theorem 2.5 of [7]. Furthermore, Theorem 4.2 of [7] asserts that F is a generalized inner derivation on $\mathcal{F}(H)$, that is, there exist $S, T \in B(H)$ such that

$$F(A) = SA - AT \tag{2.19}$$

for all $A \in \mathcal{F}(H)$.

To complete the proof, it remains to show that the relation (2.19) holds for all $A \in \mathcal{A}(H)$.

We first claim that the operators N in (2.18) and T in (2.19) differ by a scalar multiple of the identity operator I. Indeed, for any $A, B \in \mathcal{F}(H)$, $F(AB) = SAB - ABT$. On the other hand, we have

$$F(A)B + Ad(B) = SAB - ATB + ANB - ABN.$$

Comparing the above two relations, we see that

$$AB(N-T) = A(N-T)B \tag{2.20}$$

holds true for all $A, B \in \mathcal{F}(H)$.

Pick $y, u \in H$ such that $\langle u, y\rangle = 1$. Now for arbitrary $x, v \in H$, the relation (2.20) becomes $x \otimes y \cdot u \otimes v \cdot (N-T) = x \otimes y \cdot (N-T) \cdot u \otimes v$. This leads to $(N-T)^* v = \langle (N-T)u, y\rangle v$ for any $v \in H$. Hence, $(N-T)^* = \langle (N-T)u, y\rangle I$, or equivalently, $N-T = \langle y, (N-T)u\rangle I$. Taking $\lambda = \langle y, (N-T)u\rangle$, we get $N - T = \lambda I$.

We now define a linear map $G : \mathcal{A}(H) \to B(H)$ as follows: $G(A) = SA - AT$ for all $A \in \mathcal{A}(H)$. We set $F_0 = F - G$, and observe that $F_0(A) = 0$ for any $A \in \mathcal{F}(H)$. Thus, it remains to show that $F_0(A) = 0$ for all $A \in \mathcal{A}(H)$.

For any $A \in \mathcal{A}(H)$, we can write

$$\begin{aligned}
& F_0(AA^*A) = F(AA^*A) - G(AA^*A) = \\
= \;& F(A)A^*A + Ad(A^*)A + AA^*d(A) - SAA^*A + AA^*AT \\
= \;& F(A)A^*A + ANA^*A - AA^*NA + AA^*NA - AA^*AN - SAA^*A + AA^*AT \\
= \;& F(A)A^*A + A(T+\lambda I)A^*A - AA^*(T+\lambda I)A + AA^*(T+\lambda I)A \\
& -AA^*A(T+\lambda I) - SAA^*A + AA^*AT = F(A)A^*A - SAA^*A + ATA^*A.
\end{aligned}$$

and

$$F_0(A)A^*A = F(A)A^*A - G(A)A^*A = F(A)A^*A - SAA^*A + ATA^*A.$$

Therefore, we have $F_0(AA^*A) = F_0(A)A^*A$ for any $A \in \mathcal{A}(H)$.

Let $A \in \mathcal{A}(H)$ and P be a rank one projection. We write $K = A - AP - PA + PAP$. One can easily check that $KP = PK = K^*P = PK^* = 0$ and $F_0(K) = F_0(A)$. We have

$$\begin{aligned}
F_0(A)K^*K = F_0(K)K^*K = F_0(KK^*K) = F_0(KK^*K + P) = \\
= F_0((K+P)(K+P)^*(K+P)) = F_0(K+P)(K+P)^*(K+P) = \\
= F_0(A)(K^* + P)(K+P) = F_0(A)K^*K + F_0(A)P,
\end{aligned}$$

implying that $F_0(A)P = 0$. Since P is arbitrary, it follows that $F_0(A) = 0$ for all $A \in \mathcal{A}(H)$. This completes the proof of the theorem. □

As an immediate consequence of Theorem 2.1, we have the following corollary.

Corollary 2.1 ([13], Theorem 1). *Let H be a complex Hilbert space, and let $\mathcal{A}(H) \subseteq \mathcal{B}(H)$ be a standard operator algebra, which is closed under the adjoint operation. Suppose there exists a linear mapping $d : \mathcal{A}(H) \to \mathcal{B}(H)$ satisfying the relation*

$$d(AA^*A) = d(A)A^*A + Ad(A^*)A + AA^*d(A)$$

for all $A \in \mathcal{A}(H)$. Then $d(A) = TA - AT$ for all $A \in \mathcal{A}(H)$ and some $T \in \mathcal{B}(H)$, which means that d is an inner derivation.

The proof of the following theorem is similar to that of Lemma of [10]. For the sake of completeness, we include it here.

Theorem 2.2. *Let H be a complex Hilbert space, and let $\mathcal{A}(H) \subseteq \mathcal{B}(H)$ be a standard operator algebra, which is closed under the adjoint operation. Further, let $\phi : \mathcal{A}(H) \to \mathcal{B}(H)$ be a linear mapping satisfying*

$$\phi(AA^*A) = \phi(A)A^*A \tag{2.21}$$

for all $A \in \mathcal{A}(H)$. Then ϕ is a left centralizer and there exists a linear operator $C \in \mathcal{L}(H)$ such that for all $A \in \mathcal{A}(H)$ $\phi(A) = CA$.

Proof. Let $A \in \mathcal{F}(H)$ and P be a finite rank projection such that $AP = PA = A$. Substituting $A + P$ for A in relation (2.21), we obtain

$$\begin{aligned} &\phi(A^2 + A^*A + AA^* + 2A + A^*) \\ = \;&\phi(A)A + \phi(P)A^*A + \phi(A)A^* + \phi(A)P + \phi(P)A + \phi(P)A^*. \end{aligned}$$

Replacing A by $A + P$ and $A - P$ respectively in the above relation, we can get

$$\phi(2A + A^*) = \phi(A)P + \phi(P)A + \phi(P)A^*. \tag{2.22}$$

Replacing A by iA in (2.22), we get

$$\phi(2iA - iA^*) = i\phi(A)P + i\phi(P)A - i\phi(P)A^*.$$

It follows that

$$\phi(-2A + A^*) = -\phi(A)P - \phi(P)A + \phi(P)A. \tag{2.23}$$

Equalities (2.22) and (2.23) yield that $\phi(A^*) = \phi(P)A^*$. Replacing A^* by A results in

$$\phi(A) = \phi(P)A. \tag{2.24}$$

We now show that ϕ is a left centralizer on $\mathcal{F}(H)$, that is, $\phi(AB) = \phi(A)B$ for all $A, B \in \mathcal{F}(H)$. If H is finite dimensional, the choosing $P = I$, we get $\phi(AB) = \phi(I)AB = \phi(A)B$. If H is of infinite dimension, then we fix an element $x \in H$, and claim that for any $y \in H$, there exists an element $x_y \in H$ such that $\phi(x \otimes y) = x_y \otimes y$. Let $y_1, y_2 \in H$. If y_1 and y_2 are linearly independent, then

$$\phi(x \otimes (y_1 + y_2)) = x_{y_1+y_2} \otimes (y_1 + y_2) = x_{y_1+y_2} \otimes y_1 + x_{y_1+y_2} \otimes y_2.$$

On the other hand, we have

$$\phi(x \otimes y_1) + \phi(x \otimes y_2) = x_{y_1} \otimes y_1 + x_{y_2} \otimes y_2.$$

It follows that $x_{y_1} = x_{y_1+y_2} = x_{y_2}$. In the case where y_1 and y_2 are linearly dependent, we may find a $y_3 \in H$ such that y_1, y_3 as well as y_2, y_3 are linearly independent. Therefore, $x_{y_1} = x_{y_3} = x_{y_2}$.

Pick an element $u \in H$ such that $\langle u, y\rangle \neq 0$. Let $v \in H$ be arbitrary. We have

$$\begin{aligned} \phi(x \otimes y \cdot u \otimes v) &= \phi(\langle u, y\rangle x \otimes v) = x_{\langle u,y\rangle v} \otimes \langle u, y\rangle v \\ &= \langle u, y\rangle x_{\langle u,y\rangle v} \otimes v = \langle u, y\rangle x_y \otimes v = x_y \otimes y \cdot u \otimes v = \phi(x \otimes y) u \otimes v. \end{aligned}$$

If $\langle u, y\rangle = 0$, we have $\phi(x \otimes y \cdot u \otimes v) = 0$ and, by (2.24),

$$\phi(x \otimes y \cdot u \otimes v) = \phi(P) x \otimes y \cdot u \otimes v = 0$$

for some finite rank projection P. Now, we can conclude that for any $A, B \in \mathcal{F}(H)$ $\phi(AB) = \phi(A)B$. This implies that ϕ is a left centralizer on $\mathcal{F}(H)$.

Next, we pick $y, u \in H$ with $\langle y, u\rangle = 1$, and define $Cx = \phi(x \otimes u)y$ for any $x \in H$. Obviously, C is linear. Now for any $A \in \mathcal{F}(H)$ and $x \in H$,

$$CAx = \phi(Ax \otimes u)y = \phi(A)x \otimes u(y) = \phi(A)(\langle y, u\rangle x) = \phi(A)x.$$

Thus, $\phi(A) = CA$ for all $A \in \mathcal{F}(H)$.

To complete the proof, it remains to show that $\phi(A) = CA$ for all $A \in \mathcal{A}(H)$.

Define Φ by $\Phi(A) = CA$ for all $A \in \mathcal{A}(H)$ and let $\phi_0 = \phi - \Phi$. It is obvious that $\phi_0(A) = 0$ for all $A \in \mathcal{F}(H)$. One can check that $\phi_0(AA^*A) = \phi_0(A)A^*A$ for all $A \in \mathcal{A}(H)$. Let $A \in \mathcal{A}(H)$. Suppose that P is a finite rank projection and let $K = A - AP - PA + PAP$. We have

$$\begin{aligned} \phi_0(K)K^*K &= \phi_0(KK^*K) = \phi_0(KK^*K + p) = \phi_0((K+P)(K+P)^*(K+P)) \\ &= \phi_0(K+P)(K+P)^*(K+P). \end{aligned}$$

This leads to $\phi_0(K)P = 0$. Observing that $\phi_0(K) = \phi_0(A)$, we get $\phi_0(A)P = 0$ for any finite rank projection P. Hence, $\phi_0(A) = 0$ for all $A \in \mathcal{A}(H)$. □

The proof of the next result is just a modification of that of Theorem of [10]. We present the proof for the reader's convenience.

Theorem 2.3. *Let $\phi : \mathcal{A} \to \mathcal{A}$ be a linear mapping on a semisimple H^*-algebra $\mathcal{A}$ satisfying*

$$\phi(xx^*x) = \phi(x)x^*x \tag{2.25}$$

for all $x \in \mathcal{A}$. Then ϕ is a left centralizer.

Proof. Let $e \in \mathcal{A}$ be a projection. Replacing x by $x+e$ and $x-e$ in (2.25), respectively, and comparing the resulting equalities, we arrive at

$$\phi(ee^*x + xe^*e + ex^*e) = \phi(e)e^*x + \phi(x)e^*e + \phi(e)x^*e. \tag{2.26}$$

Let $\{\mathcal{A}_\alpha : \alpha \in \Gamma\}$ be a collection of minimal closed ideals of $\mathcal{A}$ such that their orthogonal direct sum is $\mathcal{A}$. For $\alpha \in \Gamma$ and $x \in \mathcal{A}_\alpha$, let e be a minimal projection with $e \in \mathcal{A}_\beta$ $(\alpha \neq \beta)$. It follows from (2.26) that $\phi(x)e = 0$. Thus, $\phi(x) \in \mathcal{A}_\alpha$, which implies that $\mathcal{A}_\alpha$ is invariant under ϕ. By Theorem 2.2, we conclude that ϕ is a left centralizer on $\mathcal{A}_\alpha$ for each $\alpha \in \Gamma$. Furthermore, it follows from Theorem 2.2 and Remark 1 of [9] that ϕ is continuous on $\mathcal{A}_\alpha$ for every $\alpha \in \Gamma$.

Let $\{x_n\} \subseteq \mathcal{A}$ and $y \in \mathcal{A}$ be such that

$$\lim_{n\to\infty} x_n \to 0 \quad \text{and} \quad \lim_{n\to\infty} \phi(x_n) \to y.$$

If $e \in \mathcal{A}$ is a minimal projection, from (2.26) we see that

$$0 = \lim_{n\to\infty} [\phi(e)ex_n + \phi(x_n)e + \phi(e)x_n^*e] = ye,$$

implying that $y = 0$. By Closed Graph Theorem, ϕ is continuous.

For any $x, y \in \mathcal{A}$, we write $x = \sum_{\alpha\in\Gamma} x_\alpha$ and $y = \sum_{\alpha\in\Gamma} y_\alpha$, where $x_\alpha, y_\alpha \in \mathcal{A}_\alpha$ $(\alpha \in \Gamma)$. We have

$$\begin{aligned}\phi(xy) &= \phi\Big(\sum_{\alpha\in\Gamma} x_\alpha \sum_{\alpha\in\Gamma} y_\alpha\Big) = \phi\Big(\sum_{\alpha\in\Gamma.} x_\alpha y_\alpha\Big) = \sum_{\alpha\in\Gamma} \phi(x_\alpha y_\alpha) = \sum_{\alpha\in\Gamma} \phi(x_\alpha) y_\alpha \\ &= \Big(\sum_{\alpha\in\Gamma} \phi(x_\alpha)\Big)\Big(\sum_{\alpha\in\Gamma} y_\alpha\Big) = \phi\Big(\sum_{\alpha\in\Gamma} x_\alpha\Big)\Big(\sum_{\alpha\in\Gamma} y_\alpha\Big) = \phi(x)y.\end{aligned}$$

Thus, $\phi(xy) = \phi(x)y$ for all $x, y \in \mathcal{A}$. This completes the proof. □

We conclude our paper by proving an analog of Theorem 1 on semisimple H^*-algebras.

Theorem 2.4. *Let $\mathcal{A}$ be a semisimple H^*-algebra. Suppose there exists a linear mapping $F : \mathcal{A} \to \mathcal{A}$ satisfying the relation*

$$F(xx^*x) = F(x)x^*x + xd(x^*)x + xx^*d(x)$$

for all $x \in \mathcal{A}$, where the associated linear mapping $d : \mathcal{A} \to \mathcal{A}$ satisfies the relation

$$d(xx^*x) = d(x)x^*x + xd(x^*)x + xx^*d(x)$$

for all $x \in \mathcal{A}$. Then F is a generalized derivation.

Proof. By Theorem 2 of [13], d is a linear derivation. Now, for any $x \in \mathcal{A}$, we have

$$\begin{aligned}&(F-d)(xx^*x) = F(xx^*x) - d(xx^*x) = \\ =\ & \big(F(x)x^*x + xd(x^*)x + xx^*d(x)\big) - \big(d(x)x^*x + xd(x^*)x + xx^*d(x)\big) \\ =\ & F(x)x^*x - d(x)x^*x = (F-d)(x)x^*x.\end{aligned}$$

In view of Theorem 2.3, we conclude that $F-d$ is a left centralizer. Therefore, for any $x, y \in \mathcal{A}$, using the fact that d is a derivation, we obtain

$$F(xy) = (F-d)(xy) + d(xy) = (F-d)(x)y + d(x)y + xd(y) = F(x)y + xd(y).$$

Hence, F is a generalized derivation. □

Corollary 2.2 ([13], Theorem 2). *Let $\mathcal{A}$ be a semisimple H^*-algebra. Suppose there exists a linear mapping $d : \mathcal{A} \to \mathcal{A}$ satisfying the relation*

$$d(xx^*x) = d(x)x^*x + xd(x^*)x + xx^*d(x)$$

for all $x \in \mathcal{A}$. Then d is a derivation.

Список литературы

[1] W. Ambrose, "Structure theorems for a special class of Banach algebras", Trans. Amer. Math. Soc., **57**, 364 – 386 (1945).

[2] M. Brešar, "Jordan derivations on semiprime rings", Proc. Amer. Math. Soc., **104**, 1003 – 1006 (1988).

[3] P. R. Chernoff, "Representations, automorphisms, and derivations of some operator algebras", J. Funct. Anal., **12**, 275 – 289 (1973).

[4] J. M. Cusack, "Jordan derivation on rings", Proc. Amer. Math. Soc., **53**, 321 – 324 (1975).

[5] A. Fošner and J. Vukman, "Some functional equations on standard operator algebras", Acta Math. Hungar., **118**, 299 – 306 (2008).

[6] I. N. Herstein, "Jordan derivations of prime rings", Proc. Amer. Math. Soc., **8**, 1104 – 1110 (1957).

[7] W. Jing and S. Lu, "Generalized Jordan derivations on prime rings and operator algebras", Taiwanese. J. Math., **7**, 605 – 613 (2003).

[8] J. Li and H. Pendharkar, "Derivations on certain operator Algebras", Internat. J. Math. & Math. Sci., **24**, no. 5, 345 – 349 (2000).

[9] L. Molnár, "A condition for a function to be a bounded linear operator", Indian. J. Math., **35**, 1 – 4 (1993).

[10] L. Molnár, "On centralizers of an H^*-algebra", Publ. Math. Debrecen, **46**, 89 – 95 (1995).

[11] P. P. Saworotnow and J. C. Friedell, "Trace-class for an arbitrary H^*-algebra", Proc. Amer. Math. Soc., **26**, 95 – 100 (1970).

[12] P. Šemrl, "Ring derivations on standard operator algebras", J. Funct. Anal., **112**, 318 – 324 (1993).

[13] J. Vukman, "On derivations of algebras with involution", Acta. Math. Hungar., **112**, 181 – 186 (2006).

[14] F. Feng and Z. Xiao, "Generalized Jordan derivations on semiprime rings and its applications in range inclusion problems", Mediterr. J. Math., **8**, 271 – 291 (2011).

Поступила 4 декабря 2015