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PRIORI ESTIMATES AND ASYMPTOTIC PROPERTIES OF SOLUTIONS FOR SOME FRACTIONAL ORDER ELLIPTIC EQUATIONS

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Abstract. In this paper, we obtain estimates for solutions for a class of fractional order elliptic equations in different domains and boundary conditions, and prove some regularity results. Then, we study the qualitative properties of solutions with prescribed Q-curvature. [1]

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1. Introduction

In this paper, we obtain estimates for solutions of the following fractional order elliptic equation:

$$(*)\qquad \begin{cases} (-\Delta)^{\frac{3}{2}}u(x) = Q(x)e^{3u} & \text{in } \Omega \subset R^3, \\ u = 0 & \text{in } \Omega^c, \\ (-\Delta)^{\frac{1}{2}}u = 0 & \text{on } \partial\Omega, \end{cases}$$

and investigate properties of solutions of the following fractional order elliptic equation:

$$(**)\qquad (-\Delta)^{\frac{3}{2}}u(x) = Q(x)e^{3u}, \quad x \in R^3,$$

where Ω is a bounded smooth domain, $Q(x)$ is a given function in $L^p(\Omega)$ for some $\frac{6}{5} < p \leq \infty$, and $(-\Delta)^{\frac{3}{2}}$ is interpreted as $(-\Delta) \circ (-\Delta)^{\frac{1}{2}}$. To define $(-\Delta)^{\frac{1}{2}}v$ for a function v in R^3, we require that

$$v \in L_{\frac{1}{2}}(R^3) := \left\{ v \in L^1_{loc}(R^3) : \int_{R^3} \frac{|v(x)|}{1+|x|^4}dx < \infty \right\},$$

which makes $(-\Delta)^{\frac{1}{2}}v$ to be a tempered distribution (see [13]).

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Definition 1.1. *Given a tempered distribution f in R^3, we say that u is a solution of equation $(-\Delta)^{\frac{3}{2}}u = f$ if $u \in W^{2,1}_{loc}(R^3)$, $\Delta u \in L_{\frac{1}{2}}(R^3)$, and*

$$(1.1) \qquad \int_{R^3} (-\Delta u)(-\Delta)^{\frac{1}{2}}\varphi dx = \langle f, \varphi\rangle \text{ for every } \varphi \in \mathcal{S}(R^3),$$

where $\mathcal{S}(R^3)$ is the Schwarz space of rapidly decreasing smooth functions in R^3.

In equation $(*)$, we assume that $u \in L^1(\Omega)$ and $e^{3u} \in L^{p'}(\Omega)$, where p' is the conjugate exponent of p, so that $(*)$ has a meaning in the sense of distributions (see Definition 3.1 of [5]). A first question of interest is whether one can conclude that $u \in L^\infty(\Omega)$ for $(*)$. In Section 2, we give a positive answer to this question.

Recently, a series of works have been done to prove the existence and to study the qualitative properties of solutions of the following fourth order equation:

$$\Delta^2 u = Q(x)e^{4u}, \ x \in R^4.$$

For $Q(x) = 6$, Lin [7] has given a complete classification of solutions of this equation in terms of their growth, or in terms of the behavior of Δu at ∞. Xu [17], has obtained similar results by using moving spheres methods. In [8], concentration phenomena of solutions of this equation was deeply discussed. Robert and Wei [9], have studied asymptotic behavior of solutions for a fourth order mean field equation with Dirichlet boundary condition. Martinazzi [10], and Wei and Xu [14] gave classifications of solutions for higher order Liouville's and conformally invariant equations, respectively. Concentration phenomena and asymptotic behavior of solutions for higher order Liouville's and a mean field equations was studied in [11, 12]. Based on these works, Jin et al. [6] have studied the existence and asymptotics of solutions for equation $(**)$. In Section 3, we extend the results of [6], by considering more general functions $Q(x)$. We first obtain the asymptotic behavior of solutions near infinity, and then prove that all solutions satisfy an identity, which is similar to the well-known Kazdan-Warner condition.

2. L^∞-BOUNDEDNESS FOR A SINGLE SOLUTION OF EQUATION $(-\Delta)^{\frac{3}{2}}u = Q(x)e^{3u}$

Let $\Omega \subset R^3$ be a bounded domain and let h be a solution of equation:

$$(2.1) \qquad \begin{cases} (-\Delta)^{\frac{3}{2}}h(x) = f(x) & \text{in } \Omega \subset R^3, \\ h = 0 & \text{in } \Omega^c, \\ (-\Delta)^{\frac{1}{2}}h = 0 & \text{on } \partial\Omega. \end{cases}$$

The following lemma is obtained using the arguments of Brezis-Merle [1].

Lemma 2.1. *([5]). Let $f \in L^1(\overline{\Omega})$, and let $u \in L^1(\Omega)$ be a solution of equation (2.1). Then for any $p \in \left(0, \frac{2\pi^2}{||f||_{L^1(\Omega)}}\right)$ the following inequality holds:*

$$\int_\Omega \exp^{3p|u|} dx \leq C(p, \mathrm{diam}\Omega),$$

where diamΩ *denotes the diameter of domain* Ω.

We will use Lemma 2.1 to prove the theorems that follow.

Theorem 2.1. *Let u be a solution of equation (2.1) with $f \in L^1(\Omega)$. Then for every constant $k > 0$ we have*

$$e^{ku} \in L^1(\Omega).$$

Proof. Letting $0 < \epsilon < \frac{1}{k}$, we can split the function f as $f = f_1 + f_2$ with $||f_1||_1 < \epsilon$ and $f_2 \in L^\infty(\Omega)$. Denote by u_i $(i = 1, 2)$ the solution of the equation

$$\begin{cases} (-\Delta)^{\frac{3}{2}} u_i = f_i & \text{in } \Omega, \\ u_i = 0 & \text{in } \Omega^c, \quad i = 1, 2. \\ (-\Delta)^{\frac{1}{2}} u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

By Lemma 2.1, we have $\int_\Omega \exp\left[\frac{|u_1(x)|}{||f_1||_1}\right] dx < \infty$, implying that $\int_\Omega \exp[k|u_1|]dx < \infty$. Using Theorem 1.10 of [4], we obtain $|u| \leq |u_1| + |u_2|$ and $u_2 \in L^\infty(\Omega)$, and the result follows.

Theorem 2.2. *Let $u \in L_{Loc}(\Omega)$ and $(-\Delta)^{\frac{3}{2}} u \in L_{Loc}(\Omega)$. Then for every constant $k > 0$ we have*

$$e^{ku} \in L^1_{loc}(\Omega).$$

Proof. Without loss of generality, we can assume that $\Omega = B_R(\theta)$ is a ball of radius R centered at θ. For ϵ small enough, we split $(-\Delta)^{\frac{3}{2}} u$ as $(-\Delta)^{\frac{3}{2}} u = f_1 + f_2$ with $||f_1||_1 < \epsilon$ and $f_2 \in L^\infty(\Omega)$. Write $u = u_1 + u_2 + u_3$, where u_i $(i = 1, 2, 3)$ are respectively the solutions of equations:

$$\begin{cases} (-\Delta)^{\frac{3}{2}} u_i = f_i & \text{in } B_{\frac{R}{2}}, \\ u_i = 0 & \text{in } (B_{\frac{R}{2}})^c, \quad i = 1, 2, 3. \\ (-\Delta)^{\frac{1}{2}} u_i = 0 & \text{on } \partial B_{\frac{R}{2}} \end{cases}$$

It follows from Lemma 2.1 that $e^{k|u_1|} \in L^1_{Loc}(B_R)$. Using elliptic estimates from [4], we get $|u_2|_{L^\infty(B_{\frac{R}{2}})} \leq c$, implying that $e^{k|u_2|} \in L^1_{Loc}(B_R)$.

Since $(-\Delta)^{\frac{3}{2}} u_3 = 0$, we have $|(-\Delta)^{\frac{1}{2}} u_3|_{L^\infty(B_{\frac{R}{4}})} \leq c$. Taking into account that $u \in L_{Loc}(B_R)$, we get $|u_3|_{L^\infty(B_{\frac{R}{8}})} \leq c$ implying that $e^{ku_3} \in L^1_{Loc}(B_R)$.

Finally, in view of $|u| \leq |u_1| + |u_2| + |u_3|$, the result follows.

Remark 2.1. *Note that Theorem 2.2 is a local version of Theorem 2.1.*

Theorem 2.3. *Let u be a solution of equation $(*)$ with $Q \in L^p(\Omega)$, and let $e^{3u} \in L^{p'}(\Omega)$ for some $\frac{6}{5} < p \leq \infty$. Then $u \in L^\infty(\Omega)$.*

Proof. By Theorem 2.1, we have $e^{ku} \in L^1(\Omega)$ for all k, that is, $e^u \in L^r(\Omega)$ for all $r < \infty$. It follows that $Qe^{3u} \in L^{p-\delta}$ for all $\delta > 0$ if $p < \infty$, and $Qe^{3u} \in L^r(\Omega)$ for all $r < \infty$ if $p = \infty$. Now, standard elliptic theory and Sobolev embedding theorem can be applied to conclude that $u \in L^\infty(\Omega)$.

Corollary 2.1. *Let u be a solution of equation*

$$\begin{cases} (-\Delta)^{\frac{3}{2}}u = Qe^{3u} + f(x) & \text{in } \Omega, \\ u = g_1 & \text{in } \Omega^c, \\ (-\Delta)^{\frac{1}{2}}u = g_2 & \text{on } \partial\Omega \end{cases}$$

with $Q \in L^p(\Omega)$ and $e^{3u} \in L^{p'}(\Omega)$ for some $\frac{6}{5} < p \leq \infty$, where g_1, $g_2 \in L^\infty(\partial\Omega)$ and $f \in L^q(\Omega)$ for some $q > \frac{6}{5}$. Then $u \in L^\infty(\Omega)$.

Proof. Let w be the solution of equation

$$\begin{cases} (-\Delta)^{\frac{3}{2}}w = f(x) & \text{in } \Omega, \\ w = g_1 & \text{in } \Omega^c, \\ (-\Delta)^{\frac{1}{2}}w = g_2 & \text{on } \partial\Omega. \end{cases}$$

So that $w \in L^\infty(\Omega)$. Observing that the function $\tilde{u} = u - w$ satisfies the equation

$$\begin{cases} (-\Delta)^{\frac{3}{2}}\tilde{u} = Qe^{3w}e^{3\tilde{u}}, & \text{in } \Omega, \\ \tilde{u} = 0 & \text{in } \Omega^c, \\ (-\Delta)^{\frac{1}{2}}\tilde{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

we can apply Theorem 2.3 to complete the proof.

Theorem 2.4. *Let $u \in L^1(R^3)$ be a solution of equation $(**)$ with $Q \in L^p(R^3)$, and let $e^{3u} \in L^{p'}(R^3)$ for some $\frac{6}{5} < p \leq \infty$. Then $u \in L^\infty_{Loc}(R^3)$.*

Proof. Without loss of generality, we can assume that $\Omega = B_R(\theta) \subset R^3$. We fix $\epsilon > 0$ small enough and split Qe^{3u} as $Qe^{3u} = f_1 + f_2$ with $||f_1||_1 < \epsilon$ and $f_2 \in L^\infty(B_R)$. Denote u_1, u_2 respectively, the solutions of equations:

$$\begin{cases} (-\Delta)^{\frac{3}{2}}u_i = f_i & \text{in } B_R, \\ u_i = 0 & \text{in } (B_R)^c, \quad i = 1, 2. \\ (-\Delta)^{\frac{1}{2}}u_i = 0 & \text{on } \partial B_R, \end{cases}$$

It follows from Lemma 2.1 that $e^{k|u_1|} \in L^1(B_R)$. Using elliptic estimates from [4], we get $|u_2|_{L^\infty(B_R)} \leq c$, implying that $e^{k|u_2|} \in L^1(B_R)$. Denoting $u_3 = u - u_1 - u_2$, and observing that Δu_3 is harmonic (see [6]), by the mean value theorem for harmonic functions, we obtain $|\Delta u_3|_{L^\infty(B_{\frac{R}{2}})} \leq c$, implying that $|u_3|_{L^\infty(B_{\frac{R}{4}})} \leq c$. Next, using the equation $(-\Delta)^{\frac{3}{2}}u = (Qe^{3u_1})e^{3u_2+3u_3}$ and elliptic estimates from [4], we get $||(-\Delta)^{\frac{1}{2}}u||_{L^\infty(B_{\frac{R}{8}})} \leq c$, implying that $||u||_{L^\infty(B_{\frac{R}{16}})} \leq c$, and the result follows.

From results of Brezis and Merle [1], it follows that a solution u of equation (**) is bounded from above if it satisfies the equation $-\Delta u = V(x)e^u$ and some other conditions. This result was used to study the qualitative properties and classification of solutions for some second order elliptic equations (see [2, 3, 16]). The following question arise naturally: do there exist a solution u of equation (**) with $\int_{R^3} Qe^{3u} < +\infty$ that is bounded from above? The theorem that follows contains a partial answer to this question.

Theorem 2.5. *Assume that the function $Q(x)$ in (**) is bounded away from 0 and bounded from above, and let u be a C^2 solution of equation (**) satisfying $\int_{R^3} e^{3u} < +\infty$ and $u(x) = o(|x|^2)$. Then $u^+ \in L^\infty(R^3)$.*

To prove the theorem we need a number of lemmas that follow.

Lemma 2.2. *([15]). Let u be a C^2 function on R^4 satisfying:*

(a) Qe^{4u} is in $L^1(R^4)$ with $0 < m \leq Q \leq M$ for some constants m and M;

(b) in the sense of weak derivative, u satisfies the following equation:

$$\Delta u + \frac{2}{\beta_0}\int_{R^4} \frac{Q(y)e^{4u(y)}}{|x-y|^2}dy = 0.$$

Then there is a constant $c > 0$, depending on u, such that $|\Delta u|(x) \leq c$ on R^4, where β_0 is given by $(-\Delta_x)^2\left(ln\frac{1}{|x-y|}\right) = \beta_0\delta_y(x)$.

In fact, $\beta_0 = 8\pi^2$.

Lemma 2.3. *([15]). Let u be a C^2 function on R^4 such that $0 \leq (-\Delta)u(x) \leq A$ on R^4 for some constant A, and let $\int_{R^4} Q(y)e^{4u(y)}dy = \alpha < \infty$ with $0 < m \leq Q \leq M$. Then there exists a constant B, depending only on A, m, M and α such that $u(x) \leq B$ on R^4.*

Lemma 2.4. *Let u be a solution of equation (**), and let*

$$w(x) := \frac{1}{2\pi^2}\int_{R^3} ln\frac{|x-y|}{|y|+1}Q(y)e^{3u(y)}dy.$$

Then there is a constant c such that $w(x) \leq \beta ln(|x|+1) + c$, where

$$\beta = \frac{1}{2\pi^2}\int_{R^3} Q(y)e^{3u(y)}dy.$$

Proof. For $|x| \geq 4$, we decompose $R^3 = A_1 \cup A_2$, where $A_1 = \{y||y-x| \leq \frac{|x|}{2}\}$ and $A_2 = \{y||y-x| \geq \frac{|x|}{2}\}$. For $y \in A_1$, we have $|y| \geq |x| - |x-y| \geq \frac{|x|}{2} \geq |x-y|$, which implies

$$ln\frac{|x-y|}{|y|+1} \leq 0.$$

Since $|x-y| \leq |x| + |y| \leq |x|(|y|+1)$ for $|x|, |y| \geq 2$ and $ln|x-y| \leq ln|x| + c$ for $|x| \geq 4$ and $|y| \leq 2$, we have

$$\begin{aligned} w(x) &\leq \frac{1}{2\pi^2}\int_{A_2} ln\frac{|x-y|}{|y|+1} Q(y)e^{3u(y)}dy \\ &\leq \frac{1}{2\pi^2}\left(\int_{R^3} Q(y)e^{3u(y)}dy\right) ln|x| + c = \beta ln(|x|+1) + c. \end{aligned}$$

Lemma 2.5. *Let u be a solution of equation (**) with $u(x) = o(|x|^2)$. Then $\triangle u(x)$ admits the following integral representation:*

$$\triangle u(x) = -\frac{1}{2\pi^2}\int_{R^3} \frac{Q(y)e^{3u(y)}}{|x-y|^2}dy. \tag{2.2}$$

Proof. Let $v = u + w$. It is obvious that $(-\triangle)^{\frac{3}{2}} v \equiv 0$ in R^3. Using arguments similar to that of applied in Lin [7], for any $x_0 \in R^3$ and $r > 0$, we obtain

$$6\pi^2 r^2 \exp\left(\frac{r^2}{2}\triangle v(x_0)\right) \leq e^{-3v(x_0)}\int_{|x-x_0|=r} e^{3v}d\sigma.$$

Since $v = u + w \leq u(x) + \beta ln|x| + c$, by Lemma 2.4, we have

$$r^{2-3\beta}\exp\left(\frac{\triangle v(x_0)}{2}r^2\right) \in L^1[0, +\infty].$$

Thus, $\triangle v(x_0) \leq 0$ for all $x_0 \in R^3$. By Liouville's theorem, $\triangle v(x) \equiv -c_1$ in R^3 for some constant $c_1 \geq 0$. Hence, we have

$$\triangle u(x) = -\frac{1}{2\pi^2}\int_{R^3} \frac{Q(y)e^{3u(y)}}{|x-y|^2}dy - c_1. \tag{2.3}$$

Now, we claim that $c_1 = 0$. Otherwise, we have $\triangle u(x) \leq -c_1 < 0$ for $|x| \geq R_0$ where R_0 is sufficiently large. Let $h(y) = u(y) + \epsilon|y|^2 + A(|y|^{-2} - R_0^{-1})$, where ϵ is small enough such that for $|y| > R_0$

$$\triangle h(y) = \triangle u + 8\epsilon < -\frac{c_1}{2} < 0, \tag{2.4}$$

and A is sufficiently large so that $\inf_{|y|\geq R_0} h(y)$ is achieved by some $y_0 \in R^3$ with $|y_0| > R_0$. Applying the maximum principle to (2.4) at y_0, we get a contradiction. Hence, our claim is proved.

Proof of Theorem 2.5. By Lemma 2.5 and the proofs of Lemma 2.2 and Lemma 2.3, our conclusion holds.

3. Qualitative properties of solutions of equation $(-\Delta)^{\frac{3}{2}}u = Q(x)e^{3u}$

In this section, we study the qualitative properties of solutions of equation (**). In view of our Theorem 2.5 and Chen [3], we obtain the following theorem.

Theorem 3.1. *Assume that $Q(x)$ is a positive C^1 function bounded away from 0 and from above, and u is a C^2 solution of equation (**) satisfying $\int_{R^3} e^{3u}dx < \infty$ and $u(x) = o(|x|^2)$. Then*

$$-\beta ln(|x|+1) - c \le u(x) \le -\beta ln(|x|+1) + c \quad \text{for some } \beta > 1. \tag{3.1}$$

Furthermore, the following identity holds:

$$\int_{R^3} (x, \nabla Q)e^{3u}dx = 3\pi^2\beta(\beta-2). \tag{3.2}$$

We first prove a number of lemmas.

Lemma 3.1. *Assume that u satisfies the assumptions of Theorem 3.1, then*

$$\frac{w(x)}{ln|x|} \to \beta \text{ uniformly as } |x| \to \infty,$$

where $w(x)$ and β are as in Section 2.

Proof. We need only to verify that

$$I = \int_{R^3} \frac{ln|x-y| - ln(|y|+1) - ln|x|}{ln|x|} Q(y)e^{3u(y)}dy \to 0 \text{ as } |x| \to \infty.$$

Write $I = I_1 + I_2 + I_3$, where I_1, I_2, I_3 stand for integrals over the regions $D_1 = \{y : |x-y| \le 1\}$, $D_2 = \{y : |x-y| > 1 \text{ and } |y| \le k\}$ and $D_3 = \{y : |x-y| > 1 \text{ and } |y| > k\}$, respectively. Assume that $|x| \ge 3$.

(a) To estimate I_1, we simply notice that

$$|I_1| \le C\int_{|x-y|\le 1} Q(y)e^{3u(y)}dy - \frac{1}{ln|x|}\int_{|x-y|\le 1} ln|x-y|Q(y)e^{3u(y)}dy.$$

Then by the boundedness of Qe^{3u} (see Theorem 2.5) and $\int_{R^3} Q(y)e^{3u(y)}dy$, we see that $I_1 \to 0$ as $|x| \to \infty$.

(b) For each fixed k, in region D_2, we have, as $|x| \to \infty$,

$$\frac{ln|x-y| - ln(|y|+1) - ln|x|}{ln|x|} \to 0.$$

Hence, $I_2 \to 0$.

(c) To see that $I_3 \to 0$, we use the fact that

$$\left|\frac{ln|x-y| - ln(|y|+1) - ln|x|}{ln|x|}\right| \le c$$

for $|x-y| > 1$. Then letting $k \to \infty$ the result follows.

Lemma 3.2. *Assume that u satisfies the assumptions of Theorem 3.1. Then*

$$u(x) = \frac{1}{2\pi^2}\int_{R^3} ln\frac{|y|+1}{|x-y|}Q(y)e^{3u(y)}dy + c_0,$$

where c_0 is a constant.

Proof. By Lemma 2.5, we have $\triangle(u+w) = 0$ in R^3, and by Theorem 2.5, we have $u^+ \in L^\infty$. Hence, in view of Lemma 2.4, we conclude that $u + w \leq cln|x| + c$. Since $u + w$ is harmonic function, by the gradient estimates of harmonic functions, we obtain $u(x) + w(x) \equiv c$.

Lemma 3.3. *Assume that u satisfies the assumptions of Theorem 3.1. Then $u(x) \geq -\beta ln(|x|+1) - c$ and $\beta > 1$.*

Proof. By Lemmas 2.4 and 3.2, we obtain

$$u(x) > -\beta ln(|x|+1) - c.$$

From the above inequality and $\int_{R^3} e^{3u}dx < +\infty$, we get $\beta > 1$.

Lemma 3.4. *Assume that u satisfies the assumption of Theorem 3.1. Then $u(x) \leq -\beta ln(|x|+1) + c$.*

Proof. Observe first that for $|x-y| \geq 1$, we have

$$|x| \leq |x-y|(|y|+1).$$

Hence

$$ln|x| - 2ln(|y|+1) \leq ln|x-y| - ln(|y|+1).$$

Therefore, we can write

$$\begin{aligned} w(x) &\geq \frac{1}{2\pi^2}\int_{|x-y|\geq 1}(ln|x| - 2ln(|y|+1))Q(y)e^{3u(y)}dy \\ &+ \frac{1}{2\pi^2}\int_{|x-y|\leq 1}(ln|x-y| - ln(|y|+1))Q(y)e^{3u(y)}dy \\ &\geq \beta ln|x| - \frac{ln|x|}{2\pi^2}\int_{|x-y|\leq 1}Q(y)e^{3u(y)}dy \\ &+ \frac{1}{2\pi^2}\int_{|x-y|\leq 1} ln|x-y|Q(y)e^{3u(y)}dy \\ &- \frac{1}{2\pi^2}\int_{R^3} ln(|y|+1)Q(y)e^{3u(y)}dy = \beta ln|x| + I_1 + I_2 + I_3. \end{aligned}$$

Taking into account the fact that $\frac{u(x)}{ln|x|} \to -\beta$ and $\beta > 1$, and the boundedness of $Q(x)$, we conclude that $I_1, I_2 \to 0$ as $|x| \to \infty$, and I_3 is finite. Therefore $w(x) \geq \beta ln(|x|+1) - c$. Finally, by Lemma 3.2, we have $u(x) \leq -\beta ln(|x|+1) + c$.

Proof of Theorem 3.1. The assertion (3.1) follows from Lemmas 3.3 and 3.4, while the assertion (3.2) follows from Lemma 3.2 and Theorem 1.1 of [18].

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