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POWER SERIES: LOCALIZATION OF SINGULARITIES ON THE BOUNDARY OF THE DISK OF CONVERGENCE

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Abstract. The paper contains results on localization of singularities for the power series (in terms of their coefficients) on boundary arcs of the disk of convergence.

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1. Introduction

1.1. **Notation.** The following standard notation are used throughout this paper. The letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the sets of natural, integer, real and complex numbers, respectively, with known algebraic structure and topology. For $E \subset \mathbb{C}$ the sets $\overline{E}$, E^o and ∂E will denote correspondingly the closure, the interior and the boundary of E. Also, we set:

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$;

$I_{a,b} := [a, b]$, $|I_{a,b}| = b - a \geq 0$; $I^o_{a,b} := (a, b)$, $\left|I^o_{a,b}\right| = b - a > 0$;

$D_{r,c} := \{z \in \mathbb{C} : |z - c| < r\}$ - an open disk of radius $r > 0$ and center at $c \in \mathbb{C}$;

$D_1 := \{z \in \mathbb{C} : |z| < 1\}$ - the *unit* disk; $\partial D_1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ - the *unit* circle;

$l_\theta := \{z = re^{i\theta} : r \in \mathbb{R}_+ \geq 0\}$ - a ray in direction $e^{i\theta}$ for $\theta \in \mathbb{R}$.

For $\alpha \in (0, 2\pi]$, $\beta \in [0, 2\pi)$ and $\mu \in \partial D_1$ denote:

$\Delta_\beta := \{z \in \mathbb{C} : |\arg z| \leq \beta/2\}$ - an angle of opening $\beta \in [0, 2\pi)$ and bisector $\mathbb{R}_+$.

$\gamma_{\beta,\mu} := \{\zeta \in \partial D_1 : |\arg(\zeta/\mu)| \leq \beta/2\}$ - an arc on ∂D_1 of length β centered at μ;

$\gamma^o_{\alpha,\mu} := \{\zeta \in \partial D_1 : |\arg(\zeta/\mu)| < \alpha/2\}$ - an arc on ∂D_1 of length α centered at μ.

For a closed set $E \subset \mathbb{C}$ and a domain $\Omega \subset \mathbb{C}$ denote:

$C(E)$ - the class of continuous functions $f : E \to \mathbb{C}$;

$H(\Omega)$ - the class of holomorphic functions in Ω.

1.2. **The problem 1°.** The notions of an *analytic element* (*element*, for short) and its *analytic continuation* are fundamental in the version of the analytic functions theory, proposed by Karl Weierstrass (see [1], §2). An *element*, centered at $c \in \mathbb{C}$, is a power series with coefficients $\{f_n\}_{n=0}^{\infty} \subset \mathbb{C}$, converging in a disk $D_{r,c}$:

$$f(z) = \sum_{n=0}^{\infty} f_n(z-c)^n \text{ for } z \in D_{r,c}, \tag{1.1}$$

and presenting a holomorphic function $f \in H(D_{r,c})$; conversely, any $f \in H(D_{r,c})$ has a unique expansion of form (1.1).

The main aim of the Weierstrass approach was to investigate the *global* properties of an analytic function f, using the terms of its *local* presentation by an element of form (1.1). This includes, in particular, the problems: on *possibility* of analytic continuation of an element (1.1) to a domain, containing $D_{r,c}$, on *restoration* of that continuation (if is known its *possibility*), on *localization* of possible singularities of (1.1) out of $D_{r,c}$.

Recall that a point $\mu \in \partial D_{r,c}$ is said to be a *regular* point of element (1.1) if there is a *direct* analytic continuation of (1.1) from $D_{r,c}$ to some neighborhood $D_{\rho,\mu}$ of μ; otherwise μ is called a *singular* point of (1.1). Thus, the set of regular points of (1.1) on $\partial D_{r,c}$ is its open subset or is empty. An element (1.1) has a singular point on $\partial D_{r,c}$ if and only if it is a *canonical element*, that is, if r is *the radius* of its convergence, and $\partial D_{r,c}$ is the *natural boundary* of (1.1) if all $z \in \partial D_{r,c}$ are the singular points of (1.1).

The main subject of this paper is the problem on *localization* the singularities of a *canonical* element (1.1) along $\partial D_{r,c}$. Note that it suffice to consider this problem for the *normalized* elements f with the *unit disk* D_1 *of convergence*:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \text{ for } |z| < 1, \qquad \limsup_{n\to\infty} |f_n|^{1/n} = 1. \tag{1.2}$$

The mentioned problem for an element (1.2) was a subject for a number of classical investigations, due to K. Weierstrass, J. Hadamard, E. Fabry, G. Pólya and other authors (see [1], §2). The first results, concerning conditions guaranteeing that ∂D_1 is the *natural boundary* for (1.2), were obtained by K. Weierstrass and J. Hadamard in terms of the *gap* or *lacuna* set of (1.2):

$$\mathbb{L}_f := \{n \in \mathbb{N}_0 : f_n = 0\}. \tag{1.3}$$

E. Fabry has exploited a new approach for investigating the next general problem: find conditions on coefficients f_n of (1.2) (using the terms of their *modulus* and *arguments*), guaranteeing for (1.2) the existence of a *singular* point on a given arc $\gamma \subset \partial D_1$. The solution of this problem was the aim of E. Fabry's known *General Theorem* (see [2] and [1], Theorem 2.1.1). More perfect and strongly proved version of this result has been obtained later by G. Pólya (see [3] and [1], Theorem 2.1.3), using the introduced notions of *minimal* and *maximal* densities for the subsets of

$\mathbb{N}_0$ (see [3] and [4], VI E, Section 3). The obtained solution was satisfactory in case of using the terms of the gap sets (1.3) (see Fabry-Polya theorem on gaps in [1]). Its exactness has been discussed in a number of papers (see [3] and [5], IX B), and was proved in [6]. Mention also the extension of the noted theorem, obtained in [7].

Other principal result of the Fabry's approach was the theorem on *arguments* (see [8] and [1], Theorem 2.3.1), obtained in the special case, when the boundary arc γ actually is reduced to a point. Mention also the papers [9]-[10] on this subject, extending the General Theorem and the noted concepts in different directions.

The aim of this paper is the description of some *necessary* conditions on coefficients of an element (1.2) (expressed in terms of their modulus and arguments), in order a given open arc $\gamma^o \subset \partial D_1$ be an arc of regularity for (1.2). This will allow to find some *sufficient* conditions, guaranteeing the existence on γ^o (and also on γ) of some *singular* points of (1.2), extending the results on singularities, obtained in [9] and [7]. A necessary instrument to solve the above stated problems is the so-called "Coefficient function" method, based on the idea of *interpolation* the coefficients $\{f_n\}_{n=0}^{\infty}$ of (1.2) on the set $\mathbb{N}_0$ by entire functions φ of exponential type with some special properties.

2°. Returning to the normalized element (1.2), we present some preliminary notation and terms. Associate with the coefficients $\{f_n\}_0^\infty \subset \mathbb{C}$ of (1.2) a definite sequence of their *arguments* $\{\omega_n\}_0^\infty$:

$$f_n = |f_n| e^{i\omega_n} \text{ for } n \in \mathbb{N}_0, \tag{1.4}$$

selected by setting $\Delta\omega_n := \omega_n - \omega_{n-1}$ for $n \in \mathbb{N}_0$ with $\omega_{-1} = 0$ and defining each argument ω_n for $n \in \mathbb{N}_0$ *uniquely* by induction via minimizing $|\Delta\omega_n|$ condition:

$$\begin{cases} \Delta\omega_n = 0 \text{ for } n \in \mathbb{L}_f, \\ \Delta\omega_n \in (-\pi, \pi] \text{ for } n \notin \mathbb{L}_f. \end{cases} \tag{1.5}$$

Setting now $\mathbb{N}_0 \backslash \mathbb{L}_f := \{n_k\}_{k=0}^{\infty}$ with $f_{n_k} \neq 0$ for $k \in \mathbb{N}_0$, one can present element (1.2) in the form:

$$f(z) = \sum_{k=0}^{\infty} f_{n_k} z^{n_k} \text{ for } |z| < 1, \qquad \limsup_{k\to\infty} |f_{n_k}|^{1/n_k} = 1. \tag{1.2'}$$

Then it follows from (1.2') and (1.5) that

$$\Delta\omega_{n_k} = \arg(f_{n_k}/f_{n_{k-1}}) \in (-\pi, \pi] \text{ for } k \geq 1, \tag{1.5'}$$

with $\Delta\omega_{n_0} = \omega_{n_0} = \arg f_{n_0} \in (-\pi, \pi]$.

3°. Consider now the related with (1.2) normalized element

$$f^*(z) := f(-z) = \sum_{n=0}^{\infty} f_n^* z^n \text{ for } |z| < 1 \tag{1.6}$$

with coefficients

$$f_n^* := (-1)^n f_n = |f_n|\, e^{i\omega_n^*} \text{ for } n \in \mathbb{N}_0, \tag{1.7}$$

where the *arguments* ω_n^* of f_n^* are defined uniquely by induction. Set for this $\Delta\omega_n^* := \omega_n^* - \omega_{n-1}^*$ for $n \in \mathbb{N}_0$ with $\omega_{-1}^* = 0$, assuming as in (1.5), that $\Delta\omega_n^* = 0$ for $n \in \mathbb{L}_f$ with $\Delta\omega_{n_0}^* = \omega_{n_0}^* = \arg f_{n_0}^* \in (-\pi, \pi]$. Then it follows by (1.5') and (1.7), that $\Delta\omega_{n_k}^*$ can be defined via $\Delta\omega_{n_k}$ uniquely, since either $\Delta\omega_{n_k}^* = \Delta\omega_{n_k}$, or $\Delta\omega_{n_k}^* = (-1)\Delta\omega_{n_k}$ (correspondingly if $n_k - n_{k-1}$ is an even or odd number). In the second case one can choose the sign of $\arg(-1) = \pm\pi$, by setting:

$$\Delta\omega_{n_k}^* = \begin{cases} \Delta\omega_{n_k} + \pi, \text{ if } \Delta\omega_{n_k} \in (-\pi, 0], \\ \Delta\omega_{n_k} - \pi, \text{ if } \Delta\omega_{n_k} \in (0, \pi]. \end{cases} \tag{1.8}$$

Then with the condition $\Delta\omega_n^* = 0$ for $n \in \mathbb{L}_f$ we obtain as in (1.5), that

$$\Delta\omega_n^* \in (-\pi, \pi] \text{ for } n \in \mathbb{N}_0. \tag{1.9}$$

4°. Consider now for the element f in (1.2)-(1.2') and for any interval $I_{p,q} := [p, q]$ with $p, q \in \mathbb{N}_0$ the quantity:

$$V_f(I_{p,q}) := \sum_{n \in (p,q]} |\Delta\omega_n| = \sum_{n_k \in (p,q]} |\Delta\omega_{n_k}|, \tag{1.10}$$

the variation of the coefficient arguments of f *on* $I_{p,q}$. This quantity can be defined also for any interval $I_{a,b} \subset \mathbb{R}_+$ by formula:

$$V_f(I_{a,b}) = \max\{V_f(I_{p,q}) : I_{p,q} \subset I_{a,b}\}, \tag{1.11}$$

assuming $V_f(I_{a,b}) = 0$, if there is no any $I_{p,q} \subset I_{a,b}$ with $p, q \in \mathbb{N}_0$, $p < q$. Then by (1.5) and (1.10)-(1.11), we have

$$V_f(I_{a,b}) \le \pi\, |I_{a,b}|. \tag{1.12}$$

Introduce also the quantity:

$$\nu_f(a, b) = V_f(I_{a,b}) / |I_{a,b}| \in [0, \pi], \tag{1.13}$$

the mean variation of the coefficient arguments of f in (1.2) on $I_{a,b}$.

Definition 1.1. *The introduced in (1.10)-(1.13) quantities $V_f(I_{a,b})$ with $\nu_f(a, b)$ for the element f in (1.2) can be defined also for the element $z \to f(-z) = f^*(z)$ in (1.6)-(1.7), replacing there the arguments $\{\omega_n\}_0^\infty$ by $\{\omega_n^*\}_0^\infty$, defined in (1.7)-(1.9) for f^*, and correspondingly replacing in (1.10)-(1.13), $V_f(I_{a,b})$ by $V_f^*(I_{a,b})$ and $\nu_f(a, b)$ by $\nu_f^*(a, b)$.*

The rest of the paper is organized as follows: Sections 2 and 3 contain the descriptions of some necessary auxiliary notions and results, including the above noted "Coefficient function"method with some new necessary constructions. The statements and proofs of the main results are presented in Section 4.

2. Auxiliary results

2.1. Entire functions of exponential type. *The class $\mathcal{E}$ of entire functions φ of exponential type* (see [11]-[13]) is defined by condition:

$$\sigma_\varphi :=\limsup_{|z|\to\infty}\{|z|^{-1}\log^+|\varphi(z)|\}<+\infty,$$

where σ_φ is the *exponential type* of φ. Then $\varphi,\psi\in\mathcal{E}$ implies $\varphi+\psi\in\mathcal{E}$ and $\varphi\psi\in\mathcal{E}$.

The main characteristic for the behavior of any $\varphi\in\mathcal{E}$ along a ray l_θ for $\theta\in\mathbb{R}$ is the *exponential indicator* (*indicator*, for short) function h_φ of φ (see [11]-[13]), defined by formula (excluding the case $\varphi\equiv 0$):

$$h_\varphi(\theta)=\limsup_{r\to\infty}\left\{r^{-1}\log\left|\varphi(re^{i\theta})\right|\right\}\quad \text{for } \theta\in\mathbb{R}. \tag{2.1}$$

In this terms the growth and decrease on $\mathbb{C}$ of any $\varphi\in\mathcal{E}$ is restricted by the inequality:

$$\left|\varphi(re^{i\theta})\right|<\exp\{r[h_\varphi(\theta)+\varepsilon(r)]\}\quad \text{for } re^{i\theta}\in\mathbb{C}, \tag{2.2}$$

where $\varepsilon(r)\to 0$ as $r\to+\infty$, uniformly for $\theta\in\mathbb{R}$. Below some properties of the exponential indicator $h=h_\varphi$ are presented.

(a) h_φ is a real valued, 2π - periodic and bounded by σ_φ function:$-\sigma_\varphi\le h_\varphi(\theta)\le\sigma_\varphi$ for $\theta\in\mathbb{R}$.

(b) For $\varphi,\psi\in\mathcal{E}$ and any $\theta\in\mathbb{R}$ the following inequalities are satisfied:

$$h_{\varphi\psi}(\theta)\le h_\varphi(\theta)+h_\psi(\theta) \text{ and } h_{\varphi+\psi}(\theta)\le\max\{h_\varphi(\theta),h_\psi(\theta)\}. \tag{2.3}$$

(c) h_φ has the property of *trigonometric convexity*, (see [13]): *for any triple $\theta_1<\theta<\theta_2$ with $\theta_2-\theta_1<\pi$ the following inequality is satisfied:*

$$h_\varphi(\theta)\sin(\theta_2-\theta_1)\le h_\varphi(\theta_1)\sin(\theta_2-\theta)+h_\varphi(\theta_2)\sin(\theta-\theta_1). \tag{2.4}$$

This property implies the continuity of the indicator $h_\varphi:\mathbb{R}\to[-\sigma_\varphi,\sigma_\varphi]$ and, in addition, the following property:

(d) at each point $\theta\in\mathbb{R}$, h_φ has *one sided derivatives* $h'_\varphi(\theta_-)$ (from the left) and $h'_\varphi(\theta_+)$ (from the right), satisfying:

$$h'_\varphi(\theta_-)\le h'_\varphi(\theta_+) \text{ for } \theta\in\mathbb{R}, \tag{2.5}$$

where the equality holds, that is, actually the derivative $h'_\varphi(\theta)$ exists, except may be only a *countable* set of points $\theta\in\mathbb{R}$.

Definition 2.1. *Denote by $\mathcal{E}_0$ the subclass of the functions $\varphi\in\mathcal{E}$ with the exponential indicator h_φ, satisfying the condition $h_\varphi(0)=0$.*

(e) For any $\varphi \in \mathcal{E}_0$ and $\delta \in (0, \pi)$ the following inequality is satisfied (see [7]):

$$h_\varphi(\theta) \leq c_\varphi(\delta)\,|\sin\theta| \text{ for } |\theta| \leq \delta, \tag{2.6}$$

where for $\delta \in (0, \pi/2)$

$$c_\varphi(\delta) := \frac{\max\{h_\varphi(-\delta), h_\varphi(\delta)\}}{\sin\delta} \geq 0.$$

Actually, (2.6) follows from property (c) with (2.4) for the triples $0 < \theta < \delta$ and $-\delta < \theta < 0$, while (2.5) for the triple $-\delta < 0 < \delta$ with $\delta \in (0, \pi/2)$ states that $h_\varphi(\delta) + h_\varphi(-\delta) \geq 0$, implying $c_\varphi(\delta) \geq 0$.

(f) For any $\varphi \in \mathcal{E}_0$ and $\delta \in (0, \pi)$ from (2.2) and (2.6) we obtain the asymptotic inequality

$$|\varphi(z)| < \exp\{c_\varphi(\delta)\,|\mathrm{Im}\, z| + |z|\,\varepsilon(|z|)\} \text{ for } z \in \Delta_{2\delta}, \tag{2.7}$$

where $\varepsilon(|z|) \to 0$ as $|z| \to +\infty$.

(g) It follows from properties (d)-(e) that $c_\varphi(\delta)$ in (2.6) with $\varphi \in \mathcal{E}_0$ has the limit, as $\delta \to 0$:

$$\lim_{\delta\to 0} c_\varphi(\delta) = \limsup_{\theta\to 0} \{h_\varphi(\theta)/\,|\theta|\} := m_\varphi \geq 0, \tag{2.8}$$

where

$$m_\varphi := \max\{-h'_\varphi(0_-), h'_\varphi(0_+)\}. \tag{2.9}$$

Remark 2.1. *It follows from (2.9) and (2.5) that*

$$-m_\varphi \leq h'_\varphi(0_-) \leq h'_\varphi(0_+) \leq m_\varphi, \tag{2.10}$$

so that $m_\varphi \geq 0$. Here $m_\varphi = 0 \iff h'_\varphi(0_-) = h'_\varphi(0_+) = 0$, that is, if there exists the derivative $h'_\varphi(0) = 0$. Also, (2.12) implies the following, more stronger than (2.11), relation for m_φ :

$$m_\varphi = \max\{|h'_\varphi(0_-)|, |h'_\varphi(0_+)|\}, \tag{2.9'}$$

with the equivalent to (2.8) equality$\lim_{\delta\to 0} c_\varphi(\delta) = m_\varphi = \limsup_{\theta\to 0} |h_\varphi(\theta)/\theta|$.

2.2. **The "Coefficient function" method.** 1°. This method is the main tool for analyzing the behavior of a normalized element (1.2) via *interpolation* the coefficients $\{f_n\}_{n=0}^{\infty}$ of (1.2) on the set $\mathbb{N}_0$ by a function $\varphi \in \mathcal{E}$ or $\varphi \in H(\Delta_\beta)$ of exponential growth:

$$\varphi(n) = f_n \text{ for } n \in \mathbb{N}_0. \tag{2.11}$$

The method was used mainly to establish a criterion on possibility of the *analytic continuation* of (1.2) outside the unit disk D_1 in terms of φ (see [1], §7, [14], Chapter X and [15]-[18]). In particular, there is such a criterion for the element

(1.2) on *regularity* of some open arcs of the unit circle ∂D_1, which can be applied in problems on localization of *singularities* of (1.2) on ∂D_1 (see [10] and [7]).

Next, as a "Coefficient function" for (1.2) and for the related series we will use the functions from $\mathcal{E}_0$. The following criterion (see [14] and [7], Theorem 2) on regularity for (1.2) of an *open* proper subarc of ∂D_1 is essential for us.

Criterion 2.1. *The open arc $\partial D_1 \backslash \Delta_\beta$ on ∂D_1 for $\beta \in [0, 2\pi)$ is an arc of regularity for the normalized element (1.2) if and only if there is a function $\varphi \in \mathcal{E}_0$, satisfying the interpolation conditions (2.11) and the condition*

$$\limsup_{\theta \to 0} \{h_\varphi(\theta) / |\theta|\} := m_\varphi \leq \beta/2. \tag{2.12}$$

Note that by Remark 2.1, $m_\varphi \geq 0$ and m_φ can be defined also by formula (2.9).

2^o. Let now $\beta := 2\pi - \alpha \in [0, 2\pi)$ in (1.15) with $\alpha \in (0, 2\pi]$, so that actually $\partial D_1 \backslash \Delta_\beta := \gamma^o_{\alpha,-1}$ is an *open* arc of ∂D_1 of length α and center -1. Then the open arc $\gamma^o_{\alpha,1} \subset \partial D_1$ of length α and center 1 will be an arc of regularity for the element f in (1.2) if and only if $\partial D_1 \backslash \Delta_\beta$ is an arc of regularity for the element $z \to f(-z)$ with coefficients $(-1)^n f_n$ for $n \in \mathbb{N}_0$, and in view of Criterion 2.1 and Remark 2.1 we get the following result.

Corollary 2.1. *The open arc $\gamma^o_{\alpha,1} \subset \partial D_1$ of length $\alpha \in (0, 2\pi]$ and center 1 is an arc of regularity for the normalized element f in (1.2) if and only if there is a function $\varphi \in \mathcal{E}_0$, satisfying the interpolation conditions:*

$$(-1)^n \varphi(n) = f_n \text{ for } n \in \mathbb{N}_0, \tag{2.13}$$

and the condition (2.12) with $\beta = 2\pi - \alpha$, where m_φ can be defined also by formulas (2.11) or (2.9'). If $\alpha = 2\pi$, that is, $\beta = 0$, implying $m_\varphi = 0$, then there exists the derivative $h'_\varphi(0) = 0$.

Note that the interpolation conditions (2.13) are more adapted for the applications of Lemma 3.1 in Subsection 3.1. The next Remark 2.2 is adapting for this also the interpolation conditions (2.11) in Criterion 1.

Remark 2.2. *The interpolation conditions (2.11) can also be written as follows:*

$$(-1)^n \varphi(n) = f^*_n \quad \text{for } n \in \mathbb{N}_0, \tag{2.11'}$$

with $f^*_n := (-1)^n f_n = |f_n| e^{i\omega^*_n}$, allowing to use the selected in (1.7)-(1.9) arguments ω^*_n of f^*_n.

3^o. Consider now the open arc $\gamma^o_{\alpha,\mu} \subset \partial D_1$ of length $\alpha \in (0, 2\pi]$ and center $\mu = e^{i\lambda}$ with $\lambda \in (-\pi, \pi)$, so that $\mu \neq -1$. Then obviously $\gamma^o_{\alpha,\mu}$ will be an open arc of regularity for the element f in (1.2) if and only if $\gamma^o_{\alpha,1}$ is an arc of regularity for the normalized element $z \to f(\mu z)$ with coefficients $\{e^{i\lambda n} f_n\}_{n=0}^{+\infty}$. By Corollary 2.1, the necessary and sufficient condition for this is the existence of a function $\varphi_\lambda \in \mathcal{E}_0$ with $m_{\varphi_\lambda} \le \pi - \alpha/2$, satisfying by (2.16) the interpolation conditions:

$$(2.13') \qquad (-1)^n \varphi_\lambda(n) = e^{i\lambda n} f_n \quad \text{for} \quad n \in \mathbb{N}_0.$$

Next, consider the function $\varphi \in \mathcal{E}_0$, defined by formula

$$(2.14) \qquad \varphi(z) = \varphi_\lambda(z) \exp(-i\lambda z) \text{ for } z = re^{i\theta} \in \mathbb{C},$$

satisfying the interpolation conditions (2.13). In addition, by (2.14), we have

$$h_\varphi(\theta) = h_{\varphi_\lambda}(\theta) + \lambda \sin\theta \quad \text{for} \quad \theta \in \mathbb{R},$$

so that $\varphi \in \mathcal{E}$ with $h_\varphi(0) = h_{\varphi_\lambda}(0) = 0$, that is, $\varphi \in \mathcal{E}_0$. Since $h'_{\varphi_\lambda}(0_\pm) = h'_\varphi(0_\pm) - \lambda$, it follows from (2.9) and Remark 2.1 for φ_λ that

$$(2.15) \qquad m_{\varphi_\lambda} := \max\{\lambda - h'_\varphi(0_-), h'_\varphi(0_+) - \lambda\}.$$

Then by (2.15), either $-h'_\varphi(0_-) = m_{\varphi_\lambda} - \lambda$ with $h'_\varphi(0_+) \le m_{\varphi_\lambda} + \lambda$, or alternatively $-h'_\varphi(0_-) \le m_{\varphi_\lambda} - \lambda$ with $h'_\varphi(0_+) = m_{\varphi_\lambda} + \lambda$, so that

$$(2.16) \qquad m_\varphi = \max\{m_{\varphi_\lambda} - \lambda, m_{\varphi_\lambda} + \lambda\} = m_{\varphi_\lambda} + |\lambda|.$$

Now, by Corollary 2.1 and (2.16), the above condition (2.12) for φ_λ is equivalent to the next condition for φ :

$$(2.17) \qquad m_\varphi \le \pi - \alpha/2 + |\lambda|.$$

In addition, if $\alpha = 2\pi$, that is, $\beta = 0$, then by (2.12) it follows that $m_{\varphi_\lambda} = 0$, implying by Corollary 2.1 the existence of the derivative $h'_{\varphi_\lambda}(0) = 0$, that is, the existence of the derivative $h'_\varphi(0) = \lambda$, and by (2.16) the equation $m_\varphi = |\lambda|$.

From the above discussion we come to the following criterion.

Criterion 2.2. *The open arc $\gamma^o_{\alpha,\mu} \subset \partial D_1$ or the symmetric open arc $\gamma^o_{\alpha,-\mu} \subset \partial D_1$ of length $\alpha \in (0, 2\pi]$ and center μ or $-\mu$ with $\mu = e^{i\lambda}$ for $\lambda \in (-\pi, \pi)$ will be an arc of regularity for the element (1.2) if and only if there is a function $\varphi \in \mathcal{E}_0$, satisfying the condition (2.17) with (2.9) and the interpolation conditions: (2.13) for the center μ and (2.11) for $-\mu$. In both cases it follows for $\alpha = 2\pi$ the existence of $h'_\varphi(0) = \lambda$ with $m_\varphi = |\lambda|$.*

Actually, the Criterion 2 for the center μ follows from Corollary 1 with (2.15)-(2.17). The case with center $-\mu$ follows from the case for μ, applied to the related with (1.2) element $z \to f(-z)$ in (1.16)-(1.17) (see also Remark 2.2).

3. Two auxiliary analytic functions

3.1. The auxiliary function φ_η for a "Coefficient function" φ. A function $\psi : I_{a,b} \to \mathbb{R}$ is called *real analytic* if ψ has a locally convergent power series expansion around any center $c \in I_{a,b}$, guaranteeing for ψ a unique real analytic continuation on some *open* interval, containing $I_{a,b}$. If, in addition, $\psi(c) = 0$ for some $c \in I_{a,b}$, then it follows (excluding the case $\psi \equiv 0$ on $I_{a,b}$) the existence of some $m \in \mathbb{N}$ and a real analytic on $I_{a,b}$ function ψ_c with $\psi_c(c) \neq 0$, such that

$$\psi(x) = (x-c)^m \psi_c(x) \quad \text{for } x \in I_{a,b},$$

that is, c is a *zero* of the function ψ *of multiplicity* m. Denote by $\mathbf{n}_\psi(I_{a,b})$ the *total* number of zeros of ψ on $I_{a,b}$, taking also into account their multiplicity. The next lemma on estimation of $\mathbf{n}_\psi(I_{a,b})$ will be useful for us (see Lemma 3 in [9]).

Lemma 3.1. *Let ψ be a real analytic function on the interval $I_{p,q}$ with $p, q \in \mathbb{Z}$, and let $\mathbf{w}_\psi(I_{p,q})$ be the number of the sign changes in the finite sequence $(-1)^n\psi(n)$ for $n \in I_{p,q} \cap \mathbb{Z}$. Then $\mathbf{n}_\psi(I_{p,q}) \geq |I_{p,q}| - \mathbf{w}_\psi(I)$ with $|I_{p,q}| = q - p$.*

Now our aim is to use the above Criterion 2.2 to describe the conditions on *regularity* for normalized element (1.2) the open arc $\gamma^o_{\alpha,\mu} \subset \partial D_1$ of length $\alpha \in (0, 2\pi]$ and center $\mu = e^{i\lambda}$ for $\lambda \in (-\pi, \pi)$ in terms of the function $\varphi \in \mathcal{E}_0$, satisfying the necessary and sufficient conditions (2.13) and (2.17). Present for this the condition (2.13) in terms (1.4)-(1.5) of the modulus and arguments of the coefficients of (1.2):

$$(-1)^n \varphi(n) = f_n = |f_n| e^{i\omega_n} \quad \text{for } n \in \mathbb{N}_0. \tag{3.1}$$

Since in general φ is not real valued on any interval $I_{a,b} \subset \mathbb{R}_+$, then to apply Lemma 3.1, we need to replace φ by another function with this property and closely related to φ.

Associate with $\varphi \in \mathcal{E}_0$ from Criterion 2 the function $\varphi_\eta \in \mathcal{E}$ with the parameter $\eta \in (0, \pi)$:

$$\varphi_\eta(z) = [\varphi(z)e^{i\eta} + \tilde{\varphi}(z)e^{-i\eta}]/2 \quad \text{for } z \in \mathbb{C}, \tag{3.2}$$

where

$$\tilde{\varphi}(z) := \overline{\varphi(\overline{z})} \quad \text{for } z \in \mathbb{C},$$

so that $\tilde{\varphi} \in \mathcal{E}_0$ with $h_{\tilde{\varphi}}(\theta) = h_\varphi(-\theta)$ for $\theta \in \mathbb{R}$. Then φ_η is *real valued* on $\mathbb{R}$ for any $\eta \in (0, \pi)$:

$$\varphi_\eta(x) = \mathrm{Re}[\varphi(x)e^{i\eta}] \quad \text{for } x \in \mathbb{R}, \tag{3.3}$$

satisfying by (3.1) and (3.3) the interpolation conditions:

$$(3.4) \qquad (-1)^n \varphi_\eta(n) = \mathrm{Re}(f_n e^{i\eta}) = |f_n| \cos(\omega_n + \eta) \quad \text{for } n \in \mathbb{N}_0.$$

To estimate the growth of φ_η on $\mathbb{C}$, note that by (3.2),

$$|\varphi_\eta(z)| \le \max\{|\varphi(z)|, |\varphi(\overline{z})|\} \quad \text{for } z \in \mathbb{C},$$

and from (2.7) with $c_\varphi(\delta)$ in (2.6) it follows for $\delta \in (0, \pi)$ the asymptotic inequality

$$(3.5) \qquad |\varphi_\eta(z)| < \exp\{c_\varphi(\delta) |\mathrm{Im}\, z| + |z| \varepsilon(|z|)\} \text{ for } z \in \Delta_{2\delta},$$

where $\varepsilon(|z|) \to 0$ as $|z| \to +\infty$, uniformly for $\eta \in (0, \pi)$.

3.2. **Estimation of the number of zeros of φ_η on $I_{a,b} \subset \mathbb{R}_+$.** Apply now Lemma 3.1 to the real analytic function $\psi := \varphi_\eta \mid \mathbb{R}$ in (3.3) on any interval $I_{p,q}$ with $p, q \in \mathbb{N}_0$, noting that the quantity $\mathbf{w}_\psi(I_{p,q})$ for the function (3.3) is equal by (3.4) to the number of the *sign changes* $\mathbf{s}_f(\eta, I_{p,q})$ in the finite sequence $\{\mathrm{Re}(f_n e^{i\eta})\}$ for $n \in I_{p,q} \cap \mathbb{N}_0$ with any fixed $\eta \in (0, \pi)$, where $\{f_n\}_0^\infty$ are the coefficients of the element f in (1.2). Then for the number $\mathbf{n}_{\varphi_\eta}(I_{p,q})$ of zeros of φ_η on $I_{p,q}$, it follows with $|I_{p,q}| = q - p$ the inequality:

$$(3.6) \qquad \mathbf{n}_{\varphi_\eta}(I_{p,q}) \ge |I_{p,q}| - \mathbf{s}_f(\eta, I_{p,q}) \quad \text{for } \eta \in (0, \pi).$$

To use this estimate, we need some additional information on the character of dependence of $\mathbf{s}_f(\eta, I_{p,q})$ on the parameter $\eta \in (0, \pi)$. Consider for this with the coefficients $\{f_n\}_{n=0}^\infty$ of (1.2) also the subsequence $\{f_{n_k}\}_{k=0}^\infty$ for $\{n_k\}_{k=0}^\infty = \mathbb{N}_0 \backslash \mathbb{L}_f$ of the *nonzero* coefficients of (1.2) as in (1.2'), with the subsequence $\{\omega_{n_k}\}_{k=0}^\infty$ of their arguments, satisfying the condition (1.5'). Then, we have

$$(3.7) \qquad \mathbf{s}_\eta(f, I_{p,q}) = \sum_{n_k \in (p,q]} \mathbf{s}_f(\eta, I_{n_k}) \quad \text{for } \eta \in (0, \pi),$$

where $I_{n_k} = [n_{k-1}, n_k]$ for $k \in \mathbb{N}$, and setting

$$(3.8) \qquad e_{n_k} = \{\eta \in (0, \pi) : \mathbf{s}_f(\eta, I_{n_k}) = 1\},$$

with $\mathbf{s}_f(\eta, I_{n_k}) = 0$ for $\eta \in (0, \pi) \backslash e_{n_k}$, we have by (3.4) that $\eta \in e_{n_k} \subset (0, \pi)$ if and only if

$$(3.9) \qquad \cos(\omega_{n_{k-1}} + \eta) \cos(\omega_{n_k} + \eta) < 0.$$

2°. Our next aim is *to calculate* the length $|e_{n_k}|$ of the set e_{n_k} in (3.8). Note for this that by (3.9), the condition $\Delta\omega_{n_k} = 0$ (that is, $\omega_{n_k} = \omega_{n_{k-1}}$) implies $e_{n_k} = \varnothing$ with $|e_{n_k}| = 0$. Let us show that conversely the condition $\Delta\omega_{n_k} \ne 0$ implies that $|e_{n_k}| = |\Delta\omega_{n_k}| \in (0, \pi]$.

To this end, first note that if $\Delta\omega_{n_k} = \pi$, that is, $\omega_{n_k} = \omega_{n_{k-1}} + \pi$, then (3.9) is equivalent to the condition $\cos^2(\omega_{n_k} + \eta) > 0$, so that either $e_{n_k} = (0, \pi)$, or $e_{n_k} = (0,\pi)\backslash\{\eta_0\}$ for some $\eta_0 \in (0,\pi)$ with $\cos(\omega_{n_k} + \eta_0) = 0$, implying $|e_{n_k}| = \pi = \Delta\omega_{n_k}$ in both cases.

Consider now the case $\Delta\omega_{n_k} \neq 0$ with $\Delta\omega_{n_k} \neq \pi$, that is, if $|\Delta\omega_{n_k}| \in (0,\pi)$ by (1.5), and let us see that then always we have $|e_{n_k}| = |\Delta\omega_{n_k}|$. Setting for this$\omega'_{n_k} = \min\{\omega_{n_{k-1}}, \omega_{n_k}\}$ and $\omega''_{n_k} = \max\{\omega_{n_{k-1}}, \omega_{n_k}\}$, consider the open interval $(\omega'_{n_k}, \omega''_{n_k})$ with $|\Delta\omega_{n_k}| = \omega''_{n_k} - \omega'_{n_k} \in (0,\pi)$. Denote by X the set of zeros of cosine function, that is, $X = \{x_m\}_{m=-\infty}^{+\infty}$ with $x_m = \pi/2 + \pi m$ and $m \in \mathbb{Z}$.

Remark 3.1. *Let $|\Delta\omega_{n_k}| \in (0,\pi)$. The condition (3.9) will be satisfied for some $\eta \in (0,\pi)$ if and only if there is (a unique) $x_p \in X$ for some $p \in \mathbb{Z}$, satisfying $x_p \in I^o_\eta := (\omega'_{n_k} + \eta, \omega''_{n_k} + \eta)$ with $\left|I^o_\eta\right| = |\Delta\omega_{n_k}| < \pi$.*

The existence of x_p and its uniqueness follows from (3.9) by the mean value property of $\cos x$ for $x \in I^o_\eta$ and by condition $\left|I^o_\eta\right| < \pi$. Conversely, for any such $x_p \in X \cap I^o_\eta$ with $\left|I^o_\eta\right| < \pi$ the condition (3.9) will be satisfied.

3°. Now to calculate $|e_{n_k}|$ in case $|\Delta\omega_{n_k}| \in (0,\pi)$, there are two alternative cases: a) $(\omega'_{n_k}, \omega''_{n_k}) \cap X \neq \varnothing$ and b) $(\omega'_{n_k}, \omega''_{n_k}) \cap X = \varnothing$. For both cases there is $x_m \in X$ with $m \in \mathbb{Z}$, such that the following inequalities are satisfied:

$$x_{m-1} \leq \omega'_{n_k} < x_m < \omega''_{n_k} < x_{m+1} \tag{3.10}$$

in the case a), and

$$x_{m-1} \leq \omega'_{n_k} < \omega''_{n_k} \leq x_m. \tag{3.11}$$

in the case b).

By Remark 3.1, the condition (3.9) will be satisfied in both cases a) and b) for some $\eta \in (0,\pi)$ if there is a unique $x_p \in I^o_\eta$, satisfying the inequality:

$$x_{p-1} < \omega'_{n_k} + \eta < x_p < \omega''_{n_k} + \eta < x_{p+1}. \tag{3.12}$$

Namely, (3.12) will be satisfied in the case a) with (3.10) if and only if

$$\eta \in \begin{cases} e'_{n_k} = (0, x_m - \omega'_{n_k}) \text{ for } p = m, \\ e''_{n_k} = (x_{m+1} - \omega''_{n_k}, \pi) \text{ for } p = m+1, \end{cases}$$

so that $e_{n_k} = e'_{n_k} \cup e''_{n_k}$ with $e'_{n_k} \cap e''_{n_k} = \varnothing$ and $\left|e'_{n_k}\right| = x_m - \omega'_{n_k}$, $\left|e''_{n_k}\right| = \omega''_{n_k} - x_m$, implying $|e_{n_k}| = |\Delta\omega_{n_k}|$. In the case b) with (3.11), the inequality (3.12) will be satisfied for $p = m$ and $\eta \in e_{n_k} := (x_m - \omega''_{n_k}, x_m - \omega'_{n_k})$ with $|e_{n_k}| = |\Delta\omega_{n_k}|$.

Conclusion 3.1. *The equivalence of (3.8) and (3.9), and the above cases a) and b) imply that always $|e_{n_k}| = |\Delta\omega_{n_k}|$, so that $e_{n_k} \neq \varnothing \Leftrightarrow |\Delta\omega_{n_k}| \in (0,\pi]$ and $e_{n_k} = \varnothing \Leftrightarrow \Delta\omega_{n_k} = 0$ by openness of e_{n_k}.*

Using now (3.7)-(3.9) and Conclusion 3.1, one can calculate for an interval $I_{p,q}$ with $p, q \in \mathbb{N}_0$ the integral

$$\mathfrak{I}_f(I_{p,q}) := \int_0^\pi \mathrm{s}_f(\eta, I_{p,q})d\eta = \sum_{n_k \in (p,q]} \int_{e_{n_k}} \mathrm{s}_f(\eta, I_{n_k})d\eta,$$

so that

$$\mathfrak{I}_f(I_{p,q}) = \sum_{n_k \in (p,q]} |e_{n_k}| = \sum_{n_k \in (p,q]} |\Delta\omega_{n_k}|. \tag{3.13}$$

Then, by (1.10) it follows that

$$V_f(I_{p,q}) := \sum_{n \in (p,q]} |\Delta\omega_n| = \sum_{n_k \in (p,q]} |\Delta\omega_{n_k}| = \mathfrak{I}_f(I_{p,q}). \tag{3.14}$$

Finally, integrating the inequality (3.6) by $\eta \in (0, \pi)$ and taking into account (3.7) with (3.13)-(3.14), we obtain the inequality:

$$\int_0^\pi \mathrm{n}_{\varphi_\eta}(I_{p,q})d\eta \geq \pi[|I_{p,q}| - V_f(I_{p,q})]. \tag{3.15}$$

Corollary 3.1. *It follows from (3.15) that for any $I_{a,b} \subset \mathbb{R}_+$ with $a \in \mathbb{N}$ or $b \in \mathbb{N}$ the following inequality is satisfied:*

$$\int_0^\pi \mathrm{n}_{\varphi_\eta}(I_{a,b})d\eta \geq \max\{0, [\pi(|I_{a,b}| - 1) - V_f(I_{a,b})]\}, \tag{3.16}$$

where $V_f(I_{a,b})$ is the variation on $I_{a,b}$ of the arguments $\{\omega_n\}_0^\infty$ of coefficients $\{f_n\}_0^\infty$ of (1.2), defined in (1.11)-(1.12).

Actually, if $|I_{a,b}| \geq 1$ with $a \in \mathbb{N}$ or $b \in \mathbb{N}$, then there is $I_{p,q} \subset I_{a,b}$ with $p, q \in \mathbb{N}_0$ and $p < q$, so that $V_f(I_{p,q}) \leq V_f(I_{a,b})$ and $|I_{p,q}| \geq |I_{a,b}| - 1 \geq 0$.

Remark 3.2. *Applying Corollary 2 for the normalized element (1.6) with coefficients $\{f_n^*\}_0^\infty$ in (1.7) and their arguments $\{\omega_n^*\}_0^\infty$, defined in (1.7)-(1.9), then using the function φ_η from (3.2) for corresponding $\varphi \in \mathcal{E}_0$ in Criterion 2.2, in Corollary 3.1 instead of (3.16), we obtain the inequality:*

$$\mathrm{n}_{\varphi_\eta}(I_{a,b})d\eta \geq \max\{0, [\pi(|I_{a,b}| - 1) - V_f^*(I_{a,b})]\}, \tag{3.16'}$$

where $V_f^*(I_{a,b})$ is the variation on $I_{a,b}$ of the arguments $\{\omega_n^*\}_0^\infty$ in (1.6)-(1.9) (see Definition 1).

3.3. **An auxiliary Blaschke product. 1°**. Consider the closed disk $\overline{D}_{r,1}$ of the radius $r \in (0,1)$ and center 1 with the diameter $I(r) := I_{1-r,1+r}$. Below are presented some notation, related with $\overline{D}_{r,1}$.

We first introduce the family of closed subintervals $\{I_\tau^r\}$ of $I(r)$ for $\tau \in I^o(r)\backslash\{1\}$, by setting

$$I_\tau^r = \begin{cases} I_{\tau,1+r}, & \text{if } \tau < 1, \\ I_{1-r,\tau}, & \text{if } \tau > 1, \end{cases} \tag{3.17}$$

so that $1 \in (I_\tau^r)^o$ and $|I_\tau^r| = r + |1-\tau|$. Next, for any $\tau \in I^o(r)\backslash\{1\}$ denote by $\mathbb{D}_\tau^r$ the closed disk with diameter I_τ^r :

(3.18) $$\mathbb{D}_\tau^r := \overline{D}_{r_\tau, c_\tau} \subset \overline{D}_{r,1},$$

where the radius $r_\tau < r$ and the center $c_\tau \in (I_\tau^r)^o\backslash\{1\}$ of $\mathbb{D}_\tau^r$ can be presented with $s_\tau = sgn(1-\tau)$ by formulas:

(3.19) $$r_\tau := (r + |1-\tau|)/2 \text{ and } c_\tau := 1 + s_\tau(r - r_\tau).$$

Also, let g_τ for $\tau \in (I_\tau^r)^o\backslash\{1\}$ be the *Green's function* of $\mathbb{D}_\tau$ in (3.18)-(3.19):

(3.20) $$g_\tau(z,\zeta) = -\log|b_\tau(z,\zeta)| \text{ for } z, \zeta \in \mathbb{D}_\tau,\ z \neq \zeta,$$

where $b_\tau(z,\zeta)$ is the following *Blaschke factor* for $\mathbb{D}_\tau^r$:

(3.21) $$b_\tau(z,\zeta) := r_\tau(z-\zeta)/l_\tau(z,\overline{\zeta})$$

with $l_\tau(z,\zeta) = r_\tau^2 - (\zeta - c_\tau)(z - c_\tau)$, satisfying

(3.22) $$|b_\tau(z,\zeta)| \equiv 1 \text{ for } z \in \partial\mathbb{D}_\tau^r \text{ and } \zeta \in (\mathbb{D}_\tau^r)^o.$$

2°. We use mainly the special case $g_\tau(z) := g_\tau(z,1)$ of g_τ for $z \in \mathbb{D}_\tau\backslash\{1\}$:

(3.23) $$g_\tau(z) = \log\frac{|l_\tau(z)|}{r_\tau|z-1|} \text{ for } z \in \mathbb{D}_\tau^r\backslash\{1\},$$

where $l_\tau(z) := l_\tau(z,1)$ is a non-constant linear function of z, with real valued restriction $l_\tau(t)$ for $t \in I_\tau^r$:

(3.24) $$l_\tau(t) = r_\tau^2 + (c_\tau - 1)(t - c_\tau) = l_\tau(1) + l_\tau'(1)(t-1),$$

increasing on I_τ^r for $\tau < 1$ and *decreasing* for $\tau > 1$ by (3.19), since $l_\tau'(t) \equiv c_\tau - 1$. Thus, we have

(3.25) $$l_\tau(\tau) \le l_\tau(t) \le l_\tau(\nu_\tau) \text{ for } t \in I_\tau^r,$$

where $\nu_\tau = 1 + rs_\tau$ is the opposite to τ endpoint of I_τ^r. Using (3.18)-(3.19), we find from (3.25)

(3.26) $$l_\tau(\tau) = r_\tau|1-\tau|,\ l_\tau(\nu_\tau) = rr_\tau,\quad l_\tau(1) = r|1-\tau|,\ |l_\tau'(1)| = r - r_\tau.$$

It follows from (3.25)-(3.26) that $l_\tau(t) > 0$ for $t \in I_\tau(r)$. Then from (3.23)-(3.24) we have for $t \in I_\tau^r\backslash\{1\}$ that $p_\tau(t) := (1-t)g_\tau'(t) = l_\tau(1)/l_\tau(t) > 0$, implying $g_\tau'(t) > 0$ for $t < 1$ and $g_\tau'(t) < 0$ for $t > 1$. This with $g_\tau(t) = 0$ for $t = \tau$, $t = \nu_\tau$ gives us after integration by parts the equality

(3.27) $$\mathfrak{I}_\tau := \int_{I_\tau^r} g_\tau(t)dt = \int_{I_\tau^r} p_\tau(t)dt,$$

and setting $k_\tau := l_\tau(1)/|l_\tau'(1)| > 0$, we obtain

(3.28) $$\mathfrak{I}_\tau = k_\tau s_\tau \int_{I_\tau^r} d\log l_\tau(t) = k_\tau \log\frac{r}{|1-\tau|}.$$

Now let I_τ for $\tau \in (I^r_\tau)^o \setminus \{1\}$ be the interval with endpoints τ and 1 independently from r. Then using (3.25)-(3.26), similar to (3.28), we obtain

$$\mathfrak{I}'_\tau := \int_{I_\tau} p_\tau(t)dt = k_\tau s_\tau \int_{I_\tau} d\log l_\tau(t) = k_\tau \log \frac{r}{r_\tau} < k_\tau. \tag{3.29}$$

3°. The solution of the *Dirichlet problem* for $\mathbb{D}^r_\tau$ with boundary data $\phi \in C(L_\tau)$ for $L_\tau = \partial\mathbb{D}_\tau$ can be presented by the *Poisson integral*:

$$\mathcal{P}_\tau[\phi](z) := (2\pi)^{-1} \int_{L_\tau} \phi(\zeta)\partial_\nu g_\tau(z,\zeta)\,|d\zeta| \quad \text{for} \quad z \in (\mathbb{D}^r_\tau)^o, \tag{3.30}$$

where $\partial_\nu g_\tau$ is the derivative of $g_\tau(\cdot,\zeta)$, given by (3.20)-(3.21), in direction of the *inner* normal vector ν on L_τ.

Next, if u is a *subharmonic* function on $\mathbb{D}^o_\tau$ with continuous extension on L_τ, then by the maximum principle, we have

$$u(z) \le \mathcal{P}_\tau[u](z) \text{ for } z \in \mathbb{D}^r_\tau, \tag{3.31}$$

with equality sign for all $z \in \mathbb{D}^o_\tau$, if u is *harmonic* on $\mathbb{D}^o_\tau$.

Example 3.1. *Consider the subharmonic on $\mathbb{D}_\tau$ function u_0 :*

$$u_0(z) = |\operatorname{Im} z| \quad \textit{for} \quad z \in \mathbb{D}_\tau. \tag{3.32}$$

Then the Poisson integral (3.30) with boundary data $\phi = u_0 \mid L_\tau$ satisfies for $z = 1$ the relation:

$$\mathcal{P}_\tau[u_0](1) = \pi^{-1}\mathfrak{I}_\tau, \tag{3.33}$$

where $\mathfrak{I}_\tau$ is as in (3.27)-(3.28).

Actually, the function $u_0 \in C(\mathbb{D}_\tau)$ in (3.32) is *harmonic* in both half-disks:

$$\mathbb{D}^\pm_\tau = \{z \in \mathbb{D}^r_\tau : \pm \operatorname{Im} z \ge 0\}$$

and $u_0 = 0$ with $\partial_\nu u_0 = 1$ on $I_\tau(r)$ (from both sides of I^r_τ). Also, by (3.23), the function $z \to g_\tau(z)$ is harmonic on $\mathbb{D}^r_\tau$, except the integrable singularity at $z = 1 \in \mathbb{I}_\tau$, so that

$$\iint\limits_{\mathbb{D}^+_\tau \cup \mathbb{D}^-_\tau} [u_0 \Delta g_\tau - g_\tau \Delta u_0] dV = 0. \tag{3.34}$$

Apply now to the pair of functions u_0, g_τ and to both closed half-disks $\mathbb{D}^\pm_\tau$ the Green's identity (see [19]) with Laplace operator Δ, by noting that $g_\tau = 0$ on $\partial\mathbb{D}_\tau$. Since $\partial\mathbb{D}^\pm_\tau = I^r_\tau \cup L^\pm_\tau$ with $L^\pm_\tau = L_\tau \cap \mathbb{D}^\pm_\tau$, by (3.34), (3.27) and (3.30) for $\phi = u_0 \mid L_\tau$ with $z = 1$, we have

$$0 = \int\limits_{L_\tau} u_0(\zeta)\partial_\nu g_\tau(1,\zeta)\,|d\zeta| - 2\int_{I^r_\tau} g_\tau(t)dt = 2\pi\mathcal{P}_\tau[u_0](1) - 2\mathfrak{I}_\tau.$$

implying (3.33).

4°. Consider the *Blaschke product* with the finite number of zeros $\{z_j\}_1^m \subset \mathbb{D}_\tau^o$:

$$B_\tau(z) = \prod_{j=1}^{m} b_\tau(z, z_j) \quad \text{for } z \in \mathbb{D}_\tau^r, \tag{3.35}$$

where $b_\tau(z, \zeta)$ is the Blaschke factor for $\mathbb{D}_\tau^r$, given by (3.21) and satisfying (3.22), so that

$$|B_\tau(z)| \equiv 1 \text{ for } z \in L_\tau. \tag{3.36}$$

In terms of the Green's function $g_\tau(z, \zeta)$ (see (3.20)-(3.21)), it follows from (3.36) that

$$-\log|B_\tau(z)| = \sum_{j=1}^{m} g_\tau(z, z_j) \quad \text{for } z \in \mathbb{D}_\tau^r,$$

and, in particular,

$$-\log|B_\tau(1)| = \sum_{j=1}^{m} g_\tau(z_j). \tag{3.37}$$

4. Power series: localization of singularities

4.1. Application of the auxiliary function φ_η. 1°. We start with the following definition.

Definition 4.1. *Any sequence $Q = \{q_k\}_{k=1}^\infty \subset \mathbb{N}_0$ will be called radial for the normalized element f in (1.2) if*

$$\lim_{k\to\infty} |f_{q_k}|^{1/q_k} = 1, \tag{4.1}$$

and the (non-empty) set of all such sequences Q will be denoted by $\mathcal{R}_f$ (see [7], Definition 3).

Obviously, any radial sequence $Q \in \mathcal{R}_f$ will be simultaneously a radial sequence also for the element f^* in (1.6)-(1.7), since $|f_n^*| = |f_n|$ for $n \in \mathbb{N}_0$.

Consider now the auxiliary function φ_η defined by (3.2). Then by (1.4), (3.3)-(3.4) and (4.1) with any $Q = \{q_k\}_{k=1}^\infty \in \mathcal{R}_f$ the following condition is satisfied:

$$q_k^{-1} \log|\varphi_\eta(q_k)| := q_k^{-1} \log|\cos(\omega_{q_k} + \eta)| + \varepsilon_k, \tag{4.2}$$

where $\varepsilon_k \to 0$ as $k \to \infty$.

Returning to the closed disk $\overline{D}_{r,1}$ from Subsection 3.2 with the fixed radius $r \in (0, 1)$ and the diameter $I(r) := [1 - r, 1 + r]$, we will assume further that $r \in (0, r_\delta)$ with $r_\delta = \sin\delta$ for $\delta \in (0, \pi/2)$, implying $\overline{D}_{r,1} \subset \Delta_{2\delta}$. Then it follows for $z \in \overline{D}_{r,1}$ and $k \in \mathbb{N}$ that $q_k z \in \overline{D}_{r_k,c_k} \subset \Delta_{2\delta}$ with $r_k = q_k r$, $c_k = q_k$, that is, $\overline{D}_{r_k,c_k} = q_k \overline{D}_{r,1}$, implying by (3.5) the inequality:

$$q_k^{-1} \log |\varphi_\eta(q_k z)| < c_\varphi(\delta) |\mathrm{Im}\, z| + \varepsilon_k, \tag{4.3}$$

with $\varepsilon_k \to 0$ as $k \to +\infty$, uniformly for $z \in \overline{D}_{r,1}$ and $\eta \in (0, \pi)$.

Consider the family of closed disks $\mathbb{D}_\tau \subset \overline{D}_{r,1}$ in (3.17)-(3.19) with diameter I_τ^r for $\tau \in (I_\tau^r)^o \setminus \{1\}$. For a fixed $q_k \in Q$ denote by $t_j = t_{j,k}(\tau, \eta)$, $j = 1, 2, ..., m_k$ possible finite number solutions of the equation:

$$\varphi_\eta(q_k t) = 0 \text{ for } t \in I^o(r),\ \eta \in (0, \pi), \tag{4.4}$$

the zeros of $\varphi_\eta(q_k t)$ on $I^o(r)$, taking also into account their multiplicity. Denote by $B_{\tau,\eta}$ the finite Blaschke product for the closed disk $\mathbb{D}_\tau$ (see (3.35)) with the zeros $t_{j,k} \in (I_\tau^r)^o$, satisfying by (3.36) the condition $|B_{\tau,\eta}(z)| \equiv 1$ for $z \in L_\tau := \partial \mathbb{D}_\tau$. Otherwise we will set $B_{\tau,\eta}(z) \equiv 1$ for $z \in \mathbb{D}_\tau$, if $\varphi_\eta(q_k t)$ has no zeros on $(I_\tau^r)^o$. So that in both cases we have $|B_{\tau,\eta}(z)| \equiv 1$ for $z \in L_\tau$.

Next, for a fixed $k \in \mathbb{N}$, consider the function $\psi_{\tau,\eta} \in H(\mathbb{D}_\tau)$:

$$\psi_{\tau,\eta}(z) := \varphi_\eta(q_k z)/B_{\tau,\eta}(z) \quad \text{for } z \in \mathbb{D}_\tau \subset \overline{D}_{r,1}, \tag{4.5}$$

and introduce the subharmonic function

$$u_\tau(z) := q_k^{-1} \log |\psi_{\tau,\eta}(z)| \quad \text{for } z \in \mathbb{D}_\tau^r, \tag{4.6}$$

which is continuous on L_τ and, by (4.3) and (4.5)-(4.6), satisfies the inequality

$$u_\tau(\zeta) \le c_\varphi(\delta) |\mathrm{Im}\, \zeta| + \varepsilon_k \text{ for } \zeta \in L_\tau. \tag{4.7}$$

Then from the maximum principle (3.31), applied to u_τ in (4.6) at the point $z = 1$, and from Example 1 with (3.32)-(3.33), it follows that

$$u_\tau(1) \le \mathcal{P}_\tau[u_\tau](1) \le \pi^{-1} c_\varphi(\delta) \mathfrak{I}_\tau + \varepsilon_k, \tag{4.8}$$

where $\varepsilon_k \to 0$ as $k \to +\infty$. In addition, by (4.4)-(4.5) and (3.37), we have

$$u_\tau(1) = q_k^{-1} \log |\varphi_\eta(q_k)| + \Lambda, \quad \text{with} \quad \Lambda = q_k^{-1} \sum_{t_j \in (I_\tau^r)^o} g_\tau(t_j), \tag{4.9}$$

setting $\Lambda = 0$, if the equation (4.3) has no zeros on $I^o(r)$.

2°. Next, for any $t \in \mathbb{R}_+ \setminus \{1\}$, denote by I_t the closed interval of the length $|I_t| = |t - 1|$ with the endpoints t and 1. Also, for any $q \in \mathbb{N}$, denote by qI_t the closed interval of the length $q|I_t| = q|t-1|$ with the endpoints qt and q. In this terms let us present the sum in (4.9) by an integral, if $\Lambda \ne 0$. Then the number $\mathbf{n}_{\varphi_\eta}(q_k I_t)$, the solutions $t_j \in q_k I_t$ of equation (4.4) on I_t according their multiplicity, is a function of $t \in I(r) \setminus \{1\}$, decreasing for $t < 1$ and increasing for $t > 1$. Also, taking into account that, in contrary to $\mathbf{n}_{\varphi_\eta}(q_k I_t)$, the function $g_\tau(t)$ is *increasing* for $t < 1$ and *decreasing* for $t > 1$, and, in addition, is vanishing at the endpoints of I_τ (useful for integration by parts), we obtain for any $\tau \in I^o(r)\backslash\{1\}$:

$$q_k\Lambda = \int_{I_\tau^r} g_\tau(t)\,|d\mathbf{n}_{\varphi_\eta}(q_kI_t)| = \int_{I_\tau^r} \mathbf{n}_{\varphi_\eta}(q_kI_t)\,|g'_\tau(t)|\,dt.$$

Using the notation $p_\tau(t) := (1-t)g'_\tau(t) > 0$ for $t \in I_\tau^r$ with $p_\tau(1) = 1$, we obtain the equality

$$\Lambda = \int_{I_\tau^r} [\mathbf{n}_{\varphi_\eta}(q_kI_t)/(q_k\,|I_t|)]p_\tau(t)dt,$$

which is preserving also in the case $\Lambda = 0$ with $\mathbf{n}_{\varphi_\eta}(q_kI_t) = 0$ for $t \in (I_\tau^r)^o$. This equality together with (3.2), (4.8), (4.9) gives us the asymptotic inequality:

$$\text{(4.10)}\qquad \int_{I_\tau^r} [\mathbf{n}_{\varphi_\eta}(q_kI_t)/(q_k\,|I_t|) - \pi^{-1}c_\varphi(\delta)]p_\tau(t)dt \le \varepsilon'_k,$$

where $\varepsilon'_k = \varepsilon_k - q_k^{-1}l_k(\eta)$ with $l_k(\eta) = \log|\cos(\omega_{q_k}+\eta)|$ and $\varepsilon_k \to 0$ as $k \to +\infty$. Integrating both sides of (4.11) by $\eta \in (0,\pi)$ and setting

$$\text{(4.11)}\qquad \mathfrak{I}_\varphi(q_kI_t) := \int_0^\pi \mathbf{n}_{\varphi_\eta}(q_kI_t)d\eta,$$

we come from (4.11) with $\tau \in (I_\tau^r)^o\backslash\{1\}$ to the inequality

$$\text{(4.12)}\qquad \int_{I_\tau^r} [\mathfrak{I}_\varphi(q_kI_t)/(q_k\,|I_t|) - c_\varphi(\delta)]\,p_\tau(t)dt \le \varepsilon''_k,$$

where $\varepsilon''_k := \pi\varepsilon_k - q_k^{-1}\mathfrak{I}(\omega_{q_k})$ and $\mathfrak{I}(\omega_{q_k}) = \int_0^\pi l_k(\eta)d\eta$. Now since $\mathfrak{I}(\omega_{q_k}) = \mathfrak{I}(0)$, by π - periodicity of the function $\eta \to l_k(\eta)$, it follows that $\varepsilon''_k \to 0$ as $k \to \infty$.

4.2. **Necessary metric conditions on regularity arcs of (1.2).** After the above preparation, we can formulate the main results of this paper, using some additional notation. For a *radial* sequence $Q = \{q_k\}_{k=1}^\infty \in \mathcal{R}_f$ of the element f in (1.2) consider the *variation* $V_f(q_kI_t)$ *of the arguments* $\{\omega_n\}_0^\infty$ of the coefficients $\{f_n\}_0^\infty$ of (1.2) on the interval q_kI_t for $t \in \mathbb{R}_+\backslash\{1\}$, so that $V_f(q_kI_t) = 0$, if $q_k\,|I_t| < 1$. Now for $r \in (0, r_\delta)$ we set:

$$\text{(4.13)}\qquad v_f(t,Q) = \limsup_{k\to\infty} \nu_f(q_kI_t) \in [0,\pi] \text{ for } t \in I(r)\backslash\{1\},$$

where $\nu_f(q_kI_t) = V_f(q_kI_t)/(q_k\,|I_t|)$ is the *mean variation* of the arguments $\{\omega_n\}_0^\infty$ on the interval q_kI_t (see (1.13). Next, denote

$$\text{(4.14)}\qquad v_f^-(r,Q) = \sup_{t\in[1-r,1]} v_f(t,Q) \text{ and } v_f^-(r,Q) = \sup_{t\in[1,1+r]} v_f(t,Q),$$

so that both $v_f^\pm(r,Q)$ are *non-decreasing* functions of r, having the limits as $r \to 0$:

$$\text{(4.15)}\qquad v_f^\pm(Q) = \lim_{r\to 0} v_f^\pm(r,Q) \in [0,\pi].$$

Finally, introduce the quantity:

$$\text{(4.16)}\qquad v_f(Q) := \min\{v_f^-(Q), v_f^+(Q)\} \in [0,\pi],$$

the *mean density* the variation of $\{\Delta\omega_n\}_0^\infty$ along Q.

Analogously, using Definition 2.2 and replacing in (4.13)-(4.15) the element f by the normalized element $z \to f(-z) = f^*(z)$ in (1.6) with coefficient arguments $\{\omega_n^*\}_0^\infty$ in (1.7)-(1.9), we obtain for the element (1.2) also the quantity:

$$v_f^*(Q) := \min\{v_{f^*}^-(Q), v_{f^*}^+(Q)\} \in [0, \pi], \tag{4.17}$$

the mean density the variation of the arguments $\{\Delta\omega_n^*\}_0^\infty$ along Q.

Now we are in position to state the main results of this paper.

Theorem 4.1. *Let for $\alpha \in (0, 2\pi]$ and $\mu = e^{i\lambda}$ with $\lambda \in (-\pi, \pi)$ the open arc $\gamma_{\alpha,\mu}^o \subset \partial D_1$ or the symmetric open arc $\gamma_{\alpha,-\mu}^o \subset \partial D_1$ be an arc of regularity for the element f in (1.2). Then for any radial sequence $Q \in \mathcal{R}_f$ the following inequality is satisfied:*

$$\alpha/2 \le \begin{cases} v_f(Q) + |\lambda| & for\ \gamma_{\alpha,\mu}^o, \\ v_f^*(Q) + |\lambda| & for\ \gamma_{\alpha,-\mu}^o. \end{cases} \tag{4.18}$$

Proof. From Corollary 2.2 for the interval $I_{a,b} = q_k I_t$ with $t \in I(r)\backslash\{1\}$ we have the next estimate from below for the integral $\mathfrak{I}_\varphi(q_k I_t)$, defined in (4.11):

$$\mathfrak{I}_\varphi(q_k I_t) \ge \max\{0, [\pi(q_k |I_t| - 1) - V_f(q_k I_t)]\}. \tag{4.19}$$

For $t \in I(r)\backslash\{1\}$, define the function

$$\sigma_k(t) := \begin{cases} 0 & \text{if } |I_t| \le q_k^{-1/2}, \\ 1 & \text{if } |I_t| > q_k^{-1/2}, \end{cases} \tag{4.20}$$

satisfying the condition: $\sigma_k(t) \to 1$ as $k \to \infty$ for $t \in I(r)\backslash\{1\}$, where $r \in (0, r_\delta)$ for $r_\delta = \sin\delta$ with $\delta \in (0, \pi/2)$. Then from (4.12) and (4.19)-(4.20) for any $\tau \in I^o(r)\backslash\{1\}$ we obtain the inequality:

$$[\pi - c_\varphi(\delta)] \int_{I_\tau^r} \sigma_k(t) p_\tau(t) dt \le \int_{I_\tau^r} [\nu_f(q_k I_t) + \pi q_k^{-1/2}] p_\tau(t) dt + \varepsilon_k'', \tag{4.21}$$

where $\varepsilon_k'' \to 0$ as $k \to \infty$. Applying to (4.21) the Fatou's lemma (see [20]), we come with $k \to \infty$ to the inequality:

$$[\pi - c_\varphi(\delta)]\mathfrak{I}_\tau \le \int_{I_\tau^r} v_f(t, Q) p_\tau(t) dt, \tag{4.22}$$

where

$$\mathfrak{I}_\tau = \int_{I_\tau^r} p_\tau(t) dt.$$

Then from (4.22) and (4.14), for $s_\tau = sgn(1 - \tau)$ with $\tau \in I^o(r)\backslash\{1\}$, it follows that $[\pi - c_\varphi(\delta)]\mathfrak{I}_\tau \le \mathfrak{I}'_\tau + v_f^{s_\tau}(r, Q)\mathfrak{I}_\tau$, where $\mathfrak{I}'_\tau$ is the next integral on the interval I_τ with the endpoints τ and $1:$

$$\mathfrak{I}'_\tau = \int_{I_\tau} p_\tau(t) dt.$$

Using the relations for $\mathfrak{I}_\tau$ and $\mathfrak{I}'_\tau$ from (3.28)-(3.29), we obtain $[\pi - c_\varphi(\delta)]l_r(\tau) \leq 1 + v_f^{s_\tau}(r, Q)l_r(\tau)$, where

$$l_r(\tau) := \log \frac{r}{|1-\tau|}.$$

Dividing both sides by $l_r(\tau)$ and letting $\tau \to 1$ with $\tau < 1$ and $\tau > 1$, in view of (4.13)-(4.14), we obtain

$$\pi - c_\varphi(\delta) \leq \min\{v_f^-(r, Q), v_f^+(r, Q)\} \quad \text{for } r \in (0,1). \tag{4.23}$$

Now letting here $r \to 0$, and using (4.15)-(4.16), we obtain $\pi - c_\varphi(\delta) \leq v_f(Q)$. Finally, by (2.8), $c_\varphi(\delta) \to m_\varphi$ as $\delta \to 0$, implying

$$\pi \leq v_f(Q) + m_\varphi. \tag{4.24}$$

Then by Criterion 2, $m_\varphi \leq \pi - \alpha/2 + |\lambda|$, which together with (4.24) implies (4.18) for $\gamma^o_{\alpha,\mu}$. Applying the above arguments for the normalized element $z \to f^*(z) = f(-z)$ in (1.6) with coefficients $\{f_n^*\}_0^\infty$ in (1.7), we obtain (4.19) for $\gamma^o_{\alpha,-\mu}$. □

4.3. **Some consequences from Theorem 4.1.** From the *necessary* condition (4.18) of Theorem 4.1 on regularity of the element (1.2) on the open arcs of ∂D_1 we infer the following result.

Corollary 4.1. *For any $\alpha \in (0, 2\pi]$ and $\mu = e^{i\lambda}$ with $\lambda \in (-\pi, \pi)$ the element f in (1.2) has a singular point on the open arc $\gamma^o_{\alpha,\mu} \subset \partial D_1$ or on the symmetric open arc $\gamma^o_{\alpha,-\mu} \subset \partial D_1$, if for some radial sequence $Q \in \mathcal{R}_f$, or correspondingly for $Q' \in \mathcal{R}_f$, the following inequality is satisfied:*

$$\alpha/2 > \begin{cases} v_f(Q) + |\lambda| & \text{for } \gamma^o_{\alpha,\mu}, \\ v_f^*(Q') + |\lambda| & \text{for } \gamma^o_{\alpha,-\mu}. \end{cases} \tag{4.25}$$

Concerning the singularities of the element (1.2) on *closed* arcs of ∂D_1, from Corollary 4.1 we obtain the next result.

Corollary 4.2. *For any $\beta \in [0, 2\pi)$ and $\mu = e^{i\lambda}$ with $\lambda \in (-\pi, \pi)$ the element f in (1.2) has a singular point on the arc $\gamma_{\beta,\mu} \subset \partial D_1$ or on the symmetric arc $\gamma_{\beta,-\mu} \subset \partial D_1$, if for some radial sequence $Q \in \mathcal{R}_f$, or correspondingly for $Q' \in \mathcal{R}_f$, the following inequality is satisfied:*

$$\beta/2 \geq \begin{cases} v_f(Q) + |\lambda| & \text{for } \gamma_{\beta,\mu}, \\ v_f^*(Q') + |\lambda| & \text{for } \gamma_{\beta,-\mu}. \end{cases} \tag{4.26}$$

Actually, if $\gamma_{\beta,\mu}$ contains only *regular* points of f, then the *open* arc $\gamma^o_{\alpha,\mu}$ for $\alpha = \beta + \varepsilon$ with sufficiently small $\varepsilon > 0$ will be an arc of regularity for f, which will contradict Corollary 4.1 for $\gamma^o_{\alpha,\mu}$. Of course, Corollary 4.2 is satisfied also for $\beta = 2\pi$, since the normalized element (1.2) always has a singular point on ∂D_1.

Remark 4.1. *If the inequalities in (4.25) are satisfied simultaneously for* $\gamma^o_{\alpha,\mu}$ *and* $\gamma^o_{\alpha,-\mu}$, *then the singular points on these arcs will be different for* $\alpha \in (0,\pi)$, *since* $\gamma^o_{\alpha,\mu} \cap \gamma^o_{\alpha,-\mu} = \varnothing$. *A similar remark with* $\beta \in [0,\pi)$ *is true also for the inequalities in (4.26) of Corollary 4.2.*

The next corollary contains some more concrete cases of the Corollaries 3-4.

Corollary 4.3. *For the element* f *in (1.2) with a radial sequence* $Q \in \mathcal{R}_f$ *the following assertions hold:*

1) If $\lambda = 0$ *and* $v_f(Q) < \pi$ *for some* $Q \in \mathcal{R}_f$, *then for* $\beta = 2v_f(Q) \in [0, 2\pi)$, *it follows from Corollary 4.2 that* f *has a singular point on the arc* $\gamma_{\beta,1}$, *implying for* $v_f(Q) = 0$ *with* $\beta = 0$, *that the point* 1 *is a singular point of* f. *If* $\lambda = 0$ *and* $v^*_f(Q') < \pi$ *for some* $Q' \in \mathcal{R}_f$, *then* f *has a singular point on the arc* $\gamma_{\beta,-1}$ *with* $\beta = 2v^*_f(Q')$ *and center* -1, *implying for* $v^*_f(Q') = 0$, *that the point* -1 *is a singular point of* f.

2) Let $\mu = e^{i\lambda}$ *with* $\lambda \in (-\pi,\pi)\backslash\{0\}$, *and* $2|\lambda| < \alpha \le 2\pi$. *Then for* $\delta = \alpha - 2|\lambda|$, *from Corollary 4.1 with (4.25) it follows that* $\gamma^o_{\delta,1} \subset \gamma^o_{\alpha,\mu}$ *and* $\gamma^o_{\delta,-1} \subset \gamma^o_{\alpha,-\mu}$, *implying by 1) for* $\gamma^o_{\delta,\pm 1}$, *that: a) the point* 1 *is a singular point of* $\gamma^o_{\alpha,\mu}$ *if* $v_f(Q) = 0$; *b) the point* -1 *is a singular point of* $\gamma^o_{\alpha,-\mu}$ *if* $v^*_f(Q') = 0$.

3) Let for element f *in (1.2) with radial sequences* $Q \in \mathcal{R}_f$ *and* $Q' \in \mathcal{R}_f$, *and for* $\alpha \in (0, 2\pi]$ *and* $\mu = e^{i\lambda}$ *with* $\lambda \in (-\pi,\pi)$ *the following inequalities be satisfied:*

$$\alpha/2 > v_f(Q) + |\lambda| \ \text{ and } \ \beta/2 \ge v^*_f(Q') + |\lambda|, \tag{4.27}$$

where $\beta = 2\pi - \alpha \in [0, 2\pi)$. *Then by Corollaries 4.1-4.2 and (4.27), the element* f *has (different) singular points on both complementary arcs* $\gamma^o_{\alpha,\mu}$ *and* $\gamma_{\beta,-\mu}$ *on* ∂D_1 *with* $\gamma_{\beta,-\mu} = \partial D_1 \backslash \gamma^o_{\alpha,\mu}$.

Note that in Theorem 1.1 and in Corollaries 4.1-4.2 instead of $v_f(Q)$ one can use also some other quantities of the element f in (1.2), related with any radial sequence $Q = \{q_k\}_{k=1}^\infty \in \mathcal{R}_f$, using in (4.13)-(4.17) directly the quantities $\{\Delta\omega_n\}_0^\infty$ and $\{\Delta\omega^*_n\}_0^\infty$, defined in (1.4)-(1.5) and (1.7)-(1.9).

For $r \in (0, r_\delta)$ and $q_k \in Q$ we set:

$$w^-_f(r, q_k) = \max_{n \in q_k I^-_r} |\Delta\omega_n| \ \text{ and } \ w^+_f(r, q_k) = \max_{n \in q_k I^+_r} |\Delta\omega_n|,$$

where $I^-_r = [1-r, 1]$ and $I^+_r = [1, 1+r]$. Then for $r \in (0, r_\delta)$ the functions$w^\mp_f(r, Q) = \limsup_{k\to\infty} w^\mp_f(r, q_k) \in [0, \pi]$ are both monotonic having the limits$\lim_{r\to 0} w^\mp_f(r, Q) = w^\pm_f(Q) \in [0, \pi]$, and we define

$$w_f(Q) := \min\{w^-_f(Q), w^+_f(Q)\}. \tag{4.28}$$

Replacing in this definition f in (1.2) by the related with f element f^* in (1.6)-(1.7) with the coefficient arguments $\{\omega_n^*\}_0^\infty$, we come for f to the second quantity

$$(4.29) \qquad w_f^*(Q) := w_{f^*}(Q) = \min\{w_{f^*}^-(Q), w_{f^*}^+(Q)\} \in [0, \pi].$$

The comparison of (4.28)-(4.29) and (4.16)-(4.17) shows that

$$0 \le v_f(Q) \le w_f(Q) \le \pi \quad \text{and} \quad 0 \le v_f^*(Q) \le w_f^*(Q) \le \pi.$$

Thus, replacing in Theorem 1 and in Corollaries 3-4 the quantity $v_f(Q)$ by $w_f(Q)$ and correspondingly $v_f^*(Q)$ by $w_f^*(Q)$, we will obtain some new, and perhaps intuitively more clear corollaries, including the influence of the gaps in coefficients of the element f, taking into account that if $f_n = 0$ for some $n \in \mathbb{N}_0$, then also $\Delta\omega_n = 0$ or $\Delta\omega_n^* = 0$. The converse is not true, since if $f_n/f_{n-1} > 0$, then again $\Delta\omega_n = 0$ by (1.4)-(1.5), but $f_n^*/f_{n-1}^* < 0$ with $\Delta\omega_n^* = \pi$ by (1.8).

Assuming now that the condition $\lim_{n\to\infty} |\Delta\omega_n| = 0$ or $\lim_{n\to\infty} |\Delta\omega_n^*| = 0$ is satisfied, it will follow from (4.28)-(4.29) that correspondingly $w_f(Q) = 0$ or $w_f^*(Q) = 0$ for *arbitrary* radial subsequence $Q \in \mathcal{R}_f$. This will imply in Corollary 4.3, item 1) the singularity of the points 1 or correspondingly -1 for any $Q \in \mathcal{R}$.

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