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ON A GENERAL NONLINEAR PROBLEM WITH DISTRIBUTED DELAYS

NASSER-EDDINE TATAR

King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia
E-mail: *tatarn@kfupm.edu.sa*

Abstract. The paper considers a general system of ordinary differential equations appearing in the neural network theory. The activation functions are assumed to be continuous and bounded by power type functions of the states and distributed delay terms. These activation functions are not necessarily Lipschitz continuous as it is commonly assumed in the literature. We obtain sufficient conditions for exponential decay of solutions.[1]

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1. Introduction

In this paper, we deal with the following system of equations:

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} f_{ij}\left(t, x_j(t), \int_{-\infty}^{t} K_{ij}(t,s,x_j(s))ds\right) + c_i(t), \tag{1.1}$$

$i = 1, ..., m$, with given continuous functions $x_j(t) = x_{0j}(t)$, $t \in (-\infty, 0]$, $a_i(t) \geq 0$ and $c_i(t)$, $i = 1, ..., m$, $t > 0$. The functions f_{ij} and K_{ij}, $i, j = 1, ..., m$, are assumed to be nonlinear and continuous, and satisfy some condition that will be specified later. Notice that the system (1.1) is a generalized version of much simpler systems, appearing in the neural network theory (see [5, 7-9, 12, 14, 15]).

In designing (artificial) neural networks, the researchers were mainly interested in the human brain. Neural networks consist of several simple computational elements (processors) known as "neurons which are highly interconnected and arranged in layers. The tasks of neurons is to transform the received signals from the input and transmit the outcome to the subsequent neurons.

The applications are numerous: quality control, identification of consumer characteristics, target marketing, financial health prediction, texture analysis, adaptive

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control, data segmentation, recognition of genes, medical diagnosis, signal processing, etc.

Neural networks are particularly useful for tasks a traditional computer cannot perform. Some of these tasks are, for example, detection of medical phenomena, forecasting, identification and prediction.

After appearance of the basic neural network systems, they have been extensively discussed in the literature. The goal was to generalize these systems and to discuss various issues for basic and generalized systems (see [5, 7-9, 12, 14, 15]). It seems, the most studied question is the (global) asymptotic stability of solutions. To establish this property, various conditions have been imposed on the coefficients and on the activation functions, and a lot of efforts were spent to relax these conditions. The most commonly assumed condition was the Lipschitz continuity condition for activation functions. We must, however, mention the references [1, 2, 4, 10, 11, 13], where the non-Lipschitz case was studied.

In this paper we assume that the functions f_{ij} and K_{ij} in (1.1) are (or are bounded by) continuous monotone nondecreasing functions that are not necessarily bounded or Lipschitz continuous. Even the monotonicity condition may be dropped.

The main result of this paper provides sufficiently mild sufficient conditions for solutions of the system (1.1) to converge to zero exponentially. To prove our main result, we use a generalization of the Gronwall inequality presented below in Lemma 2.1. Notice that this lemma may also be used to prove the local existence of solutions. The global existence can be derived from our theorem. Since here we are concerned in the convergence to zero (of any solution), the uniqueness is irrelevant.

The paper is organized as follows. Section 2 contains some notation, assumptions and a lemma, which is used in the proof of the main result of the paper. In Section 3 we state and prove our main result (Theorem 3.1), followed by some corollaries and remarks.

2. Preliminaries

The functions f_{ij} and K_{ij}, $i, j = 1, ..., m$, appearing in the system (1.1), are assumed to satisfy the following assumption.

Assumption (H1). For $t > 0$ and $i, j = 1, ..., m$,

$$\left|f_{ij}\left(t, x_j(t), \int_{-\infty}^{t} K_{ij}(t,s,x_j(s))ds\right)\right|$$
$$\leq b_{ij}(t)\left|x_j(t)\right|^{\alpha_{ij}}\left(|x_j(t)| + \int_{-\infty}^{t} l_{ij}(t-s)\psi_{ij}\left(|x_j(s)|\right)ds\right)^{\beta_{ij}},$$

where b_{ij} are nonnegative continuous functions, l_{ij} are nonnegative continuously differentiable functions with summable first order derivatives, ψ_{ij} are nonnegative nondecreasing continuous functions, and $\alpha_{ij}, \beta_{ij} \geq 0$, $i, j = 1, ..., m$.

Definition 2.1. *A function $f : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be in the class $H_{r,\omega}$ if it satisfies the following two conditions:*

(i) *$f(u)$ is nondecreasing and continuous for $u \geq 0$ and positive for $u > 0$;*

(ii) *$f(au) \leq r(a)\omega(u)$ for $a > 0$, $u \geq 0$, where $r(a)$ is a nonnegative continuous function in $\mathbf{R}_+$ and $\omega(u)$ is a nondecreasing continuous function in $\mathbf{R}_+$, which is positive for $u > 0$.*

Lemma 2.1. *Let $a(t)$ be a positive continuous function in $J := [\alpha, \beta)$, $k_j(t, s)$, $j = 1, ..., n$, are nonnegative continuous functions for $\alpha \leq s \leq t < \beta$, which are nondecreasing in t for any fixed s, $g_j \in H_{r_j,\omega_j}$ for $u > 0$, and $u(t)$ is a nonnegative continuous functions in J. Then the inequality*

$$u(t) \leq a(t) + \sum_{j=1}^{n} \int_{\alpha}^{t} k_j(t, s) g_j(u(s)) ds, \ t \in J,$$

implies that

$$u(t) \leq \hat{a}(t)\varphi_n(t), \ \alpha \leq t < \beta'_n,$$

where $\hat{a}(t) := \sup_{0 \leq s \leq t} a(s)$, and $\varphi_0(t) \equiv 1$,

$$\varphi_j(t) := \varphi_{j-1}(t) G_j^{-1}\left[G_j(t) + \frac{r_j[\hat{a}(t)\varphi_{j-1}(t)]}{\hat{a}(t)} \int_{\alpha}^{t} k_j(t, s) ds\right], \ j = 1, ..., n,$$

$$G_j(u) := \int_{u_j}^{u} \frac{dx}{g_j(x)}, \ u > 0 \ (u_j > 0, \ j = 1, ..., n).$$

Moreover, in this case

$$u(t) \leq \hat{a}(t)\xi_n(t), \ \alpha \leq t < \beta''_n,$$

where $\xi_0(t) \equiv 1$,

$$\xi_j(t) := \xi_{j-1}(t) G_j^{-1}\left[G_j(t) + \int_{\alpha}^{t} k_j(t, s) \frac{r_j[\hat{a}(s)\varphi_{j-1}(s)]}{\hat{a}(s)} ds\right], \ j = 1, ..., n.$$

Here β'_n and β''_n are chosen so that the functions $\varphi_j(t)$ and $\xi_j(t)$, $j = 1, ..., n$, are defined for $\alpha \leq t < \beta'_n$ and for $\alpha \leq t < \beta''_n$, respectively.

We also will need the following assumption.

Assumption (H2). Assume that $\psi_{ij}(u) \in H_{r_j,\omega_j}$ and g_l is a relabeling of $x^{\alpha_{ij}+\beta_{ij}}$ and ψ_{ij} with k_l as coefficients (with $\tilde{k}_l$ in the other case).

We will also use the following notation.

$$x(t) := \sum_{i=1}^{m} |x_i(t)|, \; x_0(t) := \sum_{i=1}^{m} |x_{0i}(t)|$$

$$c(t) := \int_0^t \exp\left[\int_0^s a(\sigma)d\sigma\right] \sum_{i=1}^{m} |c_i(s)|\, ds, \; t > 0,$$

$$G_j(u) := \int_{u_j}^{u} \frac{dx}{g_j(x)}, \; u > 0 \; (u_j > 0, \; j = 1, ..., n),$$

$$z(0) = x_0(0) + \sum_{i,j=1}^{m} \int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))\, d\sigma$$

$$\varphi_0(t) \equiv 1, \; \varphi_j(t) := \varphi_{j-1}(t) G_j^{-1}\left[G_j(t) + \frac{r_j\{[z(0) + c(t)]\, \varphi_{j-1}(t)\}}{z(0) + c(t)} \int_0^t k_j(s)ds\right],$$

$$\xi_0(t) \equiv 1, \; \xi_j(t) := \xi_{j-1}(t) G_j^{-1}\left[G_j(t) + \int_0^t \frac{k_j(s) r_j\{[z(0) + c(s)]\, \xi_{j-1}(s)\}}{z(0) + c(s)} ds\right],$$

$\tilde{\varphi}_0(t) \equiv 1$ and for $j = 1, ..., n$

$$\tilde{\varphi}_j(t) := \tilde{\varphi}_{j-1}(t) G_j^{-1}\left[G_j(t) + \frac{r_j\{[\tilde{z}(0) + c(t)]\, \tilde{\varphi}_{j-1}(t)\}}{\tilde{z}(0) + c(t)} \int_0^t \tilde{k}_j(s)ds\right],$$

$\tilde{\xi}_0(t) \equiv 1$ and for $j = 1, ..., n$,

$$\tilde{\xi}_j(t) := \tilde{\xi}_{j-1}(t) G_j^{-1}\left[G_j(t) + \int_0^t \frac{\tilde{k}_j(s) r_j\{[\tilde{z}(0) + c(s)]\, \tilde{\xi}_{j-1}(s)\}}{\tilde{z}(0) + c(s)} ds\right],$$

and $\tilde{k}_j$ differ from k_j by $l_{ij}(0) + \int_0^\infty |l'_{ij}(\sigma)|\, d\sigma$ instead of $l_{ij}(0)$.

3. The main result. Exponential convergence

In this section we state and prove our main result on the exponential convergence of solutions to zero.

Theorem 3.1. *Assume that the assumptions (H1) and (H2) hold, and*

$$\int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))\, d\sigma < \infty, \quad i, j = 1, ..., m.$$

Then the following assertions hold.

(a) *If $l'_{ij}(t) \le 0$, $i, j = 1, ..., m$, then there exists $\beta'_n > 0$ such that*

$$x(t) \le [z(0) + c(t)]\, \varphi_n(t) \exp\left[-\int_0^t a(s)ds\right], \; 0 \le t < \beta'_n,$$

and there exists $\beta''_n > 0$ such that

$$x(t) \le [z(0) + c(t)]\, \xi_n(t) \exp\left[-\int_0^t a(s)ds\right], \; 0 \le t < \beta''_n.$$

(b) If $l'_{ij}(t)$, $i,j=1,...,m$, are of arbitrary signs and

$$\int_0^\infty |l'_{ij}(s)| \int_{-s}^0 \psi_{ij}(u_0(\sigma))\, d\sigma ds < \infty,$$

then there exists $\tilde{\beta}'_n > 0$ such that

$$x(t) \le \left[x_0(0) + \sum_{i,j=1}^m \int_{-\infty}^0 l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))\, d\sigma\right] \times \tilde{\varphi}_n(t) \exp\left[-\int_0^t a(s)ds\right],\ 0 \le t < \tilde{\beta}'_n,$$

and there exists $\tilde{\beta}''_n > 0$ such that

$$x(t) \le \left[x_0(0) + \sum_{i,j=1}^m \int_{-\infty}^0 l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))\, d\sigma\right] \times \tilde{\xi}_n(t) \exp\left[-\int_0^t a(s)ds\right],\ 0 \le t < \tilde{\beta}''_n.$$

Proof. We first obtain a relation that will be used to prove the assertions (a) and (b) of the theorem.

Applying the Dini derivative to the equations in (1.1), for $t > 0$ and $i = 1,...,m$, we can write

$$D^+|x_i(t)| \le -a_i(t)|x_i(t)| + \sum_{j=1}^m \left| f_{ij}\left(t, x_j(t), \int_{-\infty}^t K_{ij}(t,s,x_j(s))ds\right)\right| + c_i(t),$$

Then, using our notation and the assumption **(H1)**, for $t > 0$ we obtain

$$\begin{aligned} &D^+x(t) \le -a(t)x(t) \\ (3.1)\quad &+ \sum_{i,j=1}^m b_{ij}(t)|x(t)|^{\alpha_{ij}} \left(|x_j(t)| + \int_{-\infty}^t l_{ij}(t-s)\psi_{ij}(x(s))\, ds\right)^{\beta_{ij}} + \sum_{i=1}^m |c_i(t)|. \end{aligned}$$

Multiplying both sides of (3.1) by $\exp\left[\int_0^t a(s)ds\right]$, for $t > 0$ we get

$$\begin{aligned} D^+\left\{x(t)\exp\left[\int_0^t a(s)ds\right]\right\} &\le \exp\left[\int_0^t a(s)ds\right] \sum_{i,j=1}^m b_{ij}(t)|x(t)|^{\alpha_{ij}} \\ &\times \left(|x_j(t)| + \int_{-\infty}^t l_{ij}(t-s)\psi_{ij}(x(s))\, ds\right)^{\beta_{ij}} + \exp\left[\int_0^t a(s)ds\right] \sum_{i=1}^m |c_i(t)|. \end{aligned}$$

Next, it follows that (see [6])

$$\begin{aligned} (3.2)\quad \tilde{x}(t) &\le x(0) + c(t) + \sum_{j=1}^m \int_0^t \left\{ \sum_{i=1}^m b_{ij}(s) \exp\left[(1-\alpha_{ij})\int_0^s a(\sigma)d\sigma\right] \tilde{x}(s)^{\alpha_{ij}} \right. \\ &\left. \times \left(\tilde{x}(s) + \int_{-\infty}^s l_{ij}(s-\sigma)\psi_{ij}(\tilde{x}(\sigma))\, d\sigma\right)^{\beta_{ij}} \right\} ds,\ t > 0, \end{aligned}$$

where

$$\tilde{x}(t) := x(t)\exp\left[\int_0^t a(s)ds\right],\ t > 0.$$

We define $\tilde{x}(t) = x(t) := x_0(t) = \sum_{i=1}^{m} |x_{0i}(t)|$ for $t \leq 0$. Let $y(t)$ denote the right hand side of (3.2) for $t > 0$, and let $y(t) := x(t)$ for $t \leq 0$. It is clear that $y(0) = x(0)$, $\tilde{x}(t) \leq y(t)$ for $t > 0$, and

$$\begin{aligned} y'(t) &= c'(t) + \sum_{i,j=1}^{m} \tilde{b}_{ij}(t)\tilde{x}(t)^{\alpha_{ij}} \left(\tilde{x}(t) + \int_{-\infty}^{t} l_{ij}(t-\sigma)\psi_{ij}\left(\tilde{x}(\sigma)\right) d\sigma\right)^{\beta_{ij}} \\ &\leq c'(t) + \sum_{i,j=1}^{m} \tilde{b}_{ij}(t)y(t)^{\alpha_{ij}} \left(y(t) + \int_{-\infty}^{t} l_{ij}(t-\sigma)\psi_{ij}\left(y(\sigma)\right) d\sigma\right)^{\beta_{ij}}, \ t > 0 \end{aligned} \tag{3.3}$$

where

$$\tilde{b}_{ij}(t) := \exp\left[(1-\alpha_{ij})\int_0^t a(\sigma)d\sigma\right] b_{ij}(t), \ t > 0.$$

Define

$$z(t) := \begin{cases} y(t) + \sum_{i,j=1}^{m} \int_{-\infty}^{t} l_{ij}(t-\sigma)\psi_{ij}\left(y(\sigma)\right) d\sigma, \ t > 0 \\ u_0(t) := x_0(t) + \sum_{i,j=1}^{m} \int_{-\infty}^{t} l_{ij}(t-\sigma)\psi_{ij}\left(x_0(\sigma)\right) d\sigma, \ t \leq 0. \end{cases} \tag{3.4}$$

Differentiating $z(t)$, given by (3.4), and using (3.3), we can write

$$\begin{aligned} z'(t) &= y'(t) + \sum_{i,j=1}^{m} l_{ij}(0)\psi_{ij}\left(y(t)\right) + \sum_{i,j=1}^{m} \int_{-\infty}^{t} l'_{ij}(t-\sigma)\psi_{ij}\left(y(\sigma)\right) d\sigma \\ &\leq c'(t) + \sum_{i,j=1}^{m} \tilde{b}_{ij}(t)y(t)^{\alpha_{ij}} \left(y(t) + \int_{-\infty}^{t} l_{ij}(t-\sigma)\psi_{ij}\left(y(\sigma)\right) d\sigma\right)^{\beta_{ij}} \\ &\quad + \sum_{i,j=1}^{m} l_{ij}(0)\psi_{ij}\left(y(t)\right) + \sum_{i,j=1}^{m} \int_{-\infty}^{t} l'_{ij}(t-\sigma)\psi_{ij}\left(y(\sigma)\right) d\sigma \\ &\leq c'(t) + \sum_{i,j=1}^{m} \tilde{b}_{ij}(t)z(t)^{\alpha_{ij}+\beta_{ij}} + \sum_{i,j=1}^{m} l_{ij}(0)\psi_{ij}\left(z(t)\right) \\ &\quad + \sum_{i,j=1}^{m} \int_{-\infty}^{t} l'_{ij}(t-\sigma)\psi_{ij}\left(y(\sigma)\right) d\sigma. \end{aligned} \tag{3.5}$$

Now we use the relation (3.5) to prove the assertions (a) and (b) of the theorem.

Proof of (a). Let $l'_{ij}(t) \leq 0$, $i,j = 1,...,m$. This case corresponds to the so-called "fading memory"situation. In this case, the relation (3.5) reduces to the following:

$$z'(t) \leq c'(t) + \sum_{i,j=1}^{m} \left[\tilde{b}_{ij}(t)z(t)^{\alpha_{ij}+\beta_{ij}} + l_{ij}(0)\psi_{ij}\left(z(t)\right)\right], \ t > 0.$$

Therefore, we have

$$z(t) \leq z(0) + c(t) + \sum_{i,j=1}^{m} \int_0^t \left[\tilde{b}_{ij}(s)z(s)^{\alpha_{ij}+\beta_{ij}} + l_{ij}(0)\psi_{ij}\left(z(s)\right)\right] ds, \ t > 0, \tag{3.6}$$

where $z(0) = x_0(0) + \sum_{i,j=1}^{m} \int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}\left(x_0(\sigma)\right) d\sigma$.

It is clear that the functions $z(s)^{\alpha_{ij}+\beta_{ij}}$ belong to H_{r_j,w_j}, and since ψ_{ij} are also assumed from this class, we can apply Lemma 2.1 to (3.6), to obtain

$$\tilde{x}(t) \le z(t) \le [z(0)+c(t)]\,\varphi_n(t),\ 0 \le t < \beta'_n \tag{3.7}$$

$$\tilde{x}(t) \le z(t) \le [z(0)+c(t)]\,\xi_n(t),\ 0 \le t < \beta''_n.$$

This completes the proof of assertion (a).

Proof of (b). Let $l'_{ij}(t)$, $i,j=1,...,m$, be of arbitrary signs. Then, in view of relation (3.5), we have

$$\begin{aligned} z'(t) &\le c'(t) + \sum_{i,j=1}^{m} \tilde{b}_{ij}(t) z(t)^{\alpha_{ij}+\beta_{ij}} + \sum_{i,j=1}^{m} l_{ij}(0)\psi_{ij}(z(t)) \\ &\quad + \sum_{i,j=1}^{m} \int_0^\infty \left|l'_{ij}(\sigma)\right| \psi_{ij}(z(t-\sigma))\,d\sigma,\ t>0. \end{aligned} \tag{3.8}$$

The integral term in (3.8) may be treated by introducing the auxiliary function:

$$\tilde{z}(t) = z(t) + \sum_{i,j=1}^{m} \int_0^\infty \left|l'_{ij}(s)\right| \int_{t-s}^{t} \psi_{ij}(z(\sigma))\,d\sigma ds,\ t \ge 0.$$

Differentiating $\tilde{z}(t)$, and using (3.8), we can write

$$\begin{aligned} \tilde{z}'(t) &= z'(t) + \sum_{i,j=1}^{m} \int_0^\infty \left|l'_{ij}(s)\right| \left[\psi_{ij}(z(t)) - \psi_{ij}(z(t-s))\right] d\sigma ds \\ &\le c'(t) + \sum_{i,j=1}^{m} \tilde{b}_{ij}(t) z(t)^{\alpha_{ij}+\beta_{ij}} + \sum_{i,j=1}^{m} l_{ij}(0)\psi_{ij}(z(t)) \\ &\quad + \sum_{i,j=1}^{m} \int_0^\infty \left|l'_{ij}(\sigma)\right| \psi_{ij}(z(t-\sigma))\,d\sigma \\ &\quad + \sum_{i,j=1}^{m} \int_0^\infty \left|l'_{ij}(s)\right| \left[\psi_{ij}(z(t)) - \psi_{ij}(z(t-s))\right] ds \\ &\le c'(t) + \sum_{i,j=1}^{m} \left\{ \tilde{b}_{ij}(t) z(t)^{\alpha_{ij}+\beta_{ij}} + \left[l_{ij}(0) + \int_0^\infty \left|l'_{ij}(s)\right| ds\right] \psi_{ij}(z(t)) \right\},\ t>0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \tilde{z}(t) &\le \tilde{z}(0) + c(t) \\ &\quad + \sum_{i,j=1}^{m} \int_0^t \left\{ \tilde{b}_{ij}(s) (\tilde{z}(s))^{\alpha_{ij}+\beta_{ij}} + \left[l_{ij}(0) + \int_0^\infty \left|l'_{ij}(\sigma)\right| d\sigma\right] \psi_{ij}(\tilde{z}(s)) \right\} ds \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} \tilde{z}(0) &= x_0(0) \\ &\quad + \sum_{i,j=1}^{m} \int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))\,d\sigma + \sum_{i,j=1}^{m} \int_0^\infty \left|l'_{ij}(s)\right| \int_{-s}^{0} \psi_{ij}(u_0(\sigma))\,d\sigma ds. \end{aligned}$$

Finally, we apply Lemma 2.1 to (3.9) and use (3.8), to obtain

$$\tilde{x}(t) \le \tilde{z}(t) \le \left[x_0(0) + \sum_{i,j=1}^{m} \int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))\,d\sigma \right] \tilde{\varphi}_n(t),\ 0 \le t < \tilde{\beta}'_n$$

and

$$\tilde{x}(t) \leq \tilde{z}(t) \leq \left[x_0(0) + \sum_{i,j=1}^{m} \int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))\, d\sigma \right] \tilde{\xi}_n(t), \; 0 \leq t < \tilde{\beta}_n''.$$

This completes the proof of assertion (b). Theorem 3.1 is proved.

Corollary 3.1. *If, in addition to the hypotheses of the theorem, β_n', β_n'', $\tilde{\beta}_n'$ and $\tilde{\beta}_n''$ are infinite, then we have global existence of solutions.*

Corollary 3.2. *If, in addition to the hypotheses of the theorem, we assume that $[z(0) + c(t)]\, \varphi_n(t)$ and $\left[x_0(0) + \sum_{i,j=1}^{m} \int_{-\infty}^{0} l_{ij}(-\sigma)\psi_{ij}(x_0(\sigma))\, d\sigma \right] \tilde{\varphi}_n(t)$ grow at most polynomially, and $\exp\left[\int_0^t a(s)ds\right] \to \infty$ as $t \to \infty$, then the solutions decay in exponential rate.*

Remarks:

1. The smallness condition in the initial data is dictated by Lemma 2.1. Indeed, it is required for existence of functions $\varphi_j(t)$, $j = 1, ..., n$. It will be superfluous, for instance, if the functions $G_j(u)$ have infinite range. However, the other conditions on the initial data in the statement of the result remain the same.

2. The classical Hopfield neural network system with distributed delays

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t) \int_{-\infty}^{t} l_{ij}(t-s)\psi_{ij}(|x_j(s)|)\, ds + c_i,$$

may be considered as a special case of ours when $\alpha_{ij} = 0$ and $\beta_{ij} = 1$, $i, j = 1, ..., m$. Regarding the asymptotic behavior, our Corollary 3.2 shows that the condition on $c(t)$

$$c(t) := \int_0^t \exp\left[\int_0^s a(\sigma)d\sigma\right] \sum_{i=1}^{m} |c_i(s)|\, ds, \; t > 0,$$

for the 'constants' c_i becomes $c_i = 0$, $i = 1, ..., m$. The convergence to zero would mean stability of the equilibrium 0 ($\psi_{ij}(0) = 0$, $i, j = 1, ..., m$).

3. The class $H_{r,\omega}$ is sufficiently large. For instance, it contains all submultiplicative functions ψ since $\psi \in H_{\psi,\psi}$. It contains also the class $\mathcal{F}$ introduced by Deo and Dongade [3]. Recall that the class $\mathcal{F}$ is formed by all nondecreasing continuous functions ψ in $\mathbf{R}_+$ such that $\psi(u) > 0$ for $u > 0$ and $\frac{1}{a}\psi(u) \leq \psi(\frac{u}{a})$, $u \geq 0$, $a \geq 1$. To see that $H_{r,\omega}$ contains $\mathcal{F}$, is it is enough to take r satisfying $r(a) = \max(1, a)$.

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