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GENERALIZED (JORDAN) LEFT DERIVATIONS ON RINGS ASSOCIATED WITH AN ELEMENT OF RINGS

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Abstract. In this paper, we introduce a new notion of generalized (Jordan) left derivation on rings as follows: let R be a ring, an additive mapping $F : R \longrightarrow R$ is called a generalized (resp. Jordan) left derivation if there exists an element $w \in R$ such that $F(xy) = xF(y) + yF(x) + yxw$ (resp. $F(x^2) = 2xF(x) + x^2w$) for all $x, y \in R$. Then, some related properties and results on generalized (Jordan) left derivation of square closed Lie ideals are obtained.

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1. Introduction

Throughout the paper R will denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ will stand for the commutator $xy - yx$. We will use the commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$. A mapping $\sigma : R \longrightarrow R$ is said to be commuting if $[\sigma(x), x] = 0$ for all $x \in R$. Recall that a ring R is prime if $xRy = 0$ implies that either $x = 0$ or $y = 0$, and R is semiprime if $xRx = 0$ implies that $x = 0$. A ring R is n-torsion free, where $n > 1$ is an integer, if $nx = 0$, $x \in R$, implies that $x = 0$. An additive subgroup U of R is called a Lie ideal if $[U, R] \subseteq U$. A Lie ideal U is called square closed if $u^2 \in U$ for all $u \in U$. Note that for every u, v in square closed Lie ideal U, we have $uv + vu = (u + v)^2 - u^2 - v^2 \in U$ and $uv - vu \in U$, and hence $2uv \in U$ for all $u, v \in U$.

Following [9], an additive mapping $d : R \longrightarrow R$ we will call a derivation (resp. a Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$), for all $x, y \in R$. In particular, for a fixed $a \in R$, the mapping $I_a : R \longrightarrow R$ given by $I_a(x) = [a, x]$ is a derivation, and is called an inner derivation. Following [15], an additive mapping $H : R \longrightarrow R$ is called a left (resp. right) centralizer (multiplier) of

R if $H(xy) = H(x)y$ (resp. $H(xy) = xH(y)$), for all $x, y \in R$. An additive mapping $H : R \longrightarrow R$ is called a left (resp. right) Jordan centralizer (multiplier) of R if $H(x^2) = H(x)x$ (resp. $H(x^2) = xH(x)$), for all $x \in R$. An additive function $F : R \longrightarrow R$ is called a generalized inner derivation if $F(x) = ax + xb$, for fixed $a, b \in R$. For such a mapping F, it is easy to see that $F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y)$, for all $x, y \in R$. This observation leads to the following definition, given by Brešar in [6]: An additive mapping $F : R \longrightarrow R$ is called a generalized derivation (resp. generalized Jordan derivation) if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ (resp. $F(x^2) = F(x)x + xd(x)$), for all $x, y \in R$. Hence, the concept of generalized derivation covers both the concept of derivation and the concept of left centralizer.

In [10], another type of generalized (Jordan) derivation is defined as follows: an additive mapping $F : R \longrightarrow R$ is called a generalized derivation (resp. generalized Jordan derivation) if there exists an element $w \in R$ such that $F(xy) = F(x)y + xF(y) + xwy$ (resp. $F(x^2) = F(x)x + xF(x) + xwx$), for all $x, y \in R$.

The concepts of left derivation and Jordan left derivation were introduced by Brešar and Vukman in [7], defined as follows: an additive mapping $d : R \longrightarrow R$ is called a left derivation (resp. a left Jordan derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$), for all $x, y \in R$. Ashraf and Ali [3], generalized the notions of left and Jordan left derivations as follows: an additive mapping $F : R \longrightarrow R$ is called a generalized left derivation if there exists a left derivation $d : R \longrightarrow R$ such that

$$F(xy) = xF(y) + yd(x), \text{for all } x, y \in R, \tag{1.1}$$

and an additive mapping $F : R \longrightarrow R$ is called a generalized Jordan left derivation if there exists a Jordan left derivation $d : R \longrightarrow R$ such that

$$F(x^2) = xF(x) + xd(x), \text{for all } x \in R. \tag{1.2}$$

We denote (1.1) and (1.2) by (F, d). It is easy to see that $F : R \longrightarrow R$ is a generalized left derivation if and only if F is of the form $F = d + H$, where d is a left derivation and H is a right centralizer on R. The concept of generalized left derivation covers the concepts of left derivation and right centralizer. It is easy to see that every generalized left derivation on a ring R is a generalized Jordan left derivation. However, the converse is not true in general (see Example 1.1 of [3]). In [3] it was shown that if R is a 2-torsion free prime ring, then every generalized Jordan left derivation on R is a generalized left derivation. Further, Ali [1] showed that the above result remains valid for 2-torsion free semiprime ring R. For some properties of Jordan left derivation and generalized Jordan left derivation, we refer the reader to [1, 2, 3, 4, 7, 8].

Now, we introduce another type of a generalized left derivation and a generalized Jordan left derivation. Let R be a ring, an additive mapping $F : R \longrightarrow R$ is called a generalized left derivation if there exists an element $w \in R$ such that

$$F(xy) = xF(y) + yF(x) + yxw, \tag{1.3}$$

and F is called a generalized Jordan left derivation if there exists $w \in R$ such that

$$F(x^2) = 2xF(x) + x^2 w. \tag{1.4}$$

We denote (1.3) and (1.4) by (F, w).

Observe that if (F, w) is a generalized left derivation of type (1.3), then $F + w_r : R \longrightarrow R$ is a left derivation, where $w_r : R \longrightarrow R$ is defined as $w_r(x) = xw$. Indeed, we have $(F + w_r)(xy) = F(xy) + w_r(xy) = xF(y) + yF(x) + yxw + xyw = (xF(y) + xyw) + (yF(x) + yxw) = x(F + w_r)(y) + y(F + w_r)(x)$. Also, $(F, F + w_r)$ is a generalized left derivation of type (1.1), because $F(xy) = xF(y) + yF(x) + yxw = xF(y) + y(F + w_r)(x)$. In this case R has an identity 1. The converse is also valid, that is, if (F, d) is a generalized left derivation of type (1.1), then $(F, -F(1))$ is a generalized left derivation of type (1.3), because $F(xy) = xF(y) + yd(x) = xF(y) + y(F(x) - xF(1)) = xF(y) + yF(x) + yx(-F(1))$.

Example 1.1. *Consider the ring $(\mathbb{Z}, +, \cdot)$. For $a \in \mathbb{Z}$ we set $w = -a$ and define the map $F : \mathbb{Z} \longrightarrow \mathbb{Z}$ as $F(x) = ax$, for all $x \in \mathbb{Z}$. Then it is easy to see that (F, w) is a generalized left derivation.*

Example 1.2. *Let $M = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$. Then M with usual addition and multiplication of matrices is a ring. Suppose that $w = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ and define the map $F : M \longrightarrow M$ as $F\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then (F, w) is a generalized left derivation.*

Example 1.3. *Let $M = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$. Then M with usual addition and multiplication of matrices is a ring. Suppose that $w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and define* *the map* $F : M \longrightarrow M$ *as* $F\left(\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$. *Then* (F, w) *is a generalized left derivation.*

Example 1.4. *Let* Ω *be a power set. For all* $A, B \in \Omega$, *we define* $A + B = A\Delta B = A \cup B - A \cap B$ *and* $A \cdot B = A \cap B$. *Then* $(\Omega, +, \cdot)$ *is a ring. Suppose that* $w = \Omega$ *and define the map* $F : \Omega \longrightarrow \Omega$ *as* $F(A) = A$, *for all* $A \in \Omega$. *Then* (F, w) *is a generalized left derivation.*

2. Generalized left derivations associated with an element on rings

In this section, we prove some properties of generalized (Jordan) left derivation (F, w) on semiprime and prime rings. To this end, we first recall and prove some necessary lemmas.

Lemma 2.1. *Let* R *be a 2-torsion free ring and let* L *be a square closed Lie ideal of* R. *If* $(F, w) : R \longrightarrow R$ *is a generalized Jordan left derivation on* L, *then the following assertions hold:*

(1) $F(uv + vu) = 2uF(v) + 2vF(u) + uvw + vuw$;
(2) $F(uvu) = u^2F(v) - vuF(u) + 3uvF(u) + u^2vw + 2uvuw - vu^2w$;
(3) $F(uvz + zvu) = (uz + zu)F(v) + 3uvF(z) + 3zvF(u) - vuF(z) - vzF(u) + uzvw + zuvw + 2uvzw + 2zvuw - vuzw - vzuw$;
(4) $[u, v]u(F(u) + uw) = u[u, v](F(u) + uw)$;
(5) $[u, v](F(uv) - uF(v) - vF(u) - vuw) = 0$;
(6) $F([u, v]^2) = [u, v]F([u, v])$.

Lemma 2.2 ([13]). *Let* R *be a semiprime ring and let the relation* $axb + bxc = 0$ *hold for all* $x \in R$ *and for some* $a, b, c \in R$. *Then* $(a + c)xb = 0$ *for all* $x \in R$.

Lemma 2.3 ([14]). *Let* R *be a 2-torsion free semiprime ring and let* $F : R \longrightarrow R$ *be an additive mapping satisfying* $[[F(x), x], x] = 0$ *for all* $x \in R$. *Then* $[F(x), x] = 0$, *for all* $x \in R$.

Lemma 2.4 ([11]). *Let* R *be a prime ring and let* a, b, c *be elements of* R *such that* $arbrc = 0$ *for all* $r \in R$. *Then* $a = 0$ *or* $b = 0$ *or* $c = 0$.

Lemma 2.5 ([11]). *Let* R *be a 2-torsion free prime ring and let* d_1, d_2 *be derivations of* R *such that* d_1d_2 *is also a derivation. Then* $d_1 = 0$ *or* $d_2 = 0$.

Lemma 2.6. *Let R be a prime ring and let $(F, w) : R \longrightarrow R$ be a nonzero generalized Jordan left derivation. Further, let $a \in R$ be such that $F(a) \neq -aw$. Then $(I_a^2)^2 = (I_a(I_a))^2 = 0$, where $I_a(x) = [a, x]$ is an inner derivation associated to a.*

Proof. By Lemma 2.1 (4), we have $[a, [a, x]]F(a) + [a, [a, x]]aw = 0$, for all $x \in R$. This implies that

$$I_a^2(x)(F(a) + aw) = [a, [a, x]](F(a) + aw) = 0. \tag{2.1}$$

On the other hand, we have $I_a^2(xy) = I_a^2(x)y + 2I_a(x)I_a(y) + xI_a^2(y)$. Therefore, by (2.1), we get

$$\begin{aligned} 0 &= I_a^2(xy)(F(a) + aw) = (I_a^2(x)y + 2I_a(x)I_a(y) + xI_a^2(y))(F(a) + aw) \\ &= (I_a^2(x)y + 2I_a(x)I_a(y))(F(a) + aw). \end{aligned} \tag{2.2}$$

We replace y by $I_a(yz)$ in (2.2), and use (2.1), to get $I_a^2(x)I_a(yz)(F(a) + aw) = 0$. This implies that

$$I_a^2(x)I_a(y)z(F(a) + aw) + I_a^2(x)yI_a(z)(F(a) + aw) = 0. \tag{2.3}$$

Next, we replace z by $I_a(z)$ in (2.3), and use (2.1), to obtain

$$I_a^2(x)I_a(y)I_a(z)(F(a) + aw) = 0. \tag{2.4}$$

In (2.3), we replace y by $I_a(y)$ and use (2.4), to get

$$I_a^2(x)I_a^2(y)z(F(a) + aw) = 0.$$

So, we have $I_a^2(x)I_a^2(y) = 0$, since R is prime and $F(a) \neq -aw$. Finally, we replace y by x, to obtain $(I_a^2(x))^2 = ([a, [a, x]])^2 = 0$, for all $x \in R$. □

Lemma 2.7 ([5]). *Let R be a 2-torsion free prime ring and let L be a Lie ideal of R such that $L \not\subseteq Z(R)$. If $x, y \in R$ such that $xLy = 0$, then $x = 0$ or $y = 0$.*

Lemma 2.8 ([12]). *Let R be a 2-torsion free prime ring and let L be a nonzero Lie ideal of R. If L is commutative, that is, $[u, v] = 0$ for all $u, v \in L$, then $L \subseteq Z(R)$.*

Using arguments similar to those applied in the proof of Theorem 3.1 of [3], we can prove the following result.

Proposition 2.1. *Let R be a 2-torsion free ring and let $(F, w) : R \longrightarrow R$ be a generalized Jordan left derivation. Further, let L be a square closed Lie ideal of R such that L has a commutator which is not a left zero divisor. Then (F, w) is a generalized left derivation on L.*

Corollary 2.1. *Let R be a 2-torsion free ring such that R has a commutator which is not a left zero divisor. Also, let $(F,w) : R \longrightarrow R$ be a generalized Jordan left derivation. Then (F,w) is a generalized left derivation on R.*

Theorem 2.1. *Let R be a 2-torsion free semiprime ring and let $(F,w) : R \longrightarrow R$ be a generalized Jordan left derivation, where $w \in Z(R)$. Then F is commuting on R.*

Proof. By Lemma 2.1, we have for all $x, y \in R$,

$$F(xy + yx) = 2xF(y) + 2yF(x) + xyw + yxw, \tag{2.5}$$

$$F(xyx) = x^2F(y) + 3xyF(x) - yxF(x) + x^2yw + 2xyxw - yx^2w. \tag{2.6}$$

In (2.5), we replace y by xyx and use (2.6) to get

$$\begin{aligned} F(x^2yx + xyx^2) &= 2xF(xyx) + 2xyxF(x) + x^2yxw + xyx^2w \\ &= 2x^3F(y) + 6x^2yF(x) + 2x^3yw + 5x^2yxw - xyx^2w. \end{aligned} \tag{2.7}$$

In (2.6), we replace y by $xy + yx$ and use (2.5) to obtain

$$\begin{aligned} F(x^2yx + xyx^2) &= x^2F(xy + yx) + 3x(xy + yx)F(x) - (xy + yx)xF(x) + x^2(xy + yx)w \\ &\quad + 2x(xy + yx)xw - (xy + yx)x^2w \\ &= 2x^3F(y) + 5x^2yF(x) + 2xyxF(x) - yx^2F(x) + 2x^3yw + 4x^2yxw \\ &\quad + xyx^2w - yx^3w. \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we have for all $x, y \in R$,

$$x^2yF(x) - 2xyxF(x) + yx^2F(x) + x^2yxw - 2xyx^2w + yx^3w = 0. \tag{2.9}$$

Replace y by $F(x)y$ in (2.9) to obtain

$$\begin{aligned} &x^2F(x)yF(x) - 2xF(x)yxF(x) + F(x)yx^2F(x) + x^2F(x)yxw \\ &\quad - 2xF(x)yx^2w + F(x)yx^3w = 0. \end{aligned} \tag{2.10}$$

Left multiplication in (2.9) by $F(x)$ yields

$$\begin{aligned} &F(x)x^2yF(x) - 2F(x)xyxF(x) + F(x)yx^2F(x) + F(x)x^2yxw \\ &\quad - 2F(x)xyx^2w + F(x)yx^3w = 0. \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11), we obtain for all $x, y \in R$,

$$\begin{aligned} &[F(x), x^2]yF(x) - 2[F(x), x]yxF(x) + \\ &\quad + [F(x), x^2]yxw - 2[F(x), x]yx^2w = 0. \end{aligned} \tag{2.12}$$

Replace y by yx in (2.12), to get

$$[F(x),x^2]yxF(x)-2[F(x),x]yx^2F(x)+ \\ +[F(x),x^2]yx^2w-2[F(x),x]yx^3w=0. \tag{2.13}$$

Right multiplicaion in (2.12) by x yields

$$[F(x),x^2]yF(x)x-2[F(x),x]yxF(x)x+ \\ +[F(x),x^2]yxwx-2[F(x),x]yx^2wx=0. \tag{2.14}$$

Combining (2.13) and (2.14), and taking into account that $w \in Z(R)$, we obtain for all $x, y \in R$,

$$[F(x),x^2]y[F(x),x]+[F(x),x]y(-2x[F(x),x])=0.$$

So, by Lemma 2.2, we have $\left([F(x),x^2]-2x[F(x),x]\right)y[F(x),x]=0$, which implies that

$$[[F(x),x],x]y[F(x),x]=0. \tag{2.15}$$

By (2.15), we obtain $[[F(x),x],x]y[[F(x),x],x]=0$. Therefore, $[[F(x),x],x]=0$, since R is semiprime. So, we can apply Lemma 2.3 to conclude that $[F(x),x]=0$ for all $x \in R$. □

Theorem 2.2. *Let R be a ring and $(F,w): R \longrightarrow R$ be a nonzero generalized left derivation. Then the following assertions hold:*

(1) *If R is prime, then R is commutative or $F(x)=-xw$, for all $x \in R$.*
(2) *If R is semiprime and $w \in Z(R)$, then F maps $Z(R)$ into $Z(R)$.*

Proof. To prove assertion (1) of the theorem, observe first that for all $x, y \in R$,

$$\begin{aligned} F(x(yx)) &= xF(yx)+yxF(x)+yx^2w \\ &= xyF(x)+x^2F(y)+yxF(x)+x^2yw+yx^2w. \end{aligned} \tag{2.16}$$

On the other hand, we have

$$\begin{aligned} F((xy)x) &= xyF(x)+xF(xy)+x^2yw \\ &= xyF(x)+x^2F(y)+xyF(x)+xyxw+x^2yw. \end{aligned} \tag{2.17}$$

Comparing (2.16) and (2.17), we obtain $(xy-yx)F(x)+(xy-yx)xw=0$. So, $(xy-yx)(F(x)+xw)=0$, for all $x, y \in R$. Replace y by zy, where $z \in R$ we get

$$\begin{aligned} 0 &= (xzy-zyx)(F(x)+xw)=(xzy-zxy+zxy-zyx)(F(x)+xw) \\ &= (xz-zx)y(F(x)+xw)+z(xy-yx)(F(x)+xw)=(xz-zx)y(F(x)+xw). \end{aligned}$$

Therefore,

$$(xz - zx)y(F(x) + xw) = 0, \text{for all } x, y, z \in R. \tag{2.18}$$

The last relation implies that either $x \in Z(R)$ or $F(x) = -xw$ for all $x \in R$. Put $A = \{x \in R \mid x \in Z(R)\}$ and $B = \{x \in R \mid F(x) = -xw\}$, and observe that $R = A \cup B$. So, either $R = A$ or $R = B$, since A and B are subgroups of R. If $R = A$, then R is commutative. If $R = B$, then $F(x) = -xw$, for all $x \in R$. This completes the proof of assertion (1).

To prove assertion (2) of the theorem, we put $x + a$ instead of x in the relation (2.18), where $a \in Z(R)$, to get $0 = (xz - zx)y(F(a) + aw) + (az - za)y(F(x) + xw) = (xz - zx)y(F(a) + aw)$. So, we have $(xz - zx)y(F(a) + aw) = 0$, and replace z by $F(a)$ to obtain $(xF(a) - F(a)x)y(F(a) + aw) = 0$.

Now we replace y by yx to get $(xF(a) - F(a)x)y(xF(a) + xaw) = 0$.

Therefore, we have $(xF(a) - F(a)x)y(xF(a) + xaw) - (xF(a) - F(a)x)y(F(a)x + awx) = 0$. So, $(xF(a) - F(a)x)y(xF(a) - F(a)x) = 0$, since $a, w \in Z(R)$. Hence, $xF(a) - F(a)x = 0$, implying that $F(a) \in Z(R)$. □

Theorem 2.3. *Let R be a 6-torsion free prime ring and let $(F, w) : R \longrightarrow R$ be a nonzero generalized Jordan left derivation. If $a \in R$ is such that $a^2 = 0$, then $F(a) = -aw$.*

Proof. For $a = 0$, the statement is obvious. So, we suppose that $a \neq 0$. Then we have $0 = F(0) = F(a^2) = 2aF(a) + a^2w = 2aF(a)$, implying that $aF(a) = 0$, since R is a 2-torsion free ring. By Lemma 2.1 (2), we get for all $y \in R$,

$$\begin{aligned} F(aya) &= a^2F(y) - yaF(a) + 3ayF(a) + a^2yw + 2ayaw - ya^2w \\ &= 3ayF(a) + 2ayaw. \end{aligned} \tag{2.19}$$

So, we have for all $x, y \in R$,

$$F(a(xay + yax)a) = 3axayF(a) + 3ayaxF(a) + 2axayaw + 2ayaxaw. \tag{2.20}$$

On the other hand, by Lemma 2.1 (3) and the relation (2.19), we have

$$\begin{aligned} F(a(xay + yax)a) &= F(ax(aya) + (aya)xa) \\ &= 3axF(aya) + 3ayaxF(a) - xaF(aya) + 2axayaw + 2ayaxaw \\ &= 9axayF(a) + 3ayaxF(a) + 8axayaw + 2ayaxaw. \end{aligned} \tag{2.21}$$

Comparing (2.20) and (2.21), we obtain

$$6axayF(a) + 6axayaw = 0.$$

Thus, $(axa)y(F(a)+aw) = 0$, since R is 6-torsion free. This implies that $F(a) = -aw$, since R is prime and $a \neq 0$. □

Theorem 2.4. *Let R be a prime ring such that $char(R) \neq 2, 3$, and let $(F, w) : R \longrightarrow R$ be a nonzero generalized Jordan left derivation. If there exists a nonzero $a \in R$ such that $a^2 = 0$, then $F(x) = -xw$ for all $x \in R$.*

Proof. By the hypothesis, we have $0 = F(0) = F(a^2) = 2aF(a) + a^2w = 2aF(a)$. So, $aF(a) = 0$, since R is 2-torsion free. Therefore, by Lemma 2.1 (1), $F(aba) = F(a(ba) + (ba)a) = 2aF(ba) + 2baF(a) + ba^2w + abaw = 2aF(ba) + abaw$, for all $b \in R$. This implies that

$$F(aba) = 2aF(ba) + abaw. \tag{2.22}$$

On the other hand, by Lemma 2.1 (2), we have

$$F(aba) = a^2F(b) - baF(a) + 3abF(a) + a^2bw + 2abaw - ba^2w = 2abaw \tag{2.23}$$

Comparing (2.22) and (2.23), we get for all $b \in R$,

$$2aF(ba) = abaw. \tag{2.24}$$

By (2.24), we obtain

$$F(baba) = F((ba)^2) = 2baF(ba) + babaw = 2babaw. \tag{2.25}$$

Also, by Lemma 2.1 (2), we have

$$\begin{aligned} F(ab^2a) &= a^2F(b^2) - b^2aF(a) + 3ab^2F(a) + a^2b^2w + 2ab^2aw - b^2a^2w \\ &= 2ab^2aw. \end{aligned} \tag{2.26}$$

By (2.25) and (2.26), we find

$$2\left(F(ab^2a) + F(baba)\right) = 4ab^2aw + 4babaw. \tag{2.27}$$

On the other hand, by Lemma 2.1 (3) and (2.24), we obtain

$$\begin{aligned} 2\left(F(ab^2a) + F(baba)\right) &= 2\left(F(ab^2a + baba)\right) \\ &= 2(abaF(b) + ba^2F(b) - baF(ba) - b^2aF(a) + ababw + ba^2bw \\ &\quad +2ab^2aw + 2babaw - babaw - b^2a^2w) \\ &= 2(abaF(b) - baF(ba) + ababw + 2ab^2aw + babaw) \\ &= 2abaF(b) + babaw + 2ababw + 4ab^2aw. \end{aligned} \tag{2.28}$$

By comparing (2.27) and (2.28), we get $2abaF(b) + 2ababw = 0$. This implies that

$$aba(F(b) + bw) = 0, \tag{2.29}$$

since R is 2-torsion free. We replace b by $b+c$ in (2.29), where $c \in R$, to get

$$aba(F(c)+cw)+aca(F(b)+bw)=0 \tag{2.30}$$

Next, we replace c by $ac+ca$ in (2.30), to obtain

$$aba(F(ac+ca)+(ac+ca)w)=0.$$

Therefore, by Lemma 2.1 (1), we find

$$0=aba(2aF(c)+2cF(a)+2acw+2caw)=2abacF(a)+2abacaw.$$

So, $abac(F(a)+aw)=0$, since R is 2-torsion free. Now, Lemma 2.4 implies that

$$F(a)=-aw, \tag{2.31}$$

since $a \neq 0$. Hence, by Lemma 2.1 (2), we have

$$\begin{aligned} acaF(bab) &= aca(b^2F(a)-abF(b)+b^2aw+2babw-ab^2w) \\ &= acab^2F(a)+acab^2aw+2acababw=2acababw. \end{aligned} \tag{2.32}$$

Replace b by bab in (2.30) and use (2.32), to obtain

$$\begin{aligned} 0 &= ababa(F(c)+cw)+aca(F(bab)+babw) \\ &= ababa(F(c)+cw)+3acababw=ababa(F(c)+cw). \end{aligned}$$

Thus, $ababa(F(c)+cw)=0$. Now, Lemma 2.4 implies that $a(F(c)+cw)=0$, since $a \neq 0$. So, we have

$$aF(c)=-acw. \tag{2.33}$$

Next, replace c by c^2 in (2.33), to get $acF(c)+ac^2w=0$. Then, replace c by $c+b$, to obtain $acF(b)+abF(c)+acbw+abcw=0$. Also, replace c by ac, to find $ab(F(ac)+acw)=0$. So, we have

$$F(ac)=-acw, \tag{2.34}$$

since R is prime and $a \neq 0$. Hence, in view of (2.31), (2.33), (2.34) and Lemma 2.1 (1), we can write

$$\begin{aligned} -acw+F(ca) &= F(ac)+F(ca)=F(ac+ca)=2aF(c)+2cF(a)+acw+caw \\ &= 2aF(c)+2cF(a)+acw+caw=-acw-caw. \end{aligned}$$

Therefore,

$$F(ca)=-caw. \tag{2.35}$$

Now, we replace c by bc in (2.34) and c by cb in (2.35), to obtain $F(abc) = -abcw$ and $F(cba) = -cbaw$. Hence, by Lemma 2.1 (3), we have

$$\begin{aligned}-abcw - cbaw &= F(abc) + F(cba) = F(abc + cba)\\ &= acF(b) + caF(b) - baF(c) - bcF(a) + acbw + cabw\\ &\quad +2abcw + 2cbaw - bacw - bcaw\\ &= acF(b) + acbw + 2abcw + 2cbaw.\end{aligned}$$

Thus, $ac(F(b) + bw) = 0$, implying that $F(b) = -bw$ for all $b \in R$, since R is prime and $a \neq 0$. □

Corollary 2.2. *Let R be a prime ring such that $char(R) \neq 2, 3$. If there exists a nonzero generalized Jordan left derivation $F : R \longrightarrow R$ such that $F(a) \neq -aw$ for some $a \in R$, then R is commutative.*

Proof. Let $a \in R$ be such that $F(a) \neq -aw$. Then, by Lemma 2.6, we have $(I_a^2(x))^2 = ([a, [a, x]])^2 = 0$ for all $x \in R$. So, by Theorem 2.4, we get $I_a^2(x) = [a, [a, x]] = 0$. Now, Lemma 2.5 implies that $I_a(x) = [a, x] = 0$ for all $x \in R$. This means that $a \in Z(R)$. Put $A = \{x \in R \mid x \in Z(R)\}$ and $B = \{x \in R \mid F(x) = -xw\}$, and observe that $R = A \cup B$. So, $R = A$ or $R = B$, since A and B are subgroups of R. If $R = B$, then $F(x) = -xw$ for all $x \in R$, yielding a contradiction. Thus, $R = A = Z(R)$, implying that R is commutative. □

Theorem 2.5. *Let R be a 2-torsion free prime ring and let L be a nonzero square closed Lie ideal of R such that L has no a nonzero nilpotent element of order 2. Further, let $(F, w) : R \longrightarrow R$ be a generalized Jordan left derivation. Then (F, w) is a generalized left derivation on L.*

Proof. If L is commutative, then by Lemma 2.8, we have $L \subseteq Z(R)$. So, by Lemma 2.1 (1), we get $2F(uv) = 2\{uF(v) + vF(u) + vuw\}$ for all $u, v \in L$. Therefore, $F(uv) = uF(v) + vF(u) + vuw$, since R is a 2-torsion free ring.

Now, let L be noncommutative. Then by Lemma 2.1 (4), we have for all $u, v \in L$,

$$u^2vF(u) + vu^2F(u) - 2uvuF(u) + u^2vuw + vu^3w - 2uvu^2w = 0.$$

In the above relation, we replace u by $[u, z_0]$, where $z_0 \in L$, and use Theorem 2.4, to obtain

$$[u, z_0]^2v(F([u, z_0]) + [u, z_0]w) = 0 \text{ for all } u, v \in L.$$

So, by Lemma 2.7, we have $[u,z_0]^2=0$ or $F([u,z_0])=-[u,z_0]w$. Therefore, $[u,z_0]=0$ or $F([u,z_0])=-[u,z_0]w$ for all $u\in L$, since by assumption, L has no a nonzero nilpotent element of order 2.

Next, we put $A=\{u\in L\mid [u,z_0]=0\}$ and $B=\{u\in L\mid F([u,z_0])=-[u,z_0]w\}$, and observe that $L=A\cup B$. So, either $L=A$ or $L=B$, since A and B are subgroups of L. If $L=A$, then $uz_0=z_0u$ for all $u\in L$. So, by Theorem 2.1 of [1], we have $F(uz_0)=uF(z_0)+z_0F(u)+z_0uw$ for all $u\in L$, since R is 2-torsion free. If $L=B$, then $F([u,z_0])=-[u,z_0]w$ for all $u\in L$. Therefore

$$F(uz_0)-F(z_0u)=z_0uw-uz_0w. \tag{2.36}$$

On the other hand, we have

$$F(uz_0)+F(z_0u)=2uF(z_0)+2z_0F(u)+uz_0w+z_0uw. \tag{2.37}$$

Adding the equations in (2.36) and (2.37), we get $F(uz_0)=uF(z_0)+z_0F(u)+z_0uw$ for all $u\in L$, since R is 2-torsion free. Hence, we have $F(uz)=uF(z)+zF(u)+zuw$ for all $u,z\in L$. □

Corollary 2.3. *Let R be a 2-torsion free prime ring and let $(F,w):R\longrightarrow R$ be a generalized Jordan left derivation such that R has no a nonzero nilpotent element of order 2. Then (F,w) is a generalized left derivation on R.*

Theorem 2.6. *Let R be a 2-torsion free prime ring such that R has no a nonzero nilpotent element of order 2. Further, let $(F,w):R\longrightarrow R$ be a generalized Jordan left derivation. Then R is commutative or $F(x)=-xw$ for all $x\in R$.*

Proof. The result immediately follows from Corollary 2.3 and Theorem 2.2 (1). □

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