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NONLOCAL SEMI-LINEAR FRACTIONAL-ORDER BOUNDARY VALUE PROBLEMS WITH STRIP CONDITIONS

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Abstract. This paper is concerned with the question of existence of solutions for one-dimensional higher-order semi-linear fractional differential equations supplemented with nonlocal strip type boundary conditions. The nonlocal strip condition addresses a situation where the linear combination of the values of unknown function at two nonlocal points, located to the left and right hand sides of the strip, respectively, is proportional to its strip value. The case of Stieltjes type strip condition is also discussed. Our results, relying on some standard fixed point theorems, are supported with illustrative examples.

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1. Introduction

In this paper, we consider a new class of boundary value problems of Caputo type fractional differential equations of arbitrary order involving a nonlocal sub-strip condition given by

$$(1.1)\qquad \begin{cases} {}^cD^q x(t) = f(t, x(t)),\ n-1 < q \le n,\ n \ge 2,\ t \in [0,1], \\ x(0) = x'(0) = x''(0) = \dots = x^{(n-2)}(0) = 0, \\ a x(\zeta_1) + b x(\zeta_2) = c \displaystyle\int_\eta^\xi x(s)ds,\ 0 < \zeta_1 < \eta < \xi < \zeta_2 < 1, \end{cases}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and a, b, c are real constants.

In (1.1), the nonlocal strip condition can be interpreted as follows: the linear combination of the values of unknown function at two points ζ_1 and ζ_2, located to the left and right hand sides of the strip, respectively, is proportional to its strip value ($\int_\eta^\xi x(s)ds$). This situation has interesting applications in oil exploration (geophysics) and acoustic scattering (scattering from a strip together with some nonlocal scatterers off the strip located on a given boundary).

The interest in the study of fractional calculus mainly owes to its extensive theoretical development and widespread applications in a variety of disciplines such as biological sciences, ecology, aerodynamics, control theory, viscoelasticity, electro-dynamics of complex medium, electron-analytical chemistry, environmental issues, etc. The nonlocal characteristic of fractional-order differential and integral operators helps to trace the past history of several materials and processes, and thus fractional calculus' tools have contributed toward revolutionizing the traditional mathematical modeling techniques based on integer-order calculus. More details on the topic can be found in [1]-[7]. Fractional-order boundary value problems involving classical, nonlocal, multi-point, periodic and anti-periodic, fractional-order, and integral boundary conditions have recently been investigated by many researchers (see, [8]-[26], and references therein). The paper is organized as follows. In Section 2, we recall some preliminary concepts of fractional calculus and establish an auxiliary lemma concerning the linear variant of the problem (1.1). In Section 3, we state and prove our main existence results. We emphasize that the tools of fixed point theory employed in this section are well-known, however, their exposition in the present setting allow to explore further insight in terms of the existence criteria for solutions of the problem at hand. In Section 4, we extend the existence results, obtained in Section 3, to the case of Stieltjes type strip conditions.

2. Background material

In this section, we recall some basic definitions and tools of fractional calculus (see [1, 3]), and state two auxiliary lemmas, which will be used in the proofs of the main results of the paper.

Definition 2.1. *The fractional integral of order $q > 0$ with the lower limit zero for a function f is defined as follows:*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > 0,$$

provided the right hand-side is pointwise defined on $[0, \infty)$, where $\Gamma(q) = \int_0^\infty t^{q-1}e^{-t}dt$ is the gamma function.

Definition 2.2. *The Riemann-Liouville fractional derivative of order* $q > 0$, $n-1 < q < n$, $n \in N$, *is defined as*

$$D_{0+}^{q} f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} (t-s)^{n-q-1} f(s)ds,$$

where the function $f(t)$ *has absolutely continuous derivatives up to order* $(n-1)$.

Definition 2.3. *The Caputo derivative of order* q $(n-1 < q < n)$ *for a function* $f : [0, \infty) \to R$ *is defined by*

$${}^{c}D^{q} f(t) = D^{q} \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0.$$

Remark 2.1. *If* $f(t) \in C^{n}[0, \infty)$, *then for* $n-1 < q < n$ *we have*

$${}^{c}D^{q} f(t) = \frac{1}{\Gamma(n-q)} \int_{0}^{t} \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t), \ t > 0.$$

Lemma 2.1 (see [4, 14]). *Let* $u \in AC^{m}[0,1]$ *and* $v \in AC[0,1]$. *Then for* $\rho \in (m-1, m)$, $m \in \mathbb{N}$ *and* $t \in [0,1]$ *the following assertions hold:*

- **(a):** *the general solution of the fractional differential equation* ${}^{c}D^{\rho}u(t) = 0$ *is* $u(t) = b_0 + b_1 t + b_2 t^2 + ... + b_{m-1}t^{m-1}$, *where* $b_i \in \mathbb{R}$, $i = 0, 1, 2, ..., m-1$;.
- **(b):** $I^{\rho}\, {}^{c}D^{\rho} u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{k}(0)$;
- **(c):** ${}^{c}D^{\rho} I^{\rho} v(t) = v(t)$.

To define a solution of the problem (1.1), we consider its linear variant:

$$(2.1) \quad \begin{cases} {}^{c}D^{q}x(t) = h(t), \ n-1 < q \leq n, \ n \geq 2, \ t \in [0,1], \\ x(0) = x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0, \\ ax(\zeta_1) + bx(\zeta_2) = c \displaystyle\int_{\eta}^{\xi} x(s)ds, \ 0 < \zeta_1 < \eta < \xi < \zeta_2 < 1, \end{cases}$$

where $h : [0,1] \to \mathbb{R}$ is a given appropriate function.

Definition 2.4. *A function* $x \in AC^{n}[0,1]$ *is said to be a solution of the problem* (2.1) *on* $[0,1]$ *if it satisfies the conditions in* (2.1), *and the fractional differential equation in* (2.1) *for any* $h \in AC[0,1]$.

Lemma 2.2. *A function x is a solution of the problem (2.1) (in the sense of Definition 2.4), if and only if it satisfies the following fractional integral equation:*

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds + \frac{t^{n-1}}{A}\Big[-a\int_0^{\xi_1} \frac{(\xi_1-s)^{q-1}}{\Gamma(q)} h(s)ds \\ &- b\int_0^{\xi_2} \frac{(\xi_2-s)^{q-1}}{\Gamma(q)} h(s)ds + c\int_\eta^\xi \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)duds\Big], \end{aligned} \tag{2.2}$$

where

$$A = \Big[a\zeta_1^{n-1} + b\zeta_2^{n-1} - \frac{c}{n}\Big(\xi^n - \eta^n\Big)\Big] \neq 0. \tag{2.3}$$

Proof. By Lemma 2.1, the solution of fractional differential equation in (2.1) can be written as follows:

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds + b_0 + b_1 t + b_2 t^2 + ... + b_{n-2}t^{n-2} + b_{n-1}t^{n-1}, \tag{2.4}$$

where $b_0, b_1, ..., b_{n-1} \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions $x(0) = x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0$, we find that $b_0 = b_1 = b_2 = ... = b_{n-2} = 0$. Thus, (2.4) takes the form:

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds + b_{n-1}t^{n-1}. \tag{2.5}$$

Now applying the condition $ax(\zeta_1) + bx(\zeta_2) = c\int_\eta^\xi x(s)ds$ in (2.5), we obtain

$$\begin{aligned} b_{n-1} &= \frac{1}{A}\Big[-a\int_0^{\xi_1} \frac{(\xi_1-s)^{q-1}}{\Gamma(q)} h(s)ds - b\int_0^{\xi_2} \frac{(\xi_2-s)^{q-1}}{\Gamma(q)} h(s)ds \\ &+ c\int_\eta^\xi \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)duds\Big], \end{aligned}$$

where A is given by (2.3). Substituting b_{n-1} into (2.5) we get the solution (2.2).

Conversely, by direct computation with the aid of Lemma 2.1, we infer that $x(t)$ given by (2.2) satisfies the problem (2.1). This completes the proof.

3. Existence results

Let $\mathcal{P} = C([0,1], \mathbb{R})$ denote the Banach space of all continuous functions from [0,1] to $\mathbb{R}$ endowed with the norm :$\|x\| = sup\{|x(t)|, t \in [0,1]\}$. In view of Lemma 2.2, we define an operator $\mathcal{K}$: $\mathcal{P} \longrightarrow \mathcal{P}$ associated with problem (1.1) as follows:

$$
\begin{aligned}
(\mathcal{K}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{t^{n-1}}{A}\Big[-a\int_0^{\xi_1}\frac{(\xi_1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\
&- b\int_0^{\xi_2}\frac{(\xi_2-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + c\int_\eta^{\xi}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u))duds\Big].
\end{aligned}
\tag{3.1}
$$

Observe that the problem (1.1) has solutions if and only if the operator $\mathcal{K}$ has fixed points.

For the sake of computational convenience, we set

$$
\sigma = \frac{1}{\Gamma(q+1)} + \frac{1}{|A|}\Big[|a|\frac{\zeta_1^q}{\Gamma(q+1)} + |b|\frac{\zeta_2^q}{\Gamma(q+1)} + |c|\frac{(\xi^{q+1}-\eta^{q+1})}{\Gamma(q+2)}\Big]. \tag{3.2}
$$

Now we state an existence and uniqueness result for problem (1.1) which is based on Banach's contraction mapping principle.

Theorem 3.1. *Let $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous function satisfying the Lipschitz condition:*

(A_1): $\quad |f(t,x)-f(t,y)| \le \ell|x-y|,\ \forall t\in[0,1],\ x,y\in\mathbb{R}, \ell>0.$

Then (1.1) has a unique solution provided that $\ell\sigma<1$, where σ is given by (3.2).

Proof. We first show that the operator $\mathcal{K}$ defined by (3.1) satisfies the inclusion: $\mathcal{K}B_r\subset B_r$, where $B_r=\{x\in\mathcal{P}:\|x\|\le r\}$, $r>\sigma\alpha/(1-\sigma\ell)$, and $\alpha=\sup_{t\in[0,1]}|f(t,0)|$. For $x\in B_r$ and $t\in[0,1]$, it follows from Lipschitz condition that

$$
|f(t,x(t))| \le |f(t,x(t))-f(t,0)| + |f(t,0)| \le \ell\|x\|+\alpha \le \ell r+\alpha. \tag{3.3}
$$

In view of (3.2) and (3.3), we can write

$$
\begin{aligned}
\|(\mathcal{K}x)\| &\le \sup_{t\in[0,1]}\Big\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + \frac{t^{n-1}}{|A|}\Big[|a|\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\
&+ |b|\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + |c|\int_\eta^{\xi}\int_0^{s}\frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|duds\Big]\Big\} \\
&\le (\ell r+\alpha)\sup_{t\in[0,1]}\Big\{\frac{t^q}{\Gamma(q+1)} + \frac{t^{(n-1)}}{|A|}\Big[|a|\frac{\zeta_1^q}{\Gamma(q+1)} + |b|\frac{\zeta_2^q}{\Gamma(q+1)} + |c|\frac{(\xi^{q+1}-\eta^{q+1})}{\Gamma(q+2)}\Big]\Big\} \\
&\le (\ell r+\alpha)\Big[\frac{1}{\Gamma(q+1)} + \frac{1}{|A|}\Big(|a|\frac{\zeta_1^q}{\Gamma(q+1)} + |b|\frac{\zeta_2^q}{\Gamma(q+1)} + |c|\frac{(\xi^{q+1}-\eta^{q+1})}{\Gamma(q+2)}\Big)\Big] \\
&\le (\ell r+\alpha)\sigma \le r,
\end{aligned}
$$

showing that $\mathcal{K}B_r\subset B_r$.

Now, for $x, y \in \mathbb{R}$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
\|\mathcal{K}x - \mathcal{K}y\| \leq & \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
&+ \frac{t^{n-1}}{|A|} \Big[|a| \int_0^{\zeta_1} \frac{(\zeta_1 - s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
&+ |b| \int_0^{\zeta_2} \frac{(\zeta_2 - s)^{q-1}}{\Gamma(q)} |f(s,x(s)) - f(s,y(s))| ds \\
&+ |c| \int_\eta^\xi \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u)) - f(u,y(u))| du ds \Big] \Big\} \\
\leq & \ \ell \|x-y\| \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{t^{n-1}}{|A|} \Big[|a| \int_0^{\zeta_1} \frac{(\zeta_1 - s)^{q-1}}{\Gamma q} ds \\
&+ |b| \int_0^{\zeta_2} \frac{(\zeta_2 - s)^{q-1}}{\Gamma(q)} ds + |c| \int_\eta^\xi \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} du ds \Big] \Big\} \leq \ell\sigma \|x-y\|.
\end{aligned}
$$

Taking into account that $\ell\sigma < 1$, we conclude that the operator $\mathcal{K}$ is a contraction. Thus, by Banach's contraction mapping principle, there exists a unique solution of (1.1). This completes the proof of theorem 3.1.

Our next existence result is based on the following fixed point theorem .

Lemma 3.1 (Krasnoselskii, [28]). *Let $\mathcal{Y}_1$ be a closed, convex, bounded and nonempty subset of a Banach space $\mathcal{Y}$. Let χ_1, χ_2 be operators satisfying the conditions:*

(a) $\chi_1 y_1 + \chi_2 y_2 \in \mathcal{Y}_1$ whenever $y_1, y_2 \in \mathcal{Y}_1$;

(b) χ_1 is compact and continuous;

(c) χ_2 is a contraction mapping.

Then there exists $y \in \mathcal{Y}_1$ such that $y = \chi_1 y + \chi_2 y$.

Theorem 3.2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition (A_1), and $|f(t,x)| \leq \delta(t)$ for all $(t,x) \in [0,1] \times \mathbb{R}$ and $\delta \in C([0,1], \mathbb{R}^+)$. Then problem (1.1) has at least one solution on $[0,1]$ provided that $\ell\gamma < 1$, where*

$$
\gamma = \frac{1}{|A|} \Big(|a| \frac{\zeta_1^q}{\Gamma(q+1)} + |b| \frac{\zeta_2^q}{\Gamma(q+1)} + |c| \frac{(\xi^{q+1} - \eta^{q+1})}{\Gamma(q+2)} \Big). \tag{3.4}
$$

Proof. We fix $r \geq \|\delta\|\sigma$ and set $B_r = \{x \in \mathcal{P} : \|x\| \leq r\}$. Define the operators $\mathcal{K}_1$ and $\mathcal{K}_2$ on B_r as follows:

$$\begin{aligned}(\mathcal{K}_1)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds,\\ (\mathcal{K}_2)(t) &= \frac{t^{n-1}}{A}\Big[-a\int_0^{\xi_1} \frac{(\xi_1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - b\int_0^{\xi_2} \frac{(\xi_2-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\\ &+ c\int_\eta^\xi \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u))duds\Big].\end{aligned}$$

For $x, y \in B_r$, it is easy to show that $\|(\mathcal{K}_1 x) + (\mathcal{K}_2 y)\| \leq \|\delta\|\sigma \leq r$, where σ is as in (3.2). Hence $\mathcal{K}_1 x + \mathcal{K}_2 y \in B_r$.

Next, using the condition (A_1) and formula (3.4), we can show that the operator $\mathcal{K}_2$ is a contraction. Indeed, for $x, y \in \mathbb{R}$ and $t \in [0,1]$, we can write

$$\begin{aligned}&\|(\mathcal{K}_2 x) - (\mathcal{K}_2 y)\| \leq \sup_{t\in[0,1]}\Big\{\frac{t^{n-1}}{|A|}\Big[|a|\int_0^{\xi_1} \frac{(\xi_1-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds\\ &+|b|\int_0^{\xi_2} \frac{(\xi_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds\\ &+|c|\int_\eta^\xi \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u)) - f(u,y(u))|duds\Big]\Big\}\\ &\leq \ell\|x-y\| \sup_{t\in[0,1]}\Big\{\frac{t^{n-1}}{|A|}\Big[|a|\frac{\zeta_1^q}{\Gamma(q+1)} + b|\frac{\zeta_2^q}{\Gamma(q+1)} + |c|\frac{(\xi^{q+1}-\eta^{q+1})}{\Gamma(q+2)}\Big]\Big\}\\ &\leq \ell\|x-y\|\frac{1}{|A|}\Big[|a|\frac{\zeta_1^q}{\Gamma(q+1)} + |b|\frac{\zeta_2^q}{\Gamma(q+1)} + |c|\frac{(\xi^{q+1}-\eta^{q+1})}{\Gamma(q+2)}\Big] \leq \ell\gamma\|x-y\|.\end{aligned}$$

This shows that $\mathcal{K}_2$ is a contraction in view of the condition $\ell\gamma < 1$. The continuity of f implies that the operator $\mathcal{K}_1$ is continuous. Also, $\mathcal{K}_1$ is uniformly bounded on B_r:

$$\begin{aligned}\|\mathcal{K}_1 x\| &\leq \sup_{t\in[0,1]}\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \leq \sup_{t\in[0,1]}\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\delta(s)ds\\ &\leq \|\delta\| \sup_{t\in[0,1]}\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds \leq \frac{\|\delta\|}{\Gamma(q+1)}.\end{aligned}$$

Moreover, with $\sup_{(t,x)\in[0,1]\times B_r}|f(t,x)| = \overline{f} < \infty$ and $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}\Big|(\mathcal{K}_1 x)(t_2) - (\mathcal{K}_1 x)(t_1)\Big| &= \Big|\int_0^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Big|\\ &= \Big|\int_0^{t_1} \frac{[(t_2-s)^{q-1} - (t_1-s)^{q-1}]}{\Gamma(q)} f(s,x(s))ds\\ &+ \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Big| \leq \frac{\overline{f}}{\Gamma(q+1)}\Big(2|t_2-t_1|^q + |t_2^q - t_1^q|\Big),\end{aligned}$$

which tends to zero independent of x as $t_2 - t_1 \longrightarrow 0$. This implies that $\mathcal{K}_1$ is relatively compact on B_r, and hence, by the Arzelá-Ascoli theorem, $\mathcal{K}_1$ is compact on B_r. Thus, the assumptions of Krasonselskii's fixed point theorem (Lemma 3.1) are satisfied. Hence we can apply Lemma 3.1 to conclude that the problem (1.1) has at least one solution on $[0, 1]$. This completes the proof.

Now we are going to show the existence of solutions for problem (1.1) via the following fixed point theorem (see [28]).

Theorem 3.3. *Let $\mathcal{X}$ be a Banach space. Assume that $T : \mathcal{X} \longrightarrow \mathcal{X}$ is a completely continuous operator and the set $V = \{u \in \mathcal{X} | u = \epsilon T u, 0 < \epsilon < 1\}$ is bounded. Then T has a fixed point in $\mathcal{X}$.*

Theorem 3.4. *Assume that there exists a positive constant L_1 such that $|f(t, x)| \leq L_1$ for all $t \in [0, 1]$, $x \in \mathbb{R}$. Then problem (1.1) has at least one solution on $[0, 1]$.*

Proof. We first show that the operator $\mathcal{K}$ defined by (3.1) is completely continuous. Indeed, observe that the continuity of $\mathcal{K}$ follows from the continuity of f. Let $\mathcal{D} \subset \mathcal{P}$ be bounded. Then, it is easy to show that $|(\mathcal{K}x)(t)| \leq L_1\sigma = L_2$ for all $x \in \mathcal{D}$, where σ is given by (3.2). Furthermore, for $0 < t_1 < t_2 < 1$, we can write

$$
\begin{aligned}
&|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| \\
&\leq \Big| \int_0^{t_1} \frac{[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]}{\Gamma(q)} f(s, x(s))ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \Big| \\
&+ \Big| \frac{(t_2^{n-1} - t_1^{n-1})}{A} \Big[-a \int_0^{\xi_1} \frac{(\xi_1 - s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \\
&- b \int_0^{\xi_2} \frac{(\xi_2 - s)^{q-1}}{\Gamma(q)} f(s, x(s))ds + c \int_\eta^\xi \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u, x(u))duds \Big] \Big| \\
&\leq \frac{L_1}{\Gamma(q+1)} \Big(2|t_2 - t_1|^q + |t_2^q - t_1^q| \Big) \\
&+ \frac{L_1 |t_2^{n-1} - t_1^{n-1}|}{A} \Big(|a| \frac{\zeta_1^q}{\Gamma(q+1)} + |b| \frac{\zeta_2^q}{\Gamma(q+1)} + |c| \frac{|\xi^{q+1} - \eta^{q+1}|}{\Gamma(q+2)} \Big),
\end{aligned} \tag{3.5}
$$

which tends to zero independent of x as $t_2 - t_1 \longrightarrow 0$. Therefore, $\mathcal{K}$ is equicontinuous on $[0, 1]$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{K}$ is completely continuous.

Next, we show that the set $V = \{x \in \mathcal{P} : x = \epsilon\mathcal{K}x, 0 < \epsilon < 1\}$ is bounded. Let $x \in V$ and $t \in [0,1]$. Then we have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{t^{n-1}}{A}\Big[-a\int_0^{\xi_1}\frac{(\xi_1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds \\ &- b\int_0^{\xi_2}\frac{(\xi_2-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds + c\int_\eta^\xi\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))duds\Big]. \end{aligned}$$

As before, it can be shown that $|x(t)| = \epsilon|(\mathcal{K}x)(t)| \leq L_1\sigma = L_2$. Hence, $\|x\| \leq L_2$ for all $x \in V$ and $t \in [0,1]$, showing that the set V is bounded. Thus, we can apply Theorem 3.3 to conclude that problem (1.1) has at least one solution on $[0,1]$. This completes the proof.

In our next existence result, we make use of the Leray-Schauder nonlinear alternative for single valued maps (see [29]).

Lemma 3.2 (Leray-Schauder nonlinear alternative). *Let E_1 be a closed, convex subset of a Banach space E, and let V be an open subset of E_1 with $0 \in V$. Suppose that $\mathcal{U} : \overline{V} \longrightarrow E_1$ is a continuous, compact map (that is, $\mathcal{U}(\overline{V})$ is a relatively compact subset of E_1). Then either $\mathcal{U}$ has a fixed point in $\overline{V}$ or there is $x \in \partial V$ (the boundary of V in E_1), such that $x = \kappa\mathcal{U}(x)$ for $\kappa \in (0,1)$.*

Theorem 3.5. *Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, and let the following conditions hold:*

(A_2): *there exist a function $p \in \mathcal{C}([0,1], \mathbb{R}^+)$ and a nondecreasing function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $|f(t,x)| \leq p(t)\psi(\|x\|)$ for all $(t,x) \in [0,1] \times \mathbb{R}$;*

(A_3): *there exists a constant $M > 0$ such that*

$$M\Big\{\frac{\psi(M)\|p\|}{|A|\Gamma(q+2)}\Big[(q+1)\Big(|A| + |a\zeta_1^q| + |b\zeta_2^q|\Big) + |c(\xi^{q+1} - \eta^{q+1})|\Big]\Big\}^{-1} > 1.$$

Then problem (1.1) has at least one solution on $[0,1]$.

Proof. We consider the operator $\mathcal{K} : \mathcal{P} \longrightarrow \mathcal{P}$ defined by (3.1), and show that it maps bounded sets into bounded sets in $\mathcal{P}$. For a positive number r, let $B_r = \{x \in \mathcal{P} : \|x\| \leq r\}$ be a bounded set in $\mathcal{P}$. Then, in view of condition (A_2), for $x \in B_r$ and

$t \in [0, 1]$, we have

$$\begin{aligned} |(\mathcal{K}x)(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)ds + \frac{t^{n-1}}{|A|}\Big[|a|\int_0^{\xi_1} \frac{(\xi_1 - s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)ds \\ &+ |b|\int_0^{\xi_2} \frac{(\xi_2 - s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)ds + |c|\int_\eta^\xi \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} p(u)\psi(\|x\|)duds\Big] \\ &\leq \frac{\psi(r)\|p\|}{|A|\Gamma(q+2)}\Big[(q+1)\Big(|A| + |a|\zeta_1^q + |b|\zeta_2^q\Big) + |c|(\xi^{q+1} - \eta^{q+1})\Big]. \end{aligned}$$

Next, it will be shown that $\mathcal{K}$ maps bounded sets into equicontinuous sets of $\mathcal{P}$. Let $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$ and $x \in B_r$. Then, we have

$$\begin{aligned} &|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| \\ &\leq \psi(r)\|p\|\Big[\frac{2|t_2 - t_1|^q + |t_2^q - t_1^q|}{\Gamma(q+1)} \\ &+ \frac{|t_2^{(n-1)} - t_1^{(n-1)}|}{|A|\Gamma(q+2)}\Big\{(q+1)\Big(|A| + |a|\zeta_1^q + |b|\zeta_2^q\Big) + |c|(\xi^{q+1} - \eta^{q+1})\Big\}\Big]. \end{aligned}$$

Clearly, the right-hand side of the above inequality tends to zero independent of $x \in B_r$ as $t_2 \longrightarrow t_1$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{K}$ is completely continuous.

Let x be a solution of problem (1.1). Then, following the method employed to establish the boundedness of the operator $\mathcal{K}$, for $\lambda \in (0, 1)$ we obtain

$$\|x(t)\| = \|\lambda(\mathcal{K}x)(t)\| \leq \frac{\psi(\|x\|)\|p\|}{|A|\Gamma(q+2)}\Big[(q+1)\Big(|A| + |a|\zeta_1^q + |b|\zeta_2^q\Big) + |c|(\xi^{q+1} - \eta^{q+1})\Big],$$

which can alternatively be expressed as follows:

$$\|x\|\Big\{\frac{\psi(\|x\|)\|p\|}{|A|\Gamma(q+2)}\Big[(q+1)\Big(|A| + |a|\zeta_1^q + |b|\zeta_2^q\Big) + |c|(\xi^{q+1} - \eta^{q+1})\Big]\Big\}^{-1} \leq 1.$$

In view of condition (A_3), there exists M such that $\|x\| \neq M$. We choose $\mathcal{N} = \{x \in \mathcal{P} : \|x\| < M + 1\}$, and observe that the operator $\mathcal{K} : \overline{\mathcal{N}} \longrightarrow \mathcal{P}$ is continuous and completely continuous. Also, from the choice of $\mathcal{N}$, it follows that there is no $x \in \partial\mathcal{N}$ to satisfy $x = \lambda\mathcal{K}(x)$ for some $\lambda \in (0, 1)$. Thus, we can use Lemma 3.2, to conclude that the operator $\mathcal{K}$ has a fixed point $x \in \overline{\mathcal{N}}$, which is a solution of problem (1.1). This completes the proof.

Example 3.1. *Consider the following fractional boundary value problem:*

$$
(3.6)\qquad \begin{cases} {}^cD^q x(t) = \dfrac{x}{\sqrt{t+4}} + 5t\tan^{-1}x + e^{-t}\cos(t^2+1), t \in [0,1], \\ x(0) = x'(0) = x''(0) = x'''(0) = 0, \\ ax(\zeta_1) + bx(\zeta_2) = c\displaystyle\int_\eta^\xi x(s)ds, 0 < \zeta_1 < \eta < \xi < \zeta_2 < 1, \end{cases}
$$

Here we have $q = 9/2, a = 1/2, b = 1/3, c = 1, \zeta_1 = 1/5, \zeta_2 = 2/3, \xi = 1/2, \eta = 1/3$ and $f(t,x) = \dfrac{x}{\sqrt{t+4}} + 5t\tan^{-1}x + e^{-t}\cos(t^2+1)$. With the given data, we get $\ell = 11/2$,

$$
|A| = |a\zeta_1^{n-1} + b\zeta_2^{n-1} - \frac{c}{n}(\xi^n - \eta^n)| \simeq 0.061217,
$$

$$
\sigma = \frac{1}{\Gamma(q+1)} + \frac{1}{|A|}\Big[|a|\frac{\zeta_1^q}{\Gamma(q+1)} + |b|\frac{\zeta_2^q}{\Gamma(q+1)} + |c|\frac{(\xi^{q+1} - \eta^{q+1})}{\Gamma(q+2)}\Big] \simeq 0.037113.
$$

It is clear that $\ell\sigma < 1$. Thus, all the conditions of Theorem 3.1 are satisfied, and consequently there exists a unique solution for the problem (3.6).

Example 3.2. *Consider the problem* (3.6) *with*

$$
(3.7)\qquad f(t,x) = (2t+1)\Big(\frac{x^2}{1+x^2} + \cos x\Big).
$$

Clearly, we have $|f(t,x)| \le p(t)\psi(|x|)$ with $p(t) = (2t+1)$ and $\psi(|x|) = 2$. By the assumption:

$$
M\Big\{\frac{\psi(M)\|p\|}{|A|\Gamma(q+2)}\Big[(q+1)\Big(|A| + |a|\zeta_1^q + |b|\zeta_2^q\Big) + |c|(\xi^{q+1} - \eta^{q+1})\Big]\Big\}^{-1} > 1,
$$

we find that $M > 0.222678$. Thus, by Theorem 3.5, there exists at least one solution for problem (3.6) with $f(t,x)$ given by (3.7).

4. Existence results for Stieltjes type problem

In this section, the existence results obtained in Section 3, we extend to the case of Stieltjes type strip condition. More precisely, we consider the following boundary value problem:

$$
(4.1)\qquad \begin{cases} {}^cD^q x(t) = f(t,x),\ n-1 < q \le n,\ n \ge 2\ , t \in [0,1], \\ x(0) = x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0, \\ ax(\zeta_1) + bx(\zeta_2) = c\displaystyle\int_\eta^\xi x(s)d\varphi(s), 0 < \zeta_1 < \eta < \xi < \zeta_2 < 1, \end{cases}
$$

where $\varphi(s)$ is a function of bounded variation.

In this case, we define an operator $\mathcal{K}_s$: $\mathcal{P} \longrightarrow \mathcal{P}$ as follows: $(\mathcal{K}_s x)(t) =$

$$
\begin{aligned}
(4.2)\quad &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{t^{n-1}}{A_s}\Big[-a\int_0^{\xi_1}\frac{(\xi_1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds \\
&- b\int_0^{\xi_2}\frac{(\xi_2-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds + c\int_\eta^\xi\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))dud\varphi(s)\Big],
\end{aligned}
$$

(4.3)

where

$$
A_s = \Big[a\zeta_1^{n-1} + b\zeta_2^{n-1} - c\int_\eta^\xi s^{n-1}d\varphi(s)\Big] \neq 0. \tag{4.4}
$$

In what follows we use the notation:

(4.5)

$$
\sigma_s = \frac{1}{\Gamma(q+1)} + \frac{1}{|A_s|}\Big[|a|\frac{\zeta_1^q}{\Gamma(q+1)} + |b|\frac{\zeta_2^q}{\Gamma(q+1)} + |c|\int_\eta^\xi\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}dud\varphi(s)\Big].
$$

Theorem 4.1. *Let $f : [0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous function satisfying the Lipschitz condition: $|f(t,x)-f(t,y)| \le \ell|x-y|$ for all $t\in[0,1]$, $x,y\in\mathbb{R}$ and $\ell > 0$. Then problem* (4.1) *has a unique solution provided that $\ell\sigma_s < 1$, where σ_s is given by* (4.5).

Proof. With the help of the operator $\mathcal{K}_s$ defined by (4.2), (4.4), we can complete the proof following the method of proof of Theorem 3.1. So, we omit the details.

Remark 4.1. *The analogs of Theorems 3.2, 3.4, 3.5 for problem* (4.1) *can also be obtained by using the operator $\mathcal{K}_s$ and σ_s defined by* (4.2), (4.4) *and* (4.5), *respectively.*

Example 4.1. *Consider the following Stieltjes type fractional boundary value problem:*

$$
\begin{cases}
{}^cD^q x(t) = f(t,x),\ t\in[0,1], \\
x(0)=x'(0)=x''(0)=x'''(0)=0, ax(\zeta_1)+bx(\zeta_2) = c\int_\eta^\xi x(s)d\varphi(s).
\end{cases} \tag{4.6}
$$

Here we have $q = 9/2, a = 1/2, b = 1/3, c = 1, \zeta_1 = 1/5, \zeta_2 = 2/3, \xi = 1/2, \eta = 1/3$, $\varphi(s) = s + s^2/2$ and $f(t,x) = x/\sqrt{t+4} + 5t\tan^{-1}x + e^{-t}\cos(t^2+1)$. Using the given data, we find that $\ell = 11/2, |A_s| \simeq 0.058841$ and $\sigma_s \simeq 0.038353$. With $\ell\sigma_s < 1$, all the conditions of Theorem 4.1 are satisfied. So, there exists a unique solution for problem (4.6).

Concluding remarks

In this paper, we have studied a new class of nonlocal fractional boundary value problems of arbitrary order in presence of classical and Stieltjes type strip conditions. Our results are new and take care of some new special situations. For instance, by taking $c = 0$, we obtain the results for a boundary value problem of fractional-order $q \in (n-1, n]$ involving a nonlocal condition of the form $ax(\zeta_1) + bx(\zeta_2) = 0$ with $a/b \neq -\zeta_2^{n-1}/\zeta_1^{n-1}$. In the case where $a = 0$ (or $\zeta_1 \to 0^+$) and $\zeta_2 \to 1^-$, our results correspond to a condition of the form: $x(1) = \mu_1 \int_\eta^\xi x(s)ds$ (μ_1=constant). Letting $b = 0$ and $\zeta_1 \to 0^+$, we get the results for the condition: $\int_\eta^\xi x(s)ds = 0$. The last two observations obviously hold for Stieltjes type strip conditions as well.

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