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COMMUTATORS OF HOMOGENEOUS FRACTIONAL INTEGRALS ON HERZ-TYPE HARDY SPACES WITH VARIABLE EXPONENT

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Abstract. Let $\Omega \in L^s(S^{n-1})$, $s \geq 1$, be a homogeneous function of degree zero, and let σ $(0 < \sigma < n)$ and b be Lipschitz or BMO functions. In this paper, we establish the boundedness of the commutators $[b, T_{\Omega,\sigma}]$, generated by a homogeneous fractional integral operator $T_{\Omega,\sigma}$ and function b, on the Herz-type Hardy spaces with variable exponent.

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1. Introduction

The theory of function spaces with variable exponent has extensively been studied by researchers since the work of Kováčik and Rákosník [10]. In [3] and [16], the boundedness of some integral operators on variable L^p spaces were established. In addition, the authors of [17] have defined the Herz-type Hardy spaces with variable exponent and gave their atomic characterizations.

Given an open set $E \subset \mathbb{R}^n$ and a measurable function $p(\cdot) : E \longrightarrow [1, \infty)$. By $L^{p(\cdot)}(E)$ we denote the set of measurable functions f on E such that for some $\lambda > 0$,

$$\int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm:

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}.$$

These spaces are referred to as variable L^p spaces, since they generalize the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

The space $L^{p(\cdot)}_{\text{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(E) := \{f : f \in L^{p(\cdot)}(F) \text{ for all compact subsets } F \subset E\}.$$

Define $\mathcal{P}^0(E)$ to be the set of those functions $p(\cdot) : E \longrightarrow (0, \infty)$ for which

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in E\} > 0 \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in E\} < \infty.$$

Also, define $\mathcal{P}(E)$ to be the set of those functions $p(\cdot) : E \longrightarrow [1, \infty)$ for which

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in E\} > 1 \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in E\} < \infty.$$

Denote $p'(x) = p(x)/(p(x)-1)$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of those functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ for which the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Also, by $|A|$ and χ_A we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$, respectively. The notation $f \approx g$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 g \leq f \leq C_2 g$.

The lemmas that follow contain some important properties of the variable L^p spaces.

Lemma 1.1 ([1]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that*

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}, \quad |x-y| \leq 1/2 \tag{1.1}$$

and

$$|p(x) - p(y)| \leq \frac{C}{\log(|x| + e)}, \quad |y| \geq |x|. \tag{1.2}$$

Then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, that is, the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 1.2 ([10]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.3 ([8]). *Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$ the following inequalities hold:*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1} \text{ and}$$

$$\frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where δ_1 *and* δ_2 *are constants with* $0 < \delta_1, \delta_2 < 1$.

Throughout the paper constant δ_2 will be the same as in Lemma 1.3.

Lemma 1.4 ([8]). *Suppose* $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Then there exists a constant* $C > 0$ *such that for all balls* B *in* $\mathbb{R}^n$ *the following inequality holds:*

$$\frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C.$$

Next, we recall the definition of the Herz-type spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of all positive and non-negative integers, respectively, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.

Definition 1.1 ([8]). *Let* $\alpha \in \mathbb{R}$, $0 < p \le \infty$ *and* $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. *The homogeneous Herz space with variable exponent, denoted by* $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, *is defined by*

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

The non-homogeneous Herz space with variable exponent, denoted by $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, *is defined by*

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha p}\|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

In [17], the authors have defined the Herz-type Hardy spaces with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and gave their atomic decomposition characterizations.

Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions, and let $\mathcal{S}'(\mathbb{R}^n)$ be the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N(f)(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N(f)(x) = \sup_{\phi \in \mathcal{A}_N} |\phi^*_\nabla(f)(x)|,$$

where $A_N \stackrel{d}{=} \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta|\leq N} |x^\alpha D^\beta \phi(x)| \leq 1\}$, $N > n+1$, and ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 1.2 ([17]). *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $N > n+1$.*

(i) The homogeneous Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\}$$

and $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

(ii) The non-homogeneous Herz-type Hardy space with variable exponent $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \right\}$$

and $\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N(f)\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

For $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the largest integer less than or equal to α.

Definition 1.3 ([17]). *Let $n\delta_2 \leq \alpha < \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and let s be a non-negative integer such that $s \geq [\alpha - n\delta_2]$.*

(i) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom, if it satisfies the following conditions:

(1) $\operatorname{supp} a \subset B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq |B(0,r)|^{-\alpha/n}$.

(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0, |\beta| \leq s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, q(\cdot))$-atom of restricted type, if it satisfies the conditions (2), (3) above and

(1') $\operatorname{supp} a \subset B(0,r), r \geq 1$.

Lemma 1.5 ([17]). *Let $n\delta_2 \leq \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then $f \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ (respectively $f \in HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$) if and only if*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \left(\text{respectively } f = \sum_{k=0}^{\infty} \lambda_k a_k \right) \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha, q(\cdot))$-atom (respectively a central $(\alpha, q(\cdot))$-atom of restricted type) with support contained in B_k and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$ (respectively

$\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$). *Moreover,*

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \left(\text{respectively} \quad \|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p} \right),$$

where the infimum is taken over all above decompositions of f.

Let S^{n-1} denote the unit sphere in $\mathbb{R}^n (n \geq 2)$ equipped with normalized Lebesgue measure, and let $\Omega \in L^s(S^{n-1})$, $s \geq 1$, be a homogeneous function of degree zero. For $0 < \sigma < n$, the homogeneous fractional integral operator $T_{\Omega,\sigma}$ is defined by

$$T_{\Omega,\sigma} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\sigma}} f(y) dy.$$

Muckenhoupt and Wheeden [12] have established the weighted (L^p, L^q) boundedness of the operator $T_{\Omega,\sigma}$ with power weights. Recently, Tan and Liu [15] gave the $(L^{p(\cdot)}, L^{q(\cdot)})$, $(H^{p(\cdot)}, L^{q(\cdot)})$ and $(H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}, \dot{K}^{\alpha,p_2}_{q_2(\cdot)})$ boundedness of $T_{\Omega,\sigma}$.

Let b be a locally integrable function, the commutator of a homogeneous fractional integral operator $T_{\Omega,\sigma}$, generated by b and denoted by $[b, T_{\Omega,\sigma}]$, is defined by

$$[b, T_{\Omega,\sigma}]f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\sigma}} (b(x) - b(y)) f(y) dy.$$

Motivated by the results of [5] and [15], in this paper we study the boundedness of the commutator $[b, T_{\Omega,\sigma}]$ for a homogeneous fractional integral operator $T_{\Omega,\sigma}$ on the Herz-type Hardy spaces with variable exponent.

2. A BMO ESTIMATE FOR THE COMMUTATOR OF HOMOGENEOUS FRACTIONAL INTEGRAL OPERATOR

Let us first recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where $f_Q = |Q|^{-1} \int_Q f(y) dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, and $|Q|$ denotes the Lebesgue measure of Q.

A nonnegative locally integrable function $\omega(x)$ on $\mathbb{R}^n$ is said to belong to the class $A(p,q)$ $(1 < p, q < \infty)$, if there is a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q \omega(x)^{-p'} dx \right)^{1/p'} \leq C < \infty,$$

where $p' = p/(p-1)$. Also, we will say that $\omega \in A_1$ if $M\omega(x) \leq C\omega(x)$ for a.e. x.

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. The weighted (L^p, L^q) boundedness of the commutator $[b, T_{\Omega,\sigma}]$ have been proved by Segovia and Torrea [14], Ding [5], and Ding and Lu [6].

Lemma 2.1 ([5]). *Suppose that* $0 < \sigma < n$, $s' < p < n/\sigma$, $1/q = 1/p - \sigma/n$, $\Omega \in L^s(\mathrm{S}^{n-1})$, *and* $\omega^{s'} \in A(p/s', q/s')$. *Then for* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $m \in \mathbb{Z}$ *there is a constant* C, *independent of* f, *such that*

$$\left(\int_{\mathbb{R}^n} |[b, T_{\Omega,\sigma}]^m f(x)\omega(x)|^q dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx\right)^{1/p}.$$

Lemma 2.2 ([3]). *Given a family* $\mathcal{F}$ *and an open set* $E \subset \mathbb{R}^n$. *Assume that for some* p_0 *and* q_0, $0 < p_0 \leq q_0 < \infty$, *and every weight* $\omega \in A_1$,

$$\left(\int_E f(x)^{q_0}\omega(x)dx\right)^{1/q_0} \leq C_0\left(\int_E g(x)^{p_0}\omega(x)^{p_0/q_0}dx\right)^{1/p_0}, \quad (f,g) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}^0(E)$ *such that* $p_0 < p_- \leq p_+ < \frac{p_0 q_0}{q_0 - p_0}$, *define the function* $q(\cdot)$ *by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad x \in E.$$

If $p(\cdot)$ *satisfies the conditions (1.1) and (1.2) of Lemma 1.1, then for all* $(f,g) \in \mathcal{F}$ *such that* $f \in L^{q(\cdot)}(E)$, *the following inequality holds:*

$$\|f\|_{L^{q(\cdot)}(E)} \leq C\|g\|_{L^{p(\cdot)}(E)}.$$

Using Lemmas 2.1 and 2.2, and arguments similar to those applied in the proof of Theorem 1.9 of [15], we can prove the $(L^{p(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, T_{\Omega,\sigma}]$.

Before stating our result, we first recall the definition of the L^s-Dini condition. We say that a function Ω satisfies the L^s-Dini condition if $\Omega \in L^s(\mathrm{S}^{n-1})$ with $s \geq 1$ is homogeneous of degree zero on $\mathbb{R}^n$, and

$$\int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta < \infty,$$

where $\omega_s(\delta)$ denotes the integral modulus of continuity of order s of Ω defined by

$$\omega_s(\delta) = \sup_{|\rho|<\delta}\left(\int_{\mathrm{S}^{n-1}} |\Omega(\rho x') - \Omega(\rho x')|^s\, dx'\right)^{1/s},$$

and ρ is a rotation in $\mathbb{R}^n$ with $|\rho| = \|\rho - I\|$.

Now we are ready to state our result concerning the boundedness of the commutator $[b, T_{\Omega,\sigma}]$ on the Herz-type Hardy spaces with variable exponent.

Theorem 2.1. *Suppose that* $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \beta \leq 1$, $0 < \sigma < n - \beta$, *and* $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies the conditions (1.1) and (1.2) of Lemma 1.1 with* $q_1^+ < n/\sigma$

and $1/q_1(x) - 1/q_2(x) = \sigma/n$. *Let* $\Omega \in L^s(\mathrm{S}^{n-1})(s > q_2^+)$ *with* $1 \le s' < q_1^-$ *and satisfy*

$$\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\beta}} d\delta < \infty.$$

Let $0 < p_1 \le p_2 < \infty$ *and* $n\delta_2 \le \alpha < n\delta_2 + \beta$ *(respectively* $0 < \max(n\delta_2, \alpha_2) \le \alpha_1 < n\delta_2 + \beta$*). Then the commutator* $[b, T_{\Omega,\sigma}]$ *is bounded from* $H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ *(respectively from* $HK^{\alpha_1,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$*) to* $\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$ *(respectively to* $K^{\alpha_2,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$*).*

To prove Theorem 2.1, we also will need the following lemmas.

Lemma 2.3 ([9]). *Let* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, k *be a positive integer and* B *be a ball in* $\mathbb{R}^n$. *Then for all* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and all* $j, i \in \mathbb{Z}$ *with* $j > i$, *we have*

$$\frac{1}{C}\|b\|_*^k \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^k \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^k,$$

$$\|(b - b_{B_i})^k \chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(j-i)^k \|b\|_*^k \|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $B_i = \{x \in \mathbb{R}^n : |x| \le 2^i\}$ *and* $B_j = \{x \in \mathbb{R}^n : |x| \le 2^j\}$.

Lemma 2.4 ([2]). *Given a set* E *and a function* $p(\cdot) \in \mathcal{P}(E)$, *let* $f : E \times E \to \mathbb{R}$ *be a measurable function (with respect to product measure), such that* $f(\cdot, y) \in L^{p(\cdot)}(E)$ *for almost every* $y \in E$. *Then the following inequality holds:*

$$\left\| \int_E f(\cdot, y) dy \right\|_{L^{p(\cdot)}(E)} \le C \int_E \|f(\cdot, y)\|_{L^{p(\cdot)}(E)} dy.$$

Lemma 2.5 ([13]). *Let* $\tilde{q}(\cdot)$ *be a variable exponent defined by* $\frac{1}{p(x)} = \frac{1}{\tilde{q}(x)} + \frac{1}{q}$ $(x \in \mathbb{R}^n)$. *Then for all measurable functions* f *and* g *we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}\|fg\|_{L^q(\mathbb{R}^n)}.$$

Lemma 2.6 ([4]). *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfy the conditions (1.1) and (1.2) of Lemma 1.1. Then for every cube (or ball)* $Q \subset \mathbb{R}^n$ *we have*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{p(x)}} & \text{if } |Q| \le 2^n \text{ and } x \in Q, \\ |Q|^{\frac{1}{p(\infty)}} & \text{if } |Q| \ge 1, \end{cases}$$

where $p(\infty) = \lim_{x\to\infty} p(x)$.

Lemma 2.7 ([7]). *Suppose that* $0 < \sigma < n$, $s > 1$, *and* Ω *satisfies the* L^s*-Dini condition. If there is a constant* $a_0 > 0$ *such that* $|y| < a_0 R$, *then the following inequality holds:*

$$\left(\int_{R<|x|<2R} \left| \frac{\Omega(x-y)}{|x-y|^{n-\sigma}} - \frac{\Omega(x)}{|x|^{n-\sigma}} \right|^s dx\right)^{1/s} \le CR^{\frac{n}{s}-(n-\sigma)} \left\{ \frac{|y|}{R} + \int_{|y|/2R<\delta<|y|/R} \frac{\omega_s(\delta)}{\delta} d\delta \right\}.$$

Proof of Theorem 2.1. We prove the theorem only for homogeneous case, because in view of embedding $K^{\alpha_1,p_2}_{q_2(\cdot)}(\mathbb{R}^n) \subset K^{\alpha_2,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$ for $0 < \alpha_2 \le \alpha_1$, proved in [18], the non-homogeneous case can be treated similarly. Let $f \in H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. By Lemma 1.5 we have $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of f), and a_j is a dyadic central $(\alpha, q_1(\cdot))$-atom with support B_j. Noting that $p_1 \le p_2$, we can write

$$
\begin{aligned}
\|[b,T_{\Omega,\sigma}](f)\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_2}\|[b,T_{\Omega,\sigma}](f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right\}^{p_1/p_2} \\
&\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\|[b,T_{\Omega,\sigma}](f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|[b,T_{\Omega,\sigma}](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\
&\quad + C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|[b,T_{\Omega,\sigma}](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\
&=: I_1 + I_2.
\end{aligned} \tag{2.1}
$$

We first estimate I_1. For each $k \in \mathbb{Z}$, $j \le k-2$ and *a.e.* $x \in A_k$, using Lemma 2.4, the Minkowski inequality and the vanishing moments of a_j, we get

$$
\begin{aligned}
&\|[b,T_{\Omega,\sigma}](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le \int_{B_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right|(b(\cdot)-b(y))\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}|a_j(y)|dy \\
&\le \int_{B_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right||b(\cdot)-b_{B_j}|\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}|a_j(y)|dy \\
&\quad + \int_{B_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right|\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}|b_{B_j}-b(y)||a_j(y)|dy \\
&=: I_{11} + I_{12}.
\end{aligned}
$$

To estimate I_{11}, we note that $s > q_2^+$, and denote $\tilde{q}_2(\cdot) > 1$ and $\frac{1}{q_2(x)} = \frac{1}{\tilde{q}_2(x)} + \frac{1}{s}$. Then by Lemmas 2.3 and 2.5 we have

$$
\begin{aligned}
&\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right||b(\cdot)-b_{B_j}|\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le \left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right|\chi_k(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}\||b(\cdot)-b_{B_j}|\chi_k(\cdot)\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right|\chi_k(\cdot)\right\|_{L^{s}(\mathbb{R}^n)}(k-j)\|b\|_*\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

When $|B_k| \le 2^n$ and $x_k \in B_k$, by Lemma 2.6 we have

$$
\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}_2(x_k)}} \approx \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}-\frac{\sigma}{n}}.
$$

When $|B_k| \geq 1$ we have

$$\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{q_2(\infty)}} \approx \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}-\frac{\sigma}{n}}.$$

So, we obtain $\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}-\frac{\sigma}{n}}$.

Meanwhile, by Lemma 2.7 we have

$$\begin{aligned}
&\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}}-\frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right|\chi_k(\cdot)\right\|_{L^s(\mathbb{R}^n)}\\
&\leq 2^{(k-1)(\frac{n}{s}-(n-\sigma))}\left\{\frac{|y|}{2^k}+\int_{|y|/2^k}^{|y|/2^{k-1}}\frac{\omega_s(\delta)}{\delta}d\delta\right\}\\
&\leq 2^{(k-1)(\frac{n}{s}-(n-\sigma))}\left(2^{j-k+1}+2^{(j-k+1)\beta}\int_0^1\frac{\omega_s(\delta)}{\delta}d\delta\right)\\
&\leq C2^{(k-1)(\frac{n}{s}-(n-\sigma))}2^{(j-k)\beta}.
\end{aligned}$$

So, using the generalized Hölder inequality, we obtain the following estimate for I_{11}:

(2.2)

$$\begin{aligned}
I_{11} &= \int_{B_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}}-\frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right||b(\cdot)-b_{B_j}|\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}|a_j(y)|dy\\
&\leq C(k-j)\|b\|_* 2^{(k-1)(\frac{n}{s}-(n-\sigma))}2^{(j-k)\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}|B_k|^{-\frac{1}{s}-\frac{\sigma}{n}}\int_{B_j}|a_j(y)|dy\\
&\leq C(k-j)\|b\|_* 2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

To estimate I_{12}, we use arguments similar to those applied for I_{11}, to obtain

$$\begin{aligned}
&\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}}-\frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right|\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\leq \left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}}-\frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right|\chi_k(\cdot)\right\|_{L^s(\mathbb{R}^n)}\|\chi_k(\cdot)\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)}\\
&\leq C\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}}-\frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right|\chi_k(\cdot)\right\|_{L^s(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{(k-1)(\frac{n}{s}-(n-\sigma))}2^{(j-k)\beta}\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)}\\
&\leq C2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

So, by Lemma 2.3 and the generalized Hölder inequality, we have

$$\begin{aligned}
I_{12} &= \int_{B_j}\left\|\left|\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-\sigma}}-\frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right|\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}|b_{B_j}-b(y)||a_j(y)|dy\\
&\leq C2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\int_{B_j}|b_{B_j}-b(y)||a_j(y)|dy \qquad (2.3)\\
&\leq C2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|(b_{B_j}-b)\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\
&\leq C\|b\|_* 2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Next, using (2.2), (2.3), and Lemmas 1.3 and 1.4, we can write

$$\begin{aligned}&\|[b,T_{\Omega,\sigma}](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\&\le C(k-j)\|b\|_*2^{-kn+(j-k)\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\\&\le C(k-j)\|b\|_*2^{(j-k)\beta}\|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}\\&\le C(k-j)2^{-j\alpha+(j-k)(\beta+n\delta_2)}\|b\|_*.\end{aligned}$$

So, we have

$$\begin{aligned}I_1&\le C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|(k-j)2^{-j\alpha+(j-k)(\beta+n\delta_2)}\right)^{p_1}\\&=C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|(k-j)2^{(j-k)(\beta+n\delta_2-\alpha)}\right)^{p_1}.\end{aligned}$$

When $1<p_1<\infty$, take $1/p_1+1/p_1'=1$. Since $\beta+n\delta_2-\alpha>0$, we can use the Hölder inequality to obtain

$$\begin{aligned}I_1&\le C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\right)\\&\quad\times\left(\sum_{j=-\infty}^{k-2}(k-j)^{p_1'}2^{(j-k)(\beta+n\delta_2-\alpha)p_1'/2}\right)^{p_1/p_1'}\\&\le C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\right)\end{aligned}$$

$$\begin{aligned}&=C\|b\|_*^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2}\right)\\&\le C\|b\|_*^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.\end{aligned}\tag{2.4}$$

When $0<p_1\le 1$, we have

$$\begin{aligned}I_1&\le C\|b\|_*^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}(k-j)^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1}\right)\\&=C\|b\|_*^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=j+2}^{\infty}(k-j)^{p_1}2^{(j-k)(\beta+n\delta_2-\alpha)p_1}\right)\\&\le C\|b\|_*^{p_1}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.\end{aligned}\tag{2.5}$$

Now we estimate I_2. To this end, observe first that by the $(L^{q_1(\cdot)}(\mathbb{R}^n), L^{q_2(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, T_{\Omega,\sigma}]$, we have

$$\begin{aligned} I_2 &\le C\|b\|_*^{p_1} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &\le C\|b\|_*^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha} \right)^{p_1}. \end{aligned}$$

If $0 < p_1 \le 1$, then we have

$$I_2 \le C\|b\|_*^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \le C\|b\|_*^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \tag{2.6}$$

If $1 < p_1 < \infty$, then we can apply the Hölder inequality to obtain

$$\begin{aligned} I_2 &\le C\|b\|_*^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^{p_1} 2^{(k-j)\alpha p_1/2} \right) \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1'/2} \right)^{p_1/p_1'} \\ &\le C\|b\|_*^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{2.7}$$

Combining (2.1) and (2.4)-(2.7) we complete the proof of the theorem. Theorem 2.1 is proved.

3. A Lipschitz estimate for the commutator of homogeneous fractional integral operator

For $0 < \gamma \le 1$, the Lipschitz space $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ is defined as follows:

$$\mathrm{Lip}_\gamma(\mathbb{R}^n) = \left\{ f : \|f\|_{\mathrm{Lip}_\gamma} = \sup_{x,y\in\mathbb{R}^n; x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\gamma} < \infty \right\}.$$

Let $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$. It is easy to see that $|[b, T_{\Omega,\sigma}]| \le C\|b\|_{\mathrm{Lip}_\gamma} |T_{\Omega,\sigma+\gamma}|$. In [15], the authors proved that the operator $T_{\Omega,\sigma}$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_2(\cdot)}(\mathbb{R}^n)$ for $1/q_1(x) - 1/q_2(x) = \sigma/n$ and $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the conditions (1.1) and (1.2) of Lemma 1.1 with $q_1^+ < n/\sigma$. So, we can state the following theorem.

Theorem 3.1. *Suppose that $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ with $0 < \gamma \le 1$ and $0 < \sigma < n - \gamma$. Let $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy the conditions (1.1) and (1.2) of Lemma 1.1 with $q_1^+ < n/(\sigma+\gamma)$, $1/q_1(x) - 1/q_2(x) = (\sigma+\gamma)/n$, and let $\Omega \in L^s(\mathrm{S}^{n-1})(s > q_2^+)$ with $1 \le s' < q_1^-$. Then the commutator $[b, T_{\Omega,\sigma}]$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_2(\cdot)}(\mathbb{R}^n)$.*

Next, we give a Lipschitz estimate for the commutator $[b, T_{\Omega,\sigma}]$ on the Herz-type Hardy spaces with variable exponent.

Theorem 3.2. *Suppose that $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ with $0 < \gamma \le 1$ and $0 < \sigma < n-\gamma$. Let $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy the conditions (1.1) and (1.2) of Lemma 1.1 with $q_1^+ < n/(\sigma+\gamma)$, $1/q_1(x) - 1/q_2(x) = (\sigma+\gamma)/n$, and let $\Omega \in L^s(\mathrm{S}^{n-1})$ $(s > q_2^+)$ with $1 \le s' < q_1^-$ and satisfy $\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\gamma}} d\delta < \infty$. Further, let $0 < p_1 \le p_2 < \infty$ and $n\delta_2 \le \alpha < n\delta_2 + \gamma$(respectively $0 < \max(n\delta_2, \alpha_2) \le \alpha_1 < n\delta_2 + \gamma$). Then the commutator $[b, T_{\Omega,\sigma}]$ maps $H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ (respectively $HK^{\alpha_1,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$) continuously into $\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$(respectively into $K^{\alpha_2,p_2}_{q_2(\cdot)}(\mathbb{R}^n)$).*

Proof. Similar to Theorem 2.1, it is enough to prove the theorem only for homogeneous case. Let $f \in H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$. Then by Lemma 1.5 we have $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of f), and a_j is a dyadic central $(\alpha, q_1(\cdot))$-atom with support B_j. So, we can write

$$
\begin{aligned}
\|[b,T_{\Omega,\sigma}](f)\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} &= \left\{\textstyle\sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \|[b,T_{\Omega,\sigma}](f)\chi_k\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right\}^{p_1/p_2} \\
&\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \|[b,T_{\Omega,\sigma}](f)\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|[b,T_{\Omega,\sigma}](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|[b,T_{\Omega,\sigma}](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} =: J_1 + J_2.
\end{aligned}
\tag{3.1}
$$

We first estimate J_1. For each $k \in \mathbb{Z}, j \le k-2$ and $a.e.$ $x \in A_k$, using Lemma 2.4, the Minkowski inequality and the vanishing moments of a_j, we get

$$
\begin{aligned}
&\|[b,T_{\Omega,\sigma}](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}} \right| (b(\cdot) - b(y))\chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy \\
&\le \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}} \right| |b(\cdot) - b(0)|\chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy \\
&\quad + \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}} \right| \chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |b(0) - b(y)||a_j(y)| dy =: J_{11} + J_{12}.
\end{aligned}
$$

To estimate J_{11}, we note that $s > q_2^+$, and denote $\tilde{q}_2(\cdot) > 1$ and $\frac{1}{q_2(x)} = \frac{1}{\tilde{q}_2(x)} + \frac{1}{s}$. Then by Lemma 2.5 we have

$$\begin{aligned}
&\left\|\left|\frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right| |b(\cdot) - b(0)|\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le \left\|\left|\frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right| \chi_k(\cdot)\right\|_{L^s(\mathbb{R}^n)} \||b(\cdot) - b(0)|\chi_k(\cdot)\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\|b\|_{\mathrm{Lip}_\gamma} 2^{k\gamma} \left\|\left|\frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right| \chi_k(\cdot)\right\|_{L^s(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

When $|B_k| \le 2^n$ and $x_k \in B_k$, by Lemma 2.6 we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}_2(x_k)}} \approx \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s} - \frac{\sigma+\gamma}{n}}.$$

When $|B_k| \ge 1$ we have

$$\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{\tilde{q}_2(\infty)}} \approx \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s} - \frac{\sigma+\gamma}{n}}.$$

So, we obtain $\|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s} - \frac{\sigma+\gamma}{n}}$.

Meanwhile, by Lemma 2.7 we have

$$\begin{aligned}
&\left\|\left|\frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right| \chi_k(\cdot)\right\|_{L^s(\mathbb{R}^n)} \\
&\le 2^{(k-1)(\frac{n}{s} - (n-\sigma))} \left\{\frac{|y|}{2^k} + \int_{|y|/2^k}^{|y|/2^{k-1}} \frac{\omega_s(\delta)}{\delta} d\delta\right\} \\
&\le 2^{(k-1)(\frac{n}{s} - (n-\sigma))} \left(2^{j-k+1} + 2^{(j-k+1)\gamma} \int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta\right) \\
&\le C 2^{(k-1)(\frac{n}{s} - (n-\sigma))} 2^{(j-k)\gamma}.
\end{aligned}$$

So, using the generalized Hölder inequality, we obtain the following estimate for J_{11}:

(3.2)

$$\begin{aligned}
J_{11} &= \int_{B_j} \left\|\left|\frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right| |b(\cdot) - b(0)|\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy \\
&\le C\|b\|_{\mathrm{Lip}_\gamma} 2^{k\gamma} 2^{(k-1)(\frac{n}{s} - (n-\sigma))} 2^{(j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} |B_k|^{-\frac{1}{s} - \frac{\sigma+\gamma}{n}} \int_{B_j} |a_j(y)| dy \\
&\le C\|b\|_{\mathrm{Lip}_\gamma} 2^{-kn + (j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

To estimate J_{12}, we use arguments similar to those applied for J_{11}, to obtain

$$\begin{aligned}
&\left\|\left|\frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right| \chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le \left\|\left|\frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right| \chi_k(\cdot)\right\|_{L^s(\mathbb{R}^n)} \|\chi_k(\cdot)\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\
&\le \left\|\left|\frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}}\right| \chi_k(\cdot)\right\|_{L^s(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\
&\le C 2^{(k-1)(\frac{n}{s} - (n-\sigma))} 2^{(j-k)\gamma} \|\chi_{B_k}\|_{L^{\tilde{q}_2(\cdot)}(\mathbb{R}^n)} \\
&\le C 2^{-kn + (j-k)\gamma - k\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

So, by the generalized Hölder inequality, we have

$$
\begin{aligned}
J_{12} &= \int_{B_j} \left\| \left| \frac{\Omega(\cdot - y)}{|\cdot - y|^{n-\sigma}} - \frac{\Omega(\cdot)}{|\cdot|^{n-\sigma}} \right| \chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} |b(0) - b(y)||a_j(y)|dy \\
&\leq C2^{-kn+(j-k)\gamma-k\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \int_{B_j} |b(0) - b(y)||a_j(y)|dy \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma} 2^{-kn+2(j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma} 2^{-kn+(j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.3}
$$

Next, using (3.2), (3.3), and Lemmas 1.3 and 1.4, we can write

$$
\begin{aligned}
&\|[b, T_{\Omega,\sigma}](a_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma} 2^{-kn+(j-k)\gamma} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma} 2^{(j-k)\gamma} \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \\
&\leq C2^{-j\alpha+(j-k)(\gamma+n\delta_2)} \|b\|_{\mathrm{Lip}_\gamma}.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
J_1 &\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{-j\alpha+(j-k)(\gamma+n\delta_2)} \right)^{p_1} \\
&= C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)(\gamma+n\delta_2-\alpha)} \right)^{p_1}.
\end{aligned}
$$

When $1 < p_1 < \infty$, take $1/p_1 + 1/p_1' = 1$. Since $\gamma + n\delta_2 - \alpha > 0$, we can use the Hölder inequality to obtain

$$
\begin{aligned}
J_1 &\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1/2} \right) \\
&\qquad \times \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1'/2} \right)^{p_1/p_1'} \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1/2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1/2} \right) \\
&\leq C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{aligned}
\tag{3.4}
$$

When $0 < p_1 \le 1$, we have

$$\begin{aligned} J_1 &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1} \right) \\ &= C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=j+2}^{\infty} 2^{(j-k)(\gamma+n\delta_2-\alpha)p_1} \right) \\ &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{3.5}$$

Now we estimate J_2. Observe first that by the $(L^{q_1(\cdot)}(\mathbb{R}^n), L^{q_2(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $[b, T_{\Omega,\sigma}]$, we have

$$\begin{aligned} J_2 &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha} \right)^{p_1}. \end{aligned}$$

If $0 < p_1 \le 1$, then we have

$$J_2 \le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \textstyle\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \tag{3.6}$$

If $1 < p_1 < \infty$, then we can apply the Hölder inequality to obtain

$$\begin{aligned} J_2 &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^{p_1} 2^{(k-j)\alpha p_1/2} \right) \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1'/2} \right)^{p_1/p_1'} \\ &\le C\|b\|_{\mathrm{Lip}_\gamma}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}. \end{aligned} \tag{3.7}$$

Combining (3.1) and (3.4)-(3.7) we complete the proof of the theorem. Theorem 3.1 is proved.

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