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A NOTE ON SOLUTIONS OF SOME DIFFERENTIAL-DIFFERENCE EQUATIONS

X. QI, Y. LIU AND L. YANG

University of Jinan, Jinan, Shandong, China
Shaoxing College of Arts and Sciences, Shaoxing, Zhejiang, China
Shandong University, Jinan, Shandong, China
E-mails: *xiaoguang.202@163.com, xiaogqi@mail.sdu.edu.cn; palbipul86@gmail.com, liuyongsdu@aliyun.com lzyang@sdu.edu.cn*

Abstract. This research is a continuation of the recent paper by X. Qi and L. Yang [15]. In this paper, we continue our study concerning existence of solutions of a Fermat type differential-difference equation, and improve the results obtained by K. Liu et al. in [8, 10].

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1. Introduction

A number of papers are devoted to the study of solutions of the following Fermat type functional equation:

$$f(z)^n + g(z)^n = 1. \tag{1.1}$$

For instance, in the case $n \geq 4$, Gross [3] has proved that the equation (1.1) has no transcendental meromorphic solutions. For the entire case, Montel [11] established that (1.1) has no transcendental entire solutions when $n \geq 3$. For $n = 2$, Gross [4] showed that the equation (1.1) has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, and no other solutions exist. Later on, many authors investigated the generalization of equation (1.1) (see [17, 19, 21], and references therein). Recently, many articles were focused on complex difference equations. The background for these studies lies in the recently developed difference counterparts of Nevanlinna theory (see [1, 2]). Meanwhile, the difference analogues of the Fermat type functional equations have been investigated in a number of papers (see [7, 9, 12, 13, 14, 16, 18, 22]).

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In the recent publications Liu et al. [8, 10] discussed the question of existence of entire solutions of finite order for the following Fermat type differential-difference equation:

$$(f'(z))^n + f(z+c)^m = 1. \tag{1.2}$$

Specifically, the following result was obtained.

Theorem A. *The equation (1.2) has no transcendental entire solutions of finite order, provided that $m \neq n$, where m and n are positive integers.*

In [15], the authors of this paper have obtained a similar result for more general equation.

Theorem B. *([15]) Let $f(z)$ be a transcendental meromorphic function of finite order, m and n be two positive integers such that $m \geq 2n+4$, and let $H(z)$ be a meromorphic function satisfying $\overline{N}(r, \frac{1}{H}) = S(r, f)$. Then $f(z)$ is not a solution of the following equation:*

$$(f'(z))^n + f(z+c)^m = H(z). \tag{1.3}$$

In this paper, we continue our study concerning existence of entire solutions of equation (1.3). To simplify the proofs, we transform the equation (1.3) into the following:

$$f(z)^n + (f'(z+c))^m = P(z). \tag{1.4}$$

In what follows, we assume that the reader is familiar with the elements of Nevanlinna theory (see [6, 20]). Now we are in position to state the main results of this paper.

Theorem 1.1. *Let $P(z)$ be a polynomial, and let n and m be positive integers such that $n \neq m$. Then the equation (1.4) has no transcendental entire solutions of finite order.*

The following question arises naturally: what happens if in equation (1.4), $P(z)$ is a transcendental entire function? Our next result concerns this question, where we consider the special case: $P(z) = r(z)e^{s(z)} + q(z)$.

Theorem 1.2. *Let $P(z) = r(z)e^{s(z)} + q(z)$, where $r(z)$, $s(z)$ and $q(z)$ are nonzero polynomials, If n and m are two positive integers satisfying $n > m+1$, then the equation*

$$f(z)^n + (f'(z+c))^m = r(z)e^{s(z)} + q(z) \tag{1.5}$$

has no transcendental entire solutions of finite order.

Remark. The assertion of Theorem 1.2 is not true, if $n = m + 1$ or $n < m$. For example, when $m = 1$, the following equation

$$f(z)^2 + f'(z + c) = e^{2z} + 1,$$

where $c = \ln 2 + i\pi$, has a finite order transcendental entire solution $f(z) = e^z + 1$, and the transcendental entire function $f(z) = e^z + z$ solves the equation

$$f(z) + (f'(z + c))^2 = \frac{1}{4}e^{2z} + z + 1,$$

where $c = \ln \frac{1}{2} + i\pi$.

The following result contains a partial answer as to what may happen if $n = m$ in Theorem 1.2.

Theorem 1.3. *Let $f(z)$ be an entire function of order $1 \le \sigma(f) < \infty$ and $\lambda(f) < 1$, and let $P(z) = r(z)e^{s(z)} + q(z)$, where $r(z)$, $s(z)$ and $q(z)$ are nonzero polynomials. Then $f(z)$ is not a solution of equation (1.5) when $n = m$.*

In the above stated results we considered only the finite order solutions of equation (1.4). The next result concerns the existence of infinite order solutions of (1.4).

Theorem 1.4. *Let $P(z)$ be a nonzero entire function of finite order, and let m and n be positive integers. Let $f(z)$ be an entire function satisfying the conditions: $\lambda(f) < \sigma(f) = \infty$ and the hyper-order $\sigma_2(f) < \infty$. Then $f(z)$ is not a solution of equation (1.4).*

2. Some lemmas

Lemma 2.1 ([2], Theorem 2.1). *Let $f(z)$ be a meromorphic function of finite order, and let $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f). \tag{2.1}$$

Lemma 2.2 ([1], Theorem 2.1). *Let $f(z)$ be a transcendental meromorphic function of finite order $\sigma(f)$, and let $c \in \mathbb{C}$. Then*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Thus, if f is a finite order meromorphic function, then

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.3 ([6], Theorem 2.4.2). *Let $f(z)$ be a transcendental meromorphic solution of the equation*

$$f^n A(z, f) = B(z, f),$$

where $A(z,f)$ and $B(z,f)$ are differential polynomials in f and its derivatives with small meromorphic coefficients a_λ, in the sense that $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$. If $d(B(z,f)) \le n$, then $m(r, A(z,f)) = S(r,f)$.

Lemma 2.4 ([20], Theorem 1.51). *Suppose that $f_j(z)$ $(j = 1, \dots n)$ $(n \ge 2)$ are meromorphic functions, and $g_j(z)$ $(j = 1, \dots, n)$ are entire functions satisfying the following conditions:*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.
(2) *$g_j(z) - g_k(z)$ are not constants for $1 \le j < k \le n$.*
(3) *For $1 \le j \le n$ and $1 \le h < k \le n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}$, $r \to \infty$, $r \notin E$, where $E \subset (1, \infty)$ is of finite linear measure. Then $f_j(z) \equiv 0$.*

Lemma 2.5 ([5]). *Let $f(z)$ be a transcendental entire function of infinite order and $\sigma_2(f) < \infty$. Then $f(z)$ can be represented in the form $f(z) = U(z)e^{V(z)}$, where $U(z)$ and $V(z)$ are entire functions, and $\lambda(f) = \lambda(U) = \sigma(U)$.*

3. Proof of Theorem 1.1

Suppose there is a transcendental finite order entire solution $f(z)$ of equation (1.4). We discuss the following three cases.

Case 1. If $n > m$, then rewrite the equation (1.4) as follows:

$$f(z)^m f(z)^{n-m} = P(z) - \Big(\frac{f'(z+c)}{f(z+c)}\frac{f(z+c)}{f(z)}\Big)^m f(z)^m.$$

Applying the logarithmic derivative lemma, Lemmas 2.1, 2.3 and taking into account the assumption $n > m$, we conclude that

$$T(r, f^{n-m}) = (n-m)T(r,f) = S(r,f),$$

yielding a contradiction.

Case 2. If $m > n = 1$, then we have $f(z) + (f'(z+c))^m = P(z)$. Differentiating the above equation, we get

$$f'(z) + m(f'(z+c))^{m-1}f''(z+c) = P'(z). \tag{3.1}$$

Set $f'(z+c) = F(z)$, then $f'(z) = F(z-c)$. Moreover, in view of (3.1), we have

$$F(z-c) + m(F(z))^{m-1}F'(z) = P'(z),$$

Rewrite the above equation as

$$m(F(z))^{m-1}F'(z) = P'(z) - \frac{F(z-c)}{F(z)}F(z).$$

Using Lemmas 2.1, 2.3 and taking into account the assumption $m > n = 1$, we get

$$T(r, f'(z+c)) = T(r, F') = m(r, F') = S(r, F') = S(r, f'(z+c)),$$

yielding a contradiction.

Case 3. If $m > n \geq 2$, then by the second fundamental theorem for three small target functions, we obtain

$$\begin{aligned} mT(r, f'(z+c)) &= T(r, (f'(z+c))^m) \\ &\leq \overline{N}\Big(r, \frac{1}{(f'(z+c))^m}\Big) + \overline{N}\Big(r, \frac{1}{(f'(z+c))^m - P}\Big) + S(r, f). \end{aligned}$$

Furthermore, equation (1.4) gives $mT(r, f'(z+c)) = nT(r, f) + S(r, f)$. Therefore, we can write

$$\begin{aligned} (m-1)T(r, f'(z+c)) &\leq \overline{N}\Big(r, \frac{1}{(f'(z+c))^m - P}\Big) + S(r, f) = \overline{N}(r, \frac{1}{f^n}) + S(r, f) \\ &\leq T(r, f) + S(r, f) = \frac{m}{n}T(r, f'(z+c)) + S(r, f), \end{aligned}$$

implying that $T(r, f'(z+c)) \leq (\frac{1}{m} + \frac{1}{n})T(r, f'(z+c)) + S(r, f)$, which contradicts the assumption that $m > n \geq 2$. Theorem 1.1 is proved.

4. Proof of Theorem 1.2

Let $f(z)$ be a transcendental entire solution of finite order of equation (1.5). Then differentiating (1.5) and eliminating $e^{s(z)}$, we can write

$$\begin{aligned} f(z)^{n-1}\big(nf'(z) - (s'(z) + \frac{r'(z)}{r(z)})f(z)\big) &= q'(z) - m\big(f'(z+c)\big)^{m-1}f''(z+c) \\ &\quad + (s'(z) + \frac{r'(z)}{r(z)})((f'(z+c))^m - q(z)). \end{aligned} \tag{4.1}$$

If $nf'(z) - (s'(z) + \frac{r'(z)}{r(z)})f \equiv 0$, then $f(z)^n = Ar(z)e^{s(z)}$, where A is a non-zero constant. Thus, $f(z)$ can be rewritten as

$$f(z) = h(z)e^{\frac{s(z)}{n}},$$

where $h(z)$ is a polynomial. Substituting $f(z) = h(z)e^{\frac{s(z)}{n}}$ into (1.5), we obtain

$$(A-1)r(z)e^{s(z)} + \Big((h(z+c)e^{\frac{s(z+c)}{n}})'\Big)^m - q(z) = 0. \tag{4.2}$$

Hence,

$$(A-1)r(z)e^{s(z)} + \big(G(z)e^{\frac{s(z+c)}{n}}\big)^m - q(z) = 0, \tag{4.3}$$

where $G(z) = h'(z+c) + \frac{h(z+c)s'(z+c)}{n}$. Clearly, $A \neq 1$. We set $g(z) = e^{\frac{s(z)}{n}}$, and use (4.3) and Lemma 2.2, to obtain

$$nT(r, g) = mT(r, g) + S(r, g),$$

which contradicts the assumption that $n > m+1$.

Now let $nf'(z) - (s'(z) + \frac{r'(z)}{r(z)})f \neq 0$. We rewrite the equation (4.1) as follows:

$$\begin{aligned} &f(z)^{n-1}(nf'(z) - (s'(z) + \frac{r'(z)}{r(z)})f(z)) \\ &= q'(z) - m\Big(\big(\frac{f'(z+c)}{f(z+c)}\frac{f(z+c)}{f(z)}\big)^{m-1}\frac{f''(z+c)}{f(z+c)}\frac{f(z+c)}{f(z)}\Big)f(z)^m \\ &+ (s'(z) + \frac{r'(z)}{r(z)})\Big(\big(\frac{f'(z+c)}{f(z+c)}\frac{f(z+c)}{f(z)}\big)^m f(z)^m - q(z)\Big). \end{aligned}$$

Applying the logarithmic derivative lemma and Lemmas 2.1, 2.3, and taking into account that by assumption $f(z)$ is entire, we obtain

$$T\big(r, nf'(z) - (s'(z) + \frac{r'(z)}{r(z)})f\big) = S(r,f)$$

and

$$T\big(r, f(nf'(z) - (s'(z) + \frac{r'(z)}{r(z)})f)\big) = S(r,f).$$

Therefore $T(r,f) = S(r,f)$, which is a contradiction. Theorem 1.2 is proved.

5. Proof of Theorem 1.3

Suppose that $f(z)$ is an entire solution of (1.5) satisfying $1 \leq \sigma(f) < \infty$ and $\lambda(f) < 1$. Then, by the Hadamard factorization theorem, $f(z)$ can be rewritten as $f(z) = h(z)e^{p(z)}$, where $h(z)$ is an entire function satisfying $\lambda(f) = \lambda(h) = \sigma(h) < 1$, and $p(z)$ is a polynomial. We substitute $f(z)$ into (1.5) to obtain

$$h(z)^n e^{np(z)} + \lambda(z)^n e^{np(z+c)} = r(z)e^{s(z)} + q(z), \tag{5.1}$$

where $\lambda(z) = h'(z+c) + h(z+c)p'(z+c)$. Thus, we have

$$h(z)^n e^{np(z)} + \lambda(z)^n e^{np(z+c)} - r(z)e^{s(z)} - q(z) \equiv 0. \tag{5.2}$$

We discuss the following three cases:

Case 1. If $p(z+c) - p(z) \equiv a_1$, where a_1 is a constant, then we obtain $p(z) = Bz + C$, where $B \neq 0$. Substituting $p(z) = Bz + C$ into (5.2), we get

$$e^{n(Bz+C)}(h(z)^n + \lambda(z)^n e^{na_1}) - r(z)e^{s(z)} - q(z) \equiv 0. \tag{5.3}$$

If $s(z) - n(Bz+C) \equiv a_2$, where a_2 is a constant, then by (5.3), it follows that

$$e^{n(Bz+C)}(h(z)^n + \lambda(z)^n e^{na_1}) - r(z)e^{a_2}) \equiv q(z), \tag{5.4}$$

which is impossible under the assumption that $q(z) \not\equiv 0$.

Next, if $s(z) - n(Bz+C) \not\equiv a_2$, then applying Lemma 2.4 to (5.3), we get $q(z) \equiv 0$, which is a contradiction as well. Hence, $p(z+c) - p(z) \not\equiv a_1$.

Case 2. If $np(z+c) - s(z) \equiv b_1$, where b_1 is a constant, then we get

$$(\lambda(z)^n e^{b_1} - r(z))e^{s(z)} + h(z)^n e^{np(z)} - q(z) \equiv 0. \tag{5.5}$$

Observe that $s(z) - np(z) \not\equiv b_2$. Indeed, if $s(z) - np(z) \equiv b_2$, then we have $np(z+c) - np(z) \equiv b_1 + b_2$, which is a contradiction by Case 1. When $s(z) - np(z) \not\equiv b_2$, then applying Lemma 2.4 to (5.5), we get $q(z) \equiv 0$, which is a contradiction. This shows that $np(z+c) - s(z) \not\equiv b_1$.

Case 3. Arguments, similar to those applied in Case 2, can be used to show that $np(z) - s(z) \not\equiv c_1$, where c_1 is a constant.

Combining the Cases 1 - 3, Lemma 2.4 and formula (5.2), we conclude that $q(z) \equiv 0$, which is a contradiction. Therefore, $f(z)$ is not a solution of equation (1.5). Theorem 1.3 is proved.

6. Proof of Theorem 1.4

Let $f(z)$ be an entire solution of equation (1.4) satisfying $\lambda(f) < \sigma(f) = \infty$ and $\sigma_2(f) < \infty$. Hence, we can apply Lemma 2.5 to obtain

$$f(z) = U(z)e^{V(z)}, \tag{6.1}$$

where $U(z)$ and $V(z)$ are entire functions. Moreover, since $\lambda(U) = \sigma(U) = \lambda(f) < \infty$, $\sigma(f) = \infty$, we have $\sigma(f) = \sigma(e^{V(z)}) = \infty$. Therefore, $V(z)$ is a transcendental function. Substituting (6.1) into (1.4), we obtain

$$U(z)^n e^{nV(z)} + H(z)^m e^{mV(z+c)} = P(z), \tag{6.2}$$

where $H(z) = U'(z+c) + U(z+c)V'(z+c)$. From the assumption $\sigma_2(f) < \infty$, we get $\sigma(V) < \infty$, and so $\sigma(H) < \infty$. We write (6.2) in the form

$$U(z)^n + H(z)^m e^{G(z)} = P(z)e^{-nV(z)},$$

where $G(z) := mV(z+c) - nV(z)$. If $G(z)$ is a polynomial, then we have $\sigma(U(z)^n + H(z)^m e^{G(z)}) < \infty$ and $\sigma(P(z)e^{-nV(z)}) = \infty$, which is a contradiction.

Therefore, $G(z)$ is a transcendental entire function. We write (6.2) as follows

$$U(z)^n e^{nV(z)} + H(z)^m e^{mV(z+c)} - P(z)e^0 = 0, \tag{6.3}$$

and observe that $H_1(z) = e^{G(z)}$, $H_2(z) = e^{nV(z)}$, $H_3(z) = e^{mV(z+c)}$ are entire functions of regular growth with infinite order. Therefore, for $i = 1, 2, 3$ we have $T(r, U^n) = o\{T(r, H_i)\}$, $T(r, H^m) = o\{T(r, H_i)\}$, $T(r, P) = o\{T(r, H_i)\}$.

Applying Lemma 2.5 to (6.3), we conclude that $U(z)^n \equiv H(z)^m \equiv P(z) \equiv 0$, which is a contradiction. Hence, $f(z)$ cannot be a solution of equation (1.4). Theorem 1.4 is proved.

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