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DIRICHLET AND NEUMANN PROBLEMS FOR POISSON EQUATION IN HALF LENS

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Abstract. In this article, the Green and Neumann functions are given for a half lens and the Dirichlet and Neumann problems for Poisson equation are solved. All formulas are given in explicit form.

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1. Introduction

The Laplace operator $\partial_z\partial_{\bar{z}}$ produces a second order model equation - the Poisson equation. The classical way to solve the Dirichlet and Neumann boundary value problems for Poisson equation involves application of representation formulas via the Green and Neumann functions. Explicit solutions to the Dirichlet and Neumann boundary value problems for Poisson equation are well known for some particular domains, such as, the half disc and half ring [1], the ring domain [2], the quarter ring domain [3], the equilateral triangle [4], and the unit disc [5]. Let $\Omega = \{z \in \Delta | \mathrm{Im} z > 0\}$, where Δ is a lens defined by the following formula (see [6]): $\Delta = \mathbb{D} \cap D_m(r)$, where $\mathbb{D} = \{z : |z| < 1\}$, $D_m(r) = \{z : |z - m| < r\}$, $0 < r < 1 < m$, and $r^2 + 1 = m^2$.

In this article, we provide explicit solutions and solvability conditions for the Dirichlet and Neumann problems for Poisson equation in the half lens Ω.

2. Boundary value problems for Poisson equation

In order to treat the Dirichlet and Neumann boundary value problems for second order complex partial differential equations some special kernel functions, the Green and Neumann functions, have to be constructed. These kernels then are used to solve the Dirichlet and Neumann boundary value problems for Poisson equation via the corresponding integral representation formulas for solutions. The harmonic Green function for half lens Ω is given by the following formula (see [7]):

$$G_1(z,\zeta) = \log\left|\frac{(\overline{\zeta}-z)(1-z\overline{\zeta})(\overline{\zeta}(1-mz)-(m-z))(\overline{\zeta}(m-z)-(1-mz))}{(\zeta-z)(1-z\zeta)(\zeta(1-mz)-(m-z))(\zeta(m-z)-(1-mz))}\right|^2.$$

For $z \in (m-r,1)$, that is, $z=\overline{z}$, we have

$$\begin{aligned}\partial_{\nu_z} G_1(z,\zeta) &= -i(\partial_z - \partial_{\overline{z}})G_1(z,\zeta) = 4\mathrm{Im}\Big[\tfrac{1}{\zeta-z} + \tfrac{\zeta}{1-z\zeta} + \tfrac{m\zeta-1}{\zeta(1-mz)-(m-z)} \\ &\quad + \tfrac{\zeta-m}{\zeta(m-z)-(1-mz)}\Big],\end{aligned}$$

for $z \in \partial\Omega_{\mathbb{D}}$, that is, $z\overline{z}=1$, we have

$$\begin{aligned}\partial_{\nu_z} G_1(z,\zeta) &= (z\partial_z + \overline{z}\partial_{\overline{z}})G_1(z,\zeta) = 4\mathrm{Re}\Big[\tfrac{z}{\zeta-z} + \tfrac{1}{1-z\zeta} + \tfrac{z(\zeta-m)}{\zeta(m-z)-(1-mz)} \\ &\quad + \tfrac{(m-\zeta)}{\zeta(mz-1)-(z-m)}\Big],\end{aligned}$$

and for $z \in \partial\Omega_{D_m}$, that is, $|z-m|=r$, we have

$$\begin{aligned}\partial_{\nu_z} G_1(z,\zeta) &= \Big((\tfrac{z-m}{r})\partial_z + (\tfrac{\overline{z}-m}{r})\partial_{\overline{z}}\Big)G_1(z,\zeta) = \tfrac{4}{r}\mathrm{Re}\Big[\tfrac{z-m}{\zeta-z} + \tfrac{\zeta(z-m)}{1-z\zeta} \\ &\quad + \tfrac{\zeta r^2}{\zeta(mz-1)-(z-m)} + \tfrac{r^2}{\zeta(m-z)-(1-mz)}\Big].\end{aligned}$$

The next theorem contains a representation formula for a class of functions via the Green function, which is used to solve the Dirichlet boundary value problem for Poisson equation (see [8, 9]).

Theorem 2.1. *Let $\Theta \subset \mathbb{C}$ be a regular domain, and let G_1 be the harmonic Green function for Θ. Then any $\omega \in C^2(\Theta;\mathbb{C}) \cap C^1(\overline{\Theta};\mathbb{C})$ can be represented as follows:*

$$w(z) = -\tfrac{1}{4\pi}\int_{\partial\Theta} w(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)\mathrm{d}s_\zeta - \tfrac{1}{\pi}\int_{\Theta} w_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

where ν is the outward normal derivative on $\partial\Theta$ and s is the arc length parameter.

Thus, any function $\omega \in C^2(\Omega;\mathbb{C}) \cap C^1(\overline{\Omega};\mathbb{C})$ can be represented as follows:

$$\begin{aligned}\omega(z) &= -\tfrac{1}{4\pi i}\Big[\int_{m-r}^{1}\omega(\zeta)[\partial_\zeta - \partial_{\overline{\zeta}}]G_1(z,\zeta)\mathrm{d}\zeta + \int_{\partial\Omega_{\mathbb{D}}}\omega(\zeta)[\zeta\partial_\zeta + \overline{\zeta}\partial_{\overline{\zeta}}]G_1(z,\zeta)\tfrac{\mathrm{d}\zeta}{\zeta} \\ &\quad + \int_{\partial\Omega_{D_m}} r\omega(\zeta)[(\tfrac{\zeta-m}{r})\partial_\zeta + (\tfrac{\overline{\zeta}-m}{r})\partial_{\overline{\zeta}}]G_1(z,\zeta)\tfrac{\mathrm{d}\zeta}{\zeta-m}\Big] \\ &\quad - \tfrac{1}{\pi}\int_{\Omega}\omega_{\zeta\overline{\zeta}}(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,\end{aligned}$$

where $\partial\Omega_{\mathbb{D}} = \partial\Omega \cap \partial\mathbb{D}$ and $\partial\Omega_{D_m} = \partial\Omega \cap \partial D_m(r)$, and in explicit form the Green representation formula from Theorem 2.1 for Ω is

$$\begin{aligned} \omega(z) &= \frac{1}{2\pi i}\int_{m-r}^{1} \omega(t)\Big[\frac{z-\bar z}{|t-z|^2} - \frac{z-\bar z}{|zt-1|^2} + \frac{r^2(z-\bar z)}{|t(1-mz)-(m-z)|^2} \\ &- \frac{r^2(z-\bar z)}{|t(m-z)-(1-mz)|^2}\Big]dt + \frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}} \omega(\zeta)\Big[\frac{1-|z|^2}{|\zeta-z|^2} - \frac{1-|z|^2}{|\bar\zeta-z|^2} \\ &- \frac{r^2(1-|z|^2)}{|\zeta(1-mz)-(m-z)|^2} + \frac{r^2(1-|z|^2)}{|\bar\zeta(1-mz)-(m-z)|^2}\Big]\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\partial\Omega_{D_m}} \omega(\zeta)\Big[\frac{r^2-|z-m|^2}{|\zeta-z|^2} \\ &- \frac{r^2-|z-m|^2}{|\bar\zeta-z|^2} - \frac{r^2-|z-m|^2}{|1-z\bar\zeta|^2} + \frac{r^2-|z-m|^2}{|1-z\zeta|^2}\Big]\frac{d\zeta}{\zeta-m} \\ &- \frac{1}{\pi}\int_{\Omega} \omega_{\zeta\bar\zeta}(\zeta)G_1(z,\zeta)d\xi d\eta. \end{aligned} \tag{2.1}$$

In fact formula (2.1) provides a solution to the Dirichlet problem.

Theorem 2.2. *The Dirichlet problem*

$$\omega_{z\bar z} = f \quad in\ \Omega, \quad \omega = \gamma \quad on \quad \partial\Omega, \quad \gamma(m-r) = \gamma(1) = 0,$$

with given $f \in C(\Omega,\mathbb{C})$ *and* $\gamma \in C(\partial\Omega,\mathbb{C})$ *is uniquely solvable and the solution is given by*

$$\begin{aligned} \omega(z) &= \frac{1}{2\pi i}\int_{m-r}^{1} \gamma(t)\Big[\frac{z-\bar z}{|t-z|^2} - \frac{z-\bar z}{|zt-1|^2} + \frac{r^2(z-\bar z)}{|t(1-mz)-(m-z)|^2} \\ &- \frac{r^2(z-\bar z)}{|t(m-z)-(1-mz)|^2}\Big]dt + \frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}} \gamma(\zeta)\Big[\frac{1-|z|^2}{|\zeta-z|^2} - \frac{1-|z|^2}{|\bar\zeta-z|^2} - \\ &\frac{r^2(1-|z|^2)}{|\zeta(1-mz)-(m-z)|^2} + \frac{r^2(1-|z|^2)}{|\bar\zeta(1-mz)-(m-z)|^2}\Big]\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\partial\Omega_{D_m}} \gamma(\zeta)\Big[\frac{r^2-|z-m|^2}{|\zeta-z|^2} \\ &- \frac{r^2-|z-m|^2}{|\bar\zeta-z|^2} - \frac{r^2-|z-m|^2}{|1-z\bar\zeta|^2} + \frac{r^2-|z-m|^2}{|1-z\zeta|^2}\Big]\frac{d\zeta}{\zeta-m} \\ &- \frac{1}{\pi}\int_{\Omega} f(\zeta)G_1(z,\zeta)d\xi d\eta. \end{aligned}$$

Proof. It can easily be verified that ω is a solution to the Poisson equation. So, it remains to verify that the boundary conditions are satisfied for the boundary integrals.

For $|\zeta_0| = 1$, $\zeta_0 \in \partial\Omega_{\mathbb{D}}$, $\zeta_0 \neq 1, \frac{1}{m} + i\frac{r}{m}$, $t \in (m-r,1)$, $\zeta \in \partial\Omega_{\mathbb{D}}$ and $\eta \in \partial\Omega_{D_m}$, we have

$$\begin{aligned} |t-\zeta_0|^2 &= |t\zeta_0 - 1|^2, |t(1-m\zeta_0) - (m-\zeta_0)|^2 = |t(m-\zeta_0) - (1-m\zeta_0)|^2, \\ |\zeta-\zeta_0|^2 &= |1-\bar\zeta\zeta_0|^2, |\bar\zeta-\zeta_0|^2 = |1-\zeta_0\zeta|^2, \quad 1-|\zeta_0|^2 = 0, \\ |\bar\eta-\zeta_0|^2 &\neq 0, |\eta(1-m\zeta_0) - (m-\zeta_0)|^2 \neq 0, |\bar\eta(1-m\zeta_0) - (m-\zeta_0)|^2 \neq 0. \end{aligned}$$

Hence, based on the properties of the Poisson kernel for $\mathbb{D}$ (see [10]), we can write

$$\begin{aligned}\lim_{z\to\zeta_0}\omega(z) &= \lim_{z\to\zeta_0}\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)[\tfrac{\zeta}{\zeta-z}+\tfrac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1]\tfrac{d\zeta}{\zeta}\\ &= \lim_{z\to\zeta_0}\tfrac{1}{2\pi i}\int_{\partial\mathbb{D}}\Gamma_1(\zeta)[\tfrac{\zeta}{\zeta-z}+\tfrac{\overline{\zeta}}{\overline{\zeta}-\overline{z}}-1]\tfrac{d\zeta}{\zeta}=\Gamma_1(\zeta_0)=\gamma(\zeta_0),\end{aligned}$$

where

$$\Gamma_1(\zeta)=\begin{cases}\gamma(\zeta), & \zeta\in\partial\Omega_{\mathbb{D}},\\ 0, & \zeta\in\partial\mathbb{D}\setminus\partial\Omega_{\mathbb{D}}.\end{cases}$$

For $\zeta_0\in\partial\Omega_{D_m}$, that is, $|\zeta_0-m|=r$, $\zeta_0\neq m-r,\frac{1}{m}+i\frac{r}{m}$, $t\in(m-r,1)$, $\zeta\in\partial\Omega_{D_m}$ and $\eta\in\partial\Omega_{\mathbb{D}}$, we have

$$\begin{aligned}r^2|t-\zeta_0|^2 &= |t(m-\zeta_0)-(1-m\zeta_0)|^2, r^2|\zeta_0t-1|^2=|t(1-m\zeta_0)-(m-\zeta_0)|^2\\ r^2|\zeta-\zeta_0|^2 &= |\overline{\zeta}(m-\zeta_0)-(1-m\zeta_0)|^2, r^2|\overline{\zeta}-\zeta_0|=|\zeta(1-m\zeta_0)-(m-\zeta_0)|^2\\ |\overline{\eta}-\zeta_0| &\neq 0,\quad |1-\overline{\eta}\zeta_0|\neq 0,\quad |1-\eta\zeta_0|\neq 0.\end{aligned}$$

Therefore

$$\lim_{z\to\zeta_0}\omega(z)=\lim_{z\to\zeta_0}\tfrac{1}{2\pi i}\int_{\partial D_m(r)}\Gamma_2(\zeta)(\tfrac{r^2-|z-m|^2}{|\zeta-z|^2})\tfrac{d\zeta}{\zeta-m}=\Gamma_2(\zeta_0)=\gamma(\zeta_0),$$

where

$$\Gamma_2(\zeta)=\begin{cases}\gamma(\zeta), & \zeta\in\partial\Omega_{D_m},\\ 0, & \zeta\in\partial D_m\setminus\partial\Omega_{D_m}.\end{cases}$$

Similarly, for $\zeta_0\in(m-r,1)$, that is, $\zeta_0=\overline{\zeta_0}$, $\zeta\in\partial\Omega_{\mathbb{D}}$, $\eta\in\partial\Omega_{D_m}$ and $t\in(m-r,1)$, we have

$$\begin{aligned}|\zeta-\zeta_0| &= |\overline{\zeta}-\zeta_0|,\quad |\zeta(1-m\zeta_0)-(m-\zeta_0)|^2=|\overline{\zeta}(1-m\zeta_0)-(m-\zeta_0)|^2\\ |1-\zeta_0\overline{\eta}|^2 &= |1-\zeta_0\eta|^2,\quad |\eta-\zeta_0|=|\overline{\eta}-\zeta_0|,\\ |t\zeta_0-1| &\neq 0, |t(1-m\zeta_0)-(m-\zeta_0)|^2\neq 0,\quad |t(m-\zeta_0)-(1-m\zeta_0)|^2\neq 0.\end{aligned}$$

Thus, in view of [11], we get

$$\lim_{z\to\zeta_0}\omega(z)=\lim_{z\to\zeta_0}\tfrac{1}{2\pi i}\int_{-\infty}^{+\infty}\Gamma_3(t)\tfrac{z-\overline{z}}{|t-z|^2}dt=\Gamma_3(\zeta_0)=\gamma(\zeta_0),$$

where

$$\Gamma_3(t)=\begin{cases}\gamma(t), & t\in(m-r,1),\\ 0, & t\in\mathbb{R}\setminus(m-r,1).\end{cases}$$

Here the corner points $m-r,1,\frac{1}{m}+i\frac{r}{m}$ were excluded.

To check the boundary behavior at points $m-r$ and 1, we define

$$\omega_1(z)=\tfrac{1}{2\pi i}\int_{m-r}^{1}\gamma(t)\left[\tfrac{z-\overline{z}}{|t-z|^2}-\tfrac{z-\overline{z}}{|zt-1|^2}+\tfrac{r^2(z-\overline{z})}{|t(1-mz)-(m-z)|^2}-\tfrac{r^2(z-\overline{z})}{|t(m-z)-(1-mz)|^2}\right]dt,$$

$$\omega_2(z)=\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\left[\tfrac{1-|z|^2}{|\zeta-z|^2}-\tfrac{1-|z|^2}{|\overline{\zeta}-z|^2}-\tfrac{r^2(1-|z|^2)}{|\zeta(1-mz)-(m-z)|^2}+\tfrac{r^2(1-|z|^2)}{|\overline{\zeta}(1-mz)-(m-z)|^2}\right]\tfrac{d\zeta}{\zeta}$$

and

$$\omega_3(z) = \frac{1}{2\pi i}\int_{\partial\Omega_{D_m}} \gamma(\zeta)\Big[\frac{r^2-|z-m|^2}{|\zeta-z|^2} - \frac{r^2-|z-m|^2}{|\bar\zeta-z|^2} - \frac{r^2\big(r^2-|z-m|^2\big)}{|\zeta(1-mz)-(m-z)|^2} + \frac{r^2\big(r^2-|z-m|^2\big)}{|\bar\zeta(1-mz)-(m-z)|^2}\Big]\frac{d\zeta}{\zeta-m}.$$

We first prove that $\lim_{z\to 1}\omega(z) = \gamma(1) = 0$. Indeed, using the equalities

$$\int_{m-r}^{1}\gamma(t)\frac{z-\bar z}{|zt-1|^2}dt = \int_1^{1/(m-r)}\gamma(\tfrac{1}{t})\frac{z-\bar z}{|z-t|^2}dt$$

and

$$\int_{m-r}^{1}\gamma(t)\frac{r^2(z-\bar z)}{|t(m-z)-(1-mz)|^2}dt = \int_1^{1/(m-r)}\gamma(\tfrac{1}{t})\frac{r^2(z-\bar z)}{|t(1-mz)-(m-z)|^2}dt,$$

we can write

$$\omega_1(z) = \frac{1}{2\pi i}\Big[\int_{m-r}^{1/(m-r)}\Gamma_4(t)\frac{z-\bar z}{|t-z|^2}dt + \int_{m-r}^{1/(m-r)}\Gamma_4(t)\frac{r^2(z-\bar z)}{|t(1-mz)-(m-z)|^2}dt\Big],$$

where

$$\Gamma_4(t) = \begin{cases} \gamma(t), & m-r \le t \le 1,\\ -\gamma(\frac{1}{t}), & 1 \le t \le 1/m-r.\end{cases}$$

Hence, based on the properties of the Poisson kernel for half plane, for $t_0 \in (m-r, \frac{1}{m-r})$, we have

$$\lim_{z\to t_0}\omega_1(z) = \Gamma(t_0).$$

In particular, the relation $\lim_{z\to 1}\omega_1(z) = \gamma(1) = 0$ follows, because of the continuity of Γ_4 at 1. Now we prove that $\lim_{z\to m-r}\omega_1(z) = \gamma(m-r) = 0$. Indeed, we have

$$\omega_1(z) = \frac{1}{2\pi i}\int_{-1}^{1}\Gamma_5(t)\frac{z-\bar z}{|t-z|^2}dt + \frac{1}{2\pi i}\int_{-1}^{1}\Gamma_5(t)\frac{r^2(z-\bar z)}{|t(1-mz)-(m-z)|^2}dt,$$

where

$$\Gamma_5(t) = \begin{cases} -\gamma(\frac{1-mt}{m-t}), & -1 \le t \le m-r,\\ \gamma(t), & m-r \le t \le 1.\end{cases}$$

So, for $t_0 \in (-1,1)$ we have

$$\lim_{z\to t_0}\omega_1(z) = \Gamma(t_0),$$

and, in particular, the relation $\lim_{z\to m-r}\omega_1(z) = \gamma(m-r) = 0$ follows, because of the continuity of Γ_5 at $m-r$. Next, we show that $\lim_{z\to 1}\omega_2(z) = 0$ and $\lim_{z\to m-r}\omega_2(z) = 0$. To this end, we write

$$\begin{aligned}\omega_2(z) &= \frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\frac{1-|z|^2}{|\zeta-z|^2}\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\bar\zeta)\frac{1-|z|^2}{|\zeta-z|^2}\frac{d\zeta}{\zeta}\\ &\quad -\frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\frac{r^2(1-|z|^2)}{|\zeta(1-mz)-(m-z)|^2}\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\bar\zeta)\frac{r^2(1-|z|^2)}{|\zeta(1-mz)-(m-z)|^2}\frac{d\zeta}{\zeta}\\ &= \frac{1}{2\pi i}\int_{\partial\Delta\cap\partial\mathbb{D}}\Gamma_6(\zeta)\frac{1-|z|^2}{|\zeta-z|^2}\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int_{\partial\Delta\cap\partial\mathbb{D}}\Gamma_6(\zeta)\frac{r^2(1-|z|^2)}{|\zeta(1-mz)-(m-z)|^2}\frac{d\zeta}{\zeta},\end{aligned}$$

where

$$\Gamma_6(\zeta) = \begin{cases} \gamma(\zeta), & \zeta \in \partial\Omega_{\mathbb{D}}, \\ -\gamma(\overline{\zeta}), & \zeta \in \overline{\partial\Omega_{\mathbb{D}}}, \\ 0, & \zeta \in \partial\mathbb{D} \setminus \partial\Delta. \end{cases}$$

Then, for $\zeta_0 \in \partial\mathbb{D} \cap \partial\Delta$ we have

$$\lim_{z\to\zeta_0} \omega_2(z) = \Gamma_6(\zeta_0) - \lim_{z\to\zeta_0} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \Gamma_6(\zeta)\frac{r^2(1-|z|^2)}{|\zeta(1-mz)-(m-z)|^2}\frac{d\zeta}{\zeta},$$

and, in particular, the relation $\lim_{z\to 1}\omega_2(z) = \gamma(1) = 0$, follows, because of the continuity of Γ_6 at 1. Next, in view of the equality

$$\omega_2(\tfrac{1-mz}{m-z}) = -\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \Gamma_6(\zeta)\frac{1-|z|^2}{|1-z\zeta|}\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \Gamma_6(\zeta)\frac{r^2(1-|z|^2)}{|\zeta(m-z)-(1-mz)|^2}\frac{d\zeta}{\zeta}$$

it follows that $\lim_{z\to m-r}\omega_2(\frac{1-mz}{m-z}) = -\lim_{z\to m-r}\omega_2(z)$, and hence

$$\lim_{z\to m-r} \omega_2(z) = 0.$$

Now we examine the limiting behavior of $\omega_3(z)$. We have

$$\begin{aligned}\omega_3(z) &= \frac{1}{2\pi i}\int_{\partial\Omega_{D_m}} \gamma(\zeta)\frac{r^2-|z-m|^2}{|\zeta-z|^2}\frac{d\zeta}{\zeta-m} - \frac{1}{2\pi i}\int_{\overline{\partial\Omega_{D_m}}} \gamma(\overline{\zeta})\frac{r^2-|z-m|^2}{|\zeta-z|^2}\frac{d\zeta}{\zeta-m} \\ &\quad - \frac{1}{2\pi i}\int_{\partial\Omega_{D_m}} \gamma(\zeta)\frac{r^2\left(r^2-|z-m|^2\right)}{|\zeta(1-mz)-(m-z)|^2}\frac{d\zeta}{\zeta-m} + \frac{1}{2\pi i}\int_{\overline{\partial\Omega_{D_m}}} \gamma(\overline{\zeta})\frac{r^2\left(r^2-|z-m|^2\right)}{|\zeta(1-mz)-(m-z)|^2}\frac{d\zeta}{\zeta-m}.\end{aligned}$$

Therefore

$$\omega_3(z) = \frac{1}{2\pi i}\int_{\partial D_m(r)} \Gamma_7(\zeta)\frac{r^2-|z-m|^2}{|\zeta-z|^2}\frac{d\zeta}{\zeta-m} - \frac{1}{2\pi i}\int_{\partial D_m(r)} \Gamma_7(\zeta)\frac{r^2\left(r^2-|z-m|^2\right)}{|\zeta(1-mz)-(m-z)|^2}\frac{d\zeta}{\zeta-m},$$

where

$$\Gamma_7(\zeta) = \begin{cases} \gamma(\zeta), & \zeta \in \partial\Omega_{D_m}, \\ -\gamma(\overline{\zeta}), & \zeta \in \overline{\partial\Omega_{D_m}}, \\ 0, & \zeta \in \partial D_m \setminus \partial\Delta. \end{cases}$$

Then for $\zeta_0 \in \partial\Delta \cap \partial D_m(r)$ we have

$$\lim_{z\to\zeta_0} \omega_3(z) = \Gamma_7(\zeta_0) - \lim_{z\to\zeta_0} \frac{1}{2\pi i}\int_{\partial D_m(r)} \Gamma_7(\zeta)\frac{r^2\left(r^2-|z-m|^2\right)}{|\zeta(1-mz)-(m-z)|^2}\frac{d\zeta}{\zeta-m},$$

implying that $\lim_{z\to m-r}\omega_3(z) = \Gamma_7(m-r) = 0$. From

$$\omega_3(\tfrac{1}{z}) = \frac{1}{2\pi i}\int_{\partial D_m(r)} \Gamma_7(\zeta)\frac{r^2\left(r^2-|z-m|^2\right)}{|\zeta(1-mz)-(m-z)|^2}\frac{d\zeta}{\zeta-m} - \frac{1}{2\pi i}\int_{\partial D_m(r)} \Gamma_7(\zeta)\frac{r^2-|z-m|^2}{|\zeta-z|^2}\frac{d\zeta}{\zeta-m},$$

it follows that $\lim_{z\to 1}\omega_3(\frac{1}{z}) = -\lim_{z\to 1}\omega_3(z)$, and hence $\lim_{z\to 1}\omega_3(z) = 0$. Now we show that $\lim_{z\to\frac{1}{m}+i\frac{r}{m}}\omega(z) = \gamma(\frac{1}{m}+i\frac{r}{m})$. We have

$$\begin{aligned}
1 &= \tfrac{1}{4\pi i}\int_{m-r}^{1}[u_t(z)-u_t(\overline{z})+u_t(\tfrac{1}{z})-u_t(\tfrac{1}{\overline{z}})+u_t(\tfrac{m-z}{1-mz})-u_t(\tfrac{m-\overline{z}}{1-m\overline{z}})\\
&\quad +u_t(\tfrac{1-mz}{m-z})-u_t(\tfrac{1-m\overline{z}}{m-\overline{z}})]\mathrm{d}t+\tfrac{1}{4\pi i}\int_{\partial\Omega_{\mathbb{D}}}[v_\zeta(z)+v_{\overline{\zeta}}(\overline{z})-v_{\overline{\zeta}}(z)-v_\zeta(\overline{z})\\
&\quad +v_\zeta(\tfrac{m-z}{1-mz})+v_{\overline{\zeta}}(\tfrac{m-\overline{z}}{1-m\overline{z}})-v_{\overline{\zeta}}(\tfrac{m-z}{1-mz})-v_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})]\tfrac{\mathrm{d}\zeta}{\zeta}\\
&\quad +\tfrac{1}{4\pi i}\int_{\partial\Omega_{D_m}}[w_\zeta(z)+w_{\overline{\zeta}}(\overline{z})-w_{\overline{\zeta}}(z)-w_\zeta(\overline{z})\\
&\quad +w_\zeta(\tfrac{m-z}{1-mz})+w_{\overline{\zeta}}(\tfrac{m-\overline{z}}{1-m\overline{z}})-w_{\overline{\zeta}}(\tfrac{m-z}{1-mz})-w_\zeta(\tfrac{m-\overline{z}}{1-m\overline{z}})]\tfrac{\mathrm{d}\zeta}{\zeta-m},
\end{aligned}\tag{2.2}$$

where $u_t(z) = \frac{2}{t-z}$, $v_\zeta(z) = \frac{2\zeta}{\zeta-z}$ and $w_\zeta(z) = \frac{2(\zeta-m)}{\zeta-z}$.

Multiplying (2.2) by $\gamma(\frac{1}{m}+i\frac{r}{m})$ and subtracting the resulting equation from $\omega(z)$, for $z\in\partial\Omega$ we obtain

$$\begin{aligned}
\omega(z)-\gamma(\tfrac{1}{m}+i\tfrac{r}{m}) &= \tfrac{1}{2\pi i}\int_{m-r}^{1}\tilde{\gamma}(t)[\tfrac{z-\overline{z}}{|t-z|^2}-\tfrac{z-\overline{z}}{|zt-1|^2}+\tfrac{r^2(z-\overline{z})}{|t(1-mz)-(m-z)|^2}\\
&\quad -\tfrac{r^2(z-\overline{z})}{|t(m-z)-(1-mz)|^2}]\mathrm{d}t+\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\tilde{\gamma}(\zeta)[\tfrac{1-|z|^2}{|\zeta-z|^2}-\tfrac{1-|z|^2}{|\overline{\zeta}-z|^2}-\\
&\quad \tfrac{r^2(1-|z|^2)}{|\zeta(1-mz)-(m-z)|^2}+\tfrac{r^2(1-|z|^2)}{|\overline{\zeta}(1-mz)-(m-z)|^2}]\tfrac{\mathrm{d}\zeta}{\zeta}\\
&\quad +\tfrac{1}{2\pi i}\int_{\partial\Omega_{D_m}}\tilde{\gamma}(\zeta)[\tfrac{r^2-|z-m|^2}{|\zeta-z|^2}-\tfrac{r^2-|z-m|^2}{|\overline{\zeta}-z|^2}-\tfrac{r^2-|z-m|^2}{|1-z\overline{\zeta}|^2}\\
&\quad +\tfrac{r^2-|z-m|^2}{|1-z\zeta|^2}]\tfrac{\mathrm{d}\zeta}{\zeta-m}-\tfrac{1}{\pi}\int_{\Omega}f(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta\\
&= \tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\tilde{\gamma}(\zeta)\tfrac{1-|z|^2}{|\zeta-z|^2}\tfrac{\mathrm{d}\zeta}{\zeta},
\end{aligned}$$

where $\tilde{\gamma}(\zeta) = \gamma(\zeta)-\gamma(\frac{1}{m}+i\frac{r}{m})$, and $\tilde{\gamma}(\frac{1}{m}+i\frac{r}{m}) = 0$. Thus, we have

$$\lim_{z\to\frac{1}{m}+i\frac{r}{m}}\omega(z)-\gamma(\tfrac{1}{m}+i\tfrac{r}{m}) = 0.$$

Theorem 2.2 is proved. □

Another second order representation formula, similar to the one given in Theorem 2.1 is available, where instead of Green function the harmonic Neumann function is used (see [8]).

Theorem 2.3. *Let $\Theta\subset\mathbb{C}$ be a regular domain, and let N_1 be the harmonic Neumann function for Θ. Then any $\omega\in C^2(\Theta;\mathbb{C})\cap C^1(\overline{\Theta};\mathbb{C})$ can be represented as follows:*

$$w(z) = -\frac{1}{4\pi}\int_{\partial\Theta}\{w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)-\partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta)\}\mathrm{d}s_\zeta-\frac{1}{\pi}\int_{\Theta}w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.$$

Thus, any function $\omega \in C^2(\Omega;\mathbb{C}) \cap C^1(\overline{\Omega};\mathbb{C})$ can be represented as follows:

$$\begin{aligned}
\omega(z) &= \tfrac{1}{4\pi i}\Big[\int_{m-r}^{1} N_1(z,\zeta)[\partial_\zeta - \partial_{\overline{\zeta}}]\omega(\zeta)\mathrm{d}\zeta + \int_{\partial\Omega_{\mathbb{D}}} N_1(z,\zeta)(\zeta\partial_\zeta + \overline{\zeta}\partial_{\overline{\zeta}})\omega(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta} \\
&\quad + r\int_{\partial\Omega_{D_m}} N_1(z,\zeta)[(\tfrac{\zeta-m}{r})\partial_\zeta + (\tfrac{\overline{\zeta}-m}{r})\partial_{\overline{\zeta}}]\omega(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta-m}\Big] \\
&\quad - \tfrac{1}{4\pi i}\Big[\int_{m-r}^{1} \omega(\zeta)[\partial_\zeta - \partial_{\overline{\zeta}}]N_1(z,\zeta)\mathrm{d}\zeta + \int_{\partial\Omega_{\mathbb{D}}} \omega(\zeta)[\zeta\partial_\zeta + \overline{\zeta}\partial_{\overline{\zeta}}]N_1(z,\zeta)\tfrac{\mathrm{d}\zeta}{\zeta} \\
&\quad + r\int_{\partial\Omega_{D_m}} \omega(\zeta)[(\tfrac{\zeta-m}{r})\partial_\zeta + (\tfrac{\overline{\zeta}-m}{r})\partial_{\overline{\zeta}}]N_1(z,\zeta)\tfrac{\mathrm{d}\zeta}{\zeta-m}\Big] \\
&\quad - \tfrac{1}{\pi}\int_{\Omega} w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta. \qquad (2.3)
\end{aligned}$$

It can easily be verified that the harmonic Neumann function $N_1(z,\zeta)$ for the half lens Ω is given by the following formula:

$$\begin{aligned}
N_1(z,\zeta) &= -\log\big|(\overline{\zeta}-z)(1-z\overline{\zeta})\left(\overline{\zeta}(1-mz)-(m-z)\right) \\
&\quad \left(\overline{\zeta}(m-z)-(1-mz)\right)(\zeta-z)(1-z\zeta)\left(\zeta(1-mz)-(m-z)\right) \\
&\quad \left(\zeta(m-z)-(1-mz)\right)\big|^2.
\end{aligned}$$

For $z \in \partial\Omega_{\mathbb{D}} \setminus \{1, \frac{1}{m} + i\frac{r}{m}\}$ and $\zeta \in \Omega$, we have

$$\begin{aligned}
\partial_{\nu_z} N_1(z,\zeta) &= \tfrac{z}{\overline{\zeta}-z} + \tfrac{\overline{\zeta}z}{1-z\overline{\zeta}} + \tfrac{(m\overline{\zeta}-1)z}{\overline{\zeta}(1-mz)-(m-z)} + \tfrac{(\overline{\zeta}-m)z}{\overline{\zeta}(m-z)-(1-mz)} \\
&\quad + \tfrac{z}{\zeta-z} + \tfrac{\zeta z}{1-z\zeta} + \tfrac{(m\zeta-1)z}{\zeta(1-mz)-(m-z)} + \tfrac{(\zeta-m)z}{\zeta(m-z)-(1-mz)} \\
&\quad \tfrac{\overline{z}}{\zeta-\overline{z}} + \tfrac{\zeta\overline{z}}{1-\overline{z}\zeta} + \tfrac{(m\zeta-1)\overline{z}}{\zeta(1-m\overline{z})-(m-\overline{z})} + \tfrac{(\zeta-m)\overline{z}}{\zeta(m-\overline{z})-(1-m\overline{z})} \\
&\quad + \tfrac{\overline{z}}{\overline{\zeta}-\overline{z}} + \tfrac{\overline{\zeta}\overline{z}}{1-\overline{z}\overline{\zeta}} + \tfrac{(m\overline{\zeta}-1)\overline{z}}{\overline{\zeta}(1-m\overline{z})-(m-\overline{z})} + \tfrac{(\overline{\zeta}-m)\overline{z}}{\overline{\zeta}(m-\overline{z})-(1-m\overline{z})} = -8.
\end{aligned}$$

Similarly, for $z \in (m-r,1)$, $\zeta \in \Omega$, $z \in \partial\Omega_{D_m} \setminus \{m-r, \frac{1}{m} + i\frac{r}{m}\}$ and $\zeta \in \Omega$, we get

$$\partial_{\nu_z} N_1(z,\zeta) = -i(\partial_z - \partial_{\overline{z}})N_1(z,\zeta) = 0,$$

and

$$\partial_{\nu_z} N_1(z,\zeta) = \Big((\tfrac{z-m}{r})\partial_z + (\tfrac{\overline{z}-m}{r})\partial_{\overline{z}}\Big)N_1(z,\zeta) = -\tfrac{8}{r}.$$

Now we introduce the outward normal derivatives at the corner points. Let the partial outward normal derivatives be given by the following formulas:

$$\partial^+_{\nu_z}\omega(1) = \lim_{\substack{\zeta\to 1 \\ \zeta\in\partial\Omega_{\mathbb{D}}\setminus\{1\}}} \partial_{\nu_z}\omega(\zeta), \qquad \partial^-_{\nu_z}\omega(1) = \lim_{\substack{t\to 1 \\ t\in(m-r,1)}} \partial_{\nu_z}\omega(t), \tag{2.4}$$

$$\partial^+_{\nu_z}\omega(m-r) = \lim_{\substack{t\to m-r \\ t\in(m-r,1)}} \partial_{\nu_z}\omega(t), \qquad \partial^-_{\nu_z}\omega(m-r) = \lim_{\substack{\zeta\to m-r \\ \zeta\in\partial\Omega_{D_m}\setminus\{m-r\}}} \partial_{\nu_z}\omega(\zeta), \tag{2.5}$$

and

$$\partial^+_{\nu_z}\omega(\tfrac{1}{m}+i\tfrac{r}{m}) = \lim_{\substack{\zeta\to\frac{1}{m}+i\frac{r}{m}\\ \zeta\in\partial\Omega_{D_m}\setminus\{\frac{1}{m}+i\frac{r}{m}\}}} \partial_{\nu_z}\omega(\zeta), \quad \partial^-_{\nu_z}\omega(\tfrac{1}{m}+i\tfrac{r}{m}) = \lim_{\substack{\zeta\to\frac{1}{m}+i\frac{r}{m}\\ \zeta\in\partial\Omega_{\mathrm{D}}\setminus\{\frac{1}{m}+i\frac{r}{m}\}}} \partial_{\nu_z}\omega(\zeta). \tag{2.6}$$

Definition 2.1. *If the partial outward normal derivatives (2.4)-(2.6) exist, then the outward normal derivatives at the three corner points are defined to be*

$$\partial_{\nu_z}\omega(\zeta) = \tfrac{1}{2}\big[\partial^+_{\nu_z}\omega(\zeta)+\partial^-_{\nu_z}\omega(\zeta)\big], \quad \zeta\in\{m-r, 1, \tfrac{1}{m}+i\tfrac{r}{m}\}.$$

The formula (2.3) provides a solution to the Neumann boundary value problem for Poisson equation.

Theorem 2.4. *The Neumann problem*

$$\omega_{z\bar z} = f \quad in\ \Omega, \quad \partial_\nu\omega = \gamma \quad on \quad \partial\Omega,$$
$$\frac{1}{\pi i}\int_{\partial\Omega_{\mathrm{D}}}\omega(\zeta)\tfrac{d\zeta}{\zeta} + \tfrac{1}{\pi i}\int_{\partial\Omega_{D_m}}\omega(\zeta)\tfrac{d\zeta}{\zeta-m} = c,$$

for $f\in C(\Omega;\mathbb{C})$, $\gamma\in C(\partial\Omega;\mathbb{C})$ and $c\in\mathbb{C}$ is solvable if and only if

$$\int_\Omega f(\zeta)d\xi d\eta = \tfrac{1}{4i}\int_{\partial\Omega_{\mathrm{D}}}\gamma(\zeta)\tfrac{d\zeta}{\zeta} + \tfrac{r}{4i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\tfrac{d\zeta}{\zeta-m} + \tfrac{1}{4}\int_{m-r}^1\gamma(t)dt, \tag{2.7}$$

and the weak solution is given by formula:

$$\begin{aligned}\omega(z) =& -\tfrac{1}{2\pi}\int_{m-r}^1\gamma(t)\big[\log|t-z|^2+\log|1-zt|^2+\log|t(m-z)-(1-mz)|^2\\ &+\log|t(1-mz)-(m-z)|^2\big]dt\\ &-\frac{1}{2\pi i}\int_{\partial\Omega_{\mathrm{D}}}\gamma(\zeta)\big[\log|\zeta-z|^2+\log|1-z\zeta|^2+\log|\zeta(m-z)-(1-mz)|^2\\ &+\log|\zeta(1-mz)-(m-z)|^2\big]\tfrac{d\zeta}{\zeta}\\ &-\tfrac{r}{2\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\big[\log|\zeta-z|^2+\log|1-z\zeta|^2+\log|\zeta(m-z)-(1-mz)|^2\\ &+\log|\zeta(1-mz)-(m-z)|^2\big]\tfrac{d\zeta}{\zeta-m}\\ &-\tfrac{1}{\pi}\int_\Omega f(\zeta)N_1(z,\zeta)d\xi d\eta + c.\end{aligned} \tag{2.8}$$

Proof. It follows from Theorem 2.3 that if the Neumann problem is solvable, then the solution should be of form (2.8). Now we verify that (2.8) indeed is a solution of the Neumann problem. We easily get $\omega_{z\bar z}=f$ in Ω. Next, for $z\in\Omega$, we have

$$\begin{aligned}-i(\partial_z-\partial_{\bar z})\omega(z) =& \tfrac{1}{4\pi i}\int_{m-r}^1(\partial_z-\partial_{\bar z})N_1(z,t)\gamma(t)dt - \tfrac{1}{4\pi}\int_{\partial\Omega_{\mathrm{D}}}(\partial_z-\partial_{\bar z})N_1(z,\zeta)\gamma(\zeta)\tfrac{d\zeta}{\zeta}\\ &-\tfrac{r}{4\pi}\int_{\partial\Omega_{D_m}}(\partial_z-\partial_{\bar z})N_1(z,\zeta)\gamma(\zeta)\tfrac{d\zeta}{\zeta-m} + \tfrac{i}{\pi}\int_\Omega f(\zeta)(\partial_z-\partial_{\bar z})N_1(z,\zeta)d\xi d\eta.\end{aligned}$$

and for $z \in \Omega$ and $\zeta \in \partial\Omega$ we can write

$$\begin{aligned}(\partial_z - \partial_{\overline{z}})N_1(z,\zeta) &= 2\Big[\tfrac{1}{\zeta-z} + \tfrac{\zeta}{1-z\zeta} + \tfrac{m-\zeta}{(1-mz)-\zeta(m-z)} + \tfrac{1-m\zeta}{(m-z)-\zeta(1-mz)} \\ &\quad - \tfrac{1}{\overline{\zeta}-\overline{z}} - \tfrac{\overline{\zeta}}{1-\overline{z}\overline{\zeta}} - \tfrac{m-\overline{\zeta}}{(1-m\overline{z})-\overline{\zeta}(m-\overline{z})} - \tfrac{1-m\overline{\zeta}}{(m-\overline{z})-\overline{\zeta}(1-m\overline{z})}\Big].\end{aligned}$$

Therefore, for $t_0 \in (m-r, 1)$, we have

$$\partial_{\nu_z}\omega(t_0) = \lim_{z\to t_0}\big[-i(\partial_z - \partial_{\overline{z}})\big]\omega(z) = \lim_{z\to t_0}\tfrac{1}{2\pi i}\int_{m-r}^{1}\gamma(t)\tfrac{z-\overline{z}}{|t-z|^2}\mathrm{d}t = \gamma(t_0).$$

Further, we have

$$\partial^+_{\nu_z}\omega(m-r) = \lim_{\substack{t_0\to m-r\\ t_0\in(m-r,1)}}\partial_{\nu_z}\omega(t_0) = \gamma(m-r), \tag{2.9}$$

$$\partial^-_{\nu_z}\omega(1) = \lim_{\substack{t_0\to 1\\ t_0\in(m-r,1)}}\partial_{\nu_z}\omega(t_0) = \gamma(1). \tag{2.10}$$

Similarly, we obtain

$$\begin{aligned}&(z\partial_z + \overline{z}\partial_{\overline{z}})\omega(z) = \tfrac{1}{4\pi}\int_{m-r}^{1}(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,t)\gamma(t)\mathrm{d}t \\ &+ \tfrac{1}{4\pi i}\int_{\partial\Omega_{\mathbb{D}}}(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta)\gamma(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta} + \tfrac{r}{4\pi i}\int_{\partial\Omega_{D_m}}(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta)\gamma(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta-m} \\ &- \tfrac{1}{\pi}\int_{\Omega}f(\zeta)(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.\end{aligned}$$

Taking into account that for $z \in \Omega$ and $\zeta \in \partial\Omega$

$$\begin{aligned}(z\partial_z + \overline{z}\partial_{\overline{z}})N_1(z,\zeta) &= 2\Big[\tfrac{z}{\zeta-z} + \tfrac{z\zeta}{1-z\zeta} + \tfrac{z(m-\zeta)}{(1-mz)-\zeta(m-z)} + \tfrac{z(1-m\zeta)}{(m-z)-\zeta(1-mz)} \\ &\quad + \tfrac{\overline{z}}{\overline{\zeta}-\overline{z}} + \tfrac{\overline{z}\overline{\zeta}}{1-\overline{z}\overline{\zeta}} + \tfrac{\overline{z}(m-\overline{\zeta})}{(1-m\overline{z})-\overline{\zeta}(m-\overline{z})} + \tfrac{\overline{z}(1-m\overline{\zeta})}{(m-\overline{z})-\overline{\zeta}(1-m\overline{z})}\Big],\end{aligned}$$

for $\zeta_0 \in \partial\Omega_{\mathbb{D}} \setminus \{1, \frac{1}{m} + i\frac{r}{m}\}$, we can write

$$\begin{aligned}\partial_{\nu_z}\omega(\zeta_0) &= \lim_{z\to\zeta_0}(z\partial_z + \overline{z}\partial_{\overline{z}})\omega(z) = \lim_{z\to\zeta_0}\Big\{\tfrac{1}{2\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\big[\tfrac{\zeta}{\zeta-z} + \tfrac{\overline{\zeta}}{\overline{\zeta}-\overline{z}} - 1\big]\tfrac{\mathrm{d}\zeta}{\zeta} \\ &\quad - \tfrac{2}{\pi}\int_{m-r}^{1}\gamma(t)\mathrm{d}t - \tfrac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta} - \tfrac{2r}{\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta-m} \\ &\quad + \tfrac{8}{\pi}\int_{\Omega}f(\zeta)\mathrm{d}\xi\mathrm{d}\eta\Big\} = \gamma(\zeta_0) - \tfrac{2}{\pi}\int_{m-r}^{1}\gamma(t)\mathrm{d}t \\ &\quad - \tfrac{2}{\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta} - \tfrac{2r}{\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta-m} + \tfrac{8}{\pi}\int_{\Omega}f(\zeta)\mathrm{d}\xi\mathrm{d}\eta,\end{aligned}$$

which implies the necessity and sufficiency of condition (2.7). Thus, we have

$$\partial^+_{\nu_z}\omega(1) = \lim_{\substack{\zeta_0\to 1\\ \zeta_0\in\partial\Omega_{\mathbb{D}}\setminus\{1,\frac{1}{m}+i\frac{r}{m}\}}}\partial_{\nu_z}\omega(\zeta_0) = \gamma(1), \tag{2.11}$$

$$\partial_{\nu_z}^{-}\omega(\tfrac{1}{m}+i\tfrac{r}{m}) = \lim_{\substack{\zeta_0\to\frac{1}{m}+i\frac{r}{m}\\ \zeta_0\in\partial\Omega_{\mathbb{D}}\setminus\{1,\frac{1}{m}+i\frac{r}{m}\}}} \partial_{\nu_z}\omega(\zeta_0) = \gamma(\tfrac{1}{m}+i\tfrac{r}{m}). \tag{2.12}$$

Finally, we observe that

$$\begin{aligned}\Big((\tfrac{z-m}{r})\partial_z + (\tfrac{\overline{z}-m}{r})\partial_{\overline{z}}\Big)\omega(z) &= \tfrac{1}{4\pi}\int_{m-r}^{1}\Big((\tfrac{z-m}{r})\partial_z + (\tfrac{\overline{z}-m}{r})\partial_{\overline{z}}\Big)N_1(z,t)\gamma(t)\mathrm{d}t\\ &\quad+\tfrac{1}{4\pi i}\int_{\partial\Omega_{\mathbb{D}}}\Big((\tfrac{z-m}{r})\partial_z + (\tfrac{\overline{z}-m}{r})\partial_{\overline{z}}\Big)N_1(z,\zeta)\gamma(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta}\\ &\quad+\tfrac{r}{4\pi i}\int_{\partial\Omega_{D_m}}\Big((\tfrac{z-m}{r})\partial_z + (\tfrac{\overline{z}-m}{r})\partial_{\overline{z}}\Big)N_1(z,\zeta)\gamma(\zeta)\tfrac{\mathrm{d}\zeta}{\zeta-m}\\ &\quad-\tfrac{1}{\pi}\int_{\Omega} f(\zeta)\Big((\tfrac{z-m}{r})\partial_z + (\tfrac{\overline{z}-m}{r})\partial_{\overline{z}}\Big)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.\end{aligned}$$

Next, taking into account that for $z\in\Omega$, $\zeta\in\partial\Omega$,

$$\begin{aligned}\Big((\tfrac{z-m}{r})\partial_z + (\tfrac{\overline{z}-m}{r})\partial_{\overline{z}}\Big)N_1(z,\zeta) &= \tfrac{2}{r}\Big[\tfrac{(z-m)}{\zeta-z}+\tfrac{(z-m)\zeta}{1-z\zeta}+\tfrac{(z-m)(m-\zeta)}{(1-mz)-\zeta(m-z)}\\ &+\tfrac{(z-m)(1-m\zeta)}{(m-z)-\zeta(1-mz)}+\tfrac{(\overline{z}-m)}{\overline{\zeta}-\overline{z}}+\tfrac{(\overline{z}-m)\overline{\zeta}}{1-\overline{z}\overline{\zeta}}+\tfrac{(\overline{z}-m)(m-\overline{\zeta})}{(1-m\overline{z})-\overline{\zeta}(m-\overline{z})}+\tfrac{(\overline{z}-m)(1-m\overline{\zeta})}{(m-\overline{z})-\overline{\zeta}(1-m\overline{z})}\Big],\end{aligned}$$

for $\zeta_0\in\partial\Omega_{D_m}\setminus\{m-r,\frac{1}{m}+i\frac{r}{m}\}$, we can write

$$\begin{aligned}\partial_{\nu_z}\omega(\zeta_0) &= \lim_{z\to\zeta_0}\big((\tfrac{z-m}{r})\partial_z + (\tfrac{\overline{z}-m}{r})\partial_{\overline{z}}\big)\omega(z)\\ &= \lim_{z\to\zeta_0}\Big\{\tfrac{1}{2\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\big[\tfrac{\zeta-m}{\zeta-z}+\tfrac{\overline{\zeta}-m}{\overline{\zeta}-\overline{z}}-1\big]\tfrac{\mathrm{d}\zeta}{\zeta-m}-\frac{2}{r\pi}\int_{m-r}^{1}\gamma(t)\mathrm{d}t\\ &\quad-\frac{2}{r\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta}-\frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta-m}+\frac{8}{\pi r}\int_{\Omega}f(\zeta)\mathrm{d}\xi\mathrm{d}\eta\Big\}\\ &= \gamma(\zeta_0)-\frac{2}{r\pi}\int_{m-r}^{1}\gamma(t)\mathrm{d}t-\frac{2}{r\pi i}\int_{\partial\Omega_{\mathbb{D}}}\gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\ &\quad-\frac{2}{\pi i}\int_{\partial\Omega_{D_m}}\gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta-m}+\frac{8}{\pi r}\int_{\Omega}f(\zeta)\mathrm{d}\xi\mathrm{d}\eta,\end{aligned}\tag{2.13}$$

which implies the necessity and sufficiency of condition (2.7).

Furthermore, we have

$$\partial_{\nu_z}^{-}\omega(m-r) = \lim_{\substack{\zeta_0\to m-r\\ \zeta_0\in\partial\Omega_{D_m}\setminus\{m-r,\frac{1}{m}+i\frac{r}{m}\}}} \partial_{\nu_z}\omega(\zeta) = \gamma(m-r), \tag{2.14}$$

$$\partial_{\nu_z}^{+}\omega(\tfrac{1}{m}+i\tfrac{r}{m}) = \lim_{\substack{\zeta_0\to\frac{1}{m}+i\frac{r}{m}\\ \zeta_0\in\partial\Omega_{D_m}\setminus\{m-r,\frac{1}{m}+i\frac{r}{m}\}}} \partial_{\nu_z}\omega(\zeta) = \gamma(\tfrac{1}{m}+i\tfrac{r}{m}). \tag{2.15}$$

Hence, in view of (2.9)-(2.15) and Definition 2.1, we conclude that

$$\partial_{\nu_z}\omega(\zeta)=\gamma(\zeta),\quad \zeta\in\{m-r,1,\tfrac{1}{m}+i\tfrac{r}{m}\}.$$

This completes the proof of Theorem 2.4. □

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