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RESULTS ON UNIQUENESS OF ENTIRE FUNCTIONS WHOSE DIFFERENCE POLYNOMIALS SHARE A POLYNOMIAL

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Abstract. In this paper, we use the concept of weighted sharing of values to investigate the uniqueness results when two difference polynomials of entire functions share a nonzero polynomial or a small function with a finite weight. We also investigate the situation when the original functions share the value 0 CM (counting multiplicities). The obtained results improve some recent related results of X. Li et al. [Ann. Polon. Math, 102 (2011), 111-127] and that of W. Li et al. [Bull. Malay. Math. Sci. Soc., 39 (2016), 499 - 515].

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1. Introduction. Definitions and results

In this paper, a meromorphic function means meromorphic in the complex plane. We adopt the standard notation of Nevanlinna's value distribution theory of meromorphic functions as presented in [9], [12] and [23]. By letter E we denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic function of h and by $S(r,h)$ any quantity satisfying the relation $S(r,h) = o\{T(r,h)\}$, $r \to \infty$, $r \notin E$.

Let f and g be two nonconstant meromorphic functions and let $a \in \mathbb{C} \cup \{\infty\}$. If zeros of $f-a$ and $g-a$ coincide in location and multiplicity, then we say that f and g share the value a CM (counting multiplicities). On the other hand, if zeros of $f-a$ and $g-a$ coincide only in their location, then we say that f and g share the value a IM (ignoring multiplicities). A meromorphic function α is called a small function with respect to f if $T(r,\alpha) = S(r,f)$. Throughout the paper, we denote by $\rho(f)$ the order of f (see [9], [12], [23]). We define the difference operators $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $\Delta_\eta^n f(z) = \Delta_\eta^{n-1}(\Delta_\eta f(z))$, where η is a nonzero complex number and $n \geq 2$ is an integer. In the special case where $\eta = 1$, we use the usual difference notation $\Delta_\eta f(z) = \Delta f(z)$.

A number of papers has been devoted to the uniqueness of entire and meromorphic functions whose differential polynomials share certain values or fixed points (see [5], [6], [16], [19], [20], [22], and references therein). Recently the value distribution in difference analogue has become a subject of great interest among the researchers. For instance, Halburd and Korhonen [7] established a version of Nevanlinna theory based on difference operators. The difference logarithmic derivative lemma, given by Halburd and Korhonen [8] in 2006, and by Chiang and Feng [4] in 2008, plays an important role in the study of difference analogues of Nevanlinna theory. With the development of difference analogue of Nevanlinna theory, the researchers concentrated their attention to the distribution of zeros of different types of difference polynomials and obtained the corresponding uniqueness results.

Theorem A. *(see [13]) Let f be a transcendental entire function of finite order, and let $\eta \neq 0$ be any complex constant. Then for $n \geq 2$ the function $f^n(z)f(z+\eta)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Example 1.1 ([13]). Let $f(z) = 1 + e^z$. Then the function $f(z)f(z+\pi i) - 1 = -e^{2z}$ has no zeros, showing that Theorem A does not hold if $n = 1$.

Example 1.2 ([17]). Let $f(z) = e^{-e^z}$. Then $f^2(z)f(z+\eta) - 2 = -1$ and $\rho(f) = \infty$, where η is the solution of equation $e^\eta = -2$. Evidently, the function $f^2(z)f(z+\eta) - 2$ has no zeros, showing that Theorem A does not hold if f is of infinite order.

Theorem B. *(see [18]) Let f and g be two transcendental entire functions of finite order. Let $\eta \neq 0$ be a complex constant and let $n \geq 6$ be an integer. If $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

Theorem C. *(see [17]) Let f be a transcendental entire function of finite order, and let η be a nonzero complex constant. Then for $n \geq 2$ the function $f^n(z)f(z+\eta) - P_0(z)$ has infinitely many zeros, where $P_0(z) \not\equiv 0$ is any polynomial.*

Example 1.3 ([17]). *Let $f(z) = e^{-e^z}$. Then $f^n(z)f(z+\eta) - P_0(z) = 1 - P_0(z)$ and $\rho(f) = \infty$, where η is a nonzero constant satisfying $e^\eta = -n$, $P_0(z)$ is a nonconstant polynomial, and n is a positive integer. Evidently, the function $f^n(z)f(z+\eta) - P_0(z)$ has finitely many zeros, showing that the condition $\rho(f) < \infty$ in Theorem C is necessary.*

Now the following question arises naturally.

Question 1. *Is there any uniqueness result corresponding to Theorem C ?*

Theorem D. *(see [15]) Let f and g be two distinct transcendental entire functions of finite order, and let $P_0 \not\equiv 0$ be a polynomial. Suppose that η is a nonzero complex constant and $n \geq 4$ is an integer such that $2deg(P_0) < n+1$. Also, suppose that $f^n(z)f(z+\eta) - P_0(z)$ and $g^n(z)g(z+\eta) - P_0(z)$ share 0 CM. Then the following assertions hold.*
(I) If $n \geq 4$ and $f^n(z)f(z+\eta)/P_0(z)$ is a Mobius transformation of $g^n(z)g(z+\eta)/P_0(z)$, then either
(i) $f = tg$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$, or
(ii) $f = e^Q$ and $g = te^{-Q}$, where P_0 reduces to a nonzero constant c, t is a constant such that $t^{n+1} = c^2$, and Q is a nonconstant polynomial.
(II) If $n \geq 6$, then I(i) or I(ii) holds.

Theorem E. *(see [15]) Let f and g be two transcendental entire functions of finite order, and let α ($\not\equiv 0, \infty$) be a meromorphic function such that $\rho(\alpha) < \rho(f)$. Suppose that η is a nonzero complex number, and n and m are positive integers satisfying $n \geq m+6$. If $f^n(z)(f^m(z)-1)f(z+\eta)$ and $g^n(z)(g^m(z)-1)g(z+\eta)$ share $\alpha(z)$ CM, then $f = tg$, where t is a constant such that $t^m = 1$.*

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \dots + a_0$ be a nonzero polynomial, where $a_n \neq 0$, $a_{n-1}, \dots, a_0$ are complex constants. Define $\Gamma_1 := m_1 + m_2$ and $\Gamma_2 := m_1 + 2m_2$, where m_1 is the number of simple zeros of P and m_2 is the number of multiple zeros of P. Throughout the paper we use the notation $d = \gcd(\lambda_0, \lambda_1, \dots, \lambda_n)$, where $\lambda_i = n+1$ if $a_i = 0$ and $\lambda_i = i+1$ if $a_i \neq 0$.

Theorem F. *(see [21]) Let f be a transcendental entire function of finite order and η be a fixed nonzero complex constant. Then for $n > m$ the function $P(f(z))f(z+\eta) - \alpha(z) = 0$ has infinitely many solutions, where $\alpha \not\equiv 0$ is a small function with respect to f, and m is the number of distinct zeros of P.*

Theorem G. *(see [21]) Let f and g be two transcendental entire functions of finite order, η be a nonzero complex constant, and $n > 2\Gamma_2+1$ be an integer. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share 1 CM, then one of the following cases hold:*
(i) $f = tg$, where $t^d = 1$;
(ii) f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(z+\eta) - P(w_2)w_2(z+\eta)$;

(iii) $f = e^{\alpha}$, $g = e^{\beta}$, *where* α *and* β *are two polynomials and* $\alpha + \beta = c$, *and* c *is a constant satisfying* $a_n^2 e^{(n+1)c} = 1$.

Example 1.4. *(see [21])* Let $P(z) = (z-1)^6(z+1)^6 z^{11}$, $f(z) = \sin z$, $g(z) = \cos z$ and $\eta = 2\pi$. It is easy to see that $n > 2\Gamma_2 + 1$ and $P(f(z))f(z+\eta) = P(g(z))g(z+\eta)$. Therefore $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share 1 CM. It is also clear that though f and g satisfy $R(f, g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(z+\eta) - P(w_2)w_2(z+\eta)$, we have $f \not\equiv tg$ for a constant t satisfying $t^m = 1$, where $m \in Z^+$.

Note that the functions f and g in Example 1.4 do not share 0 CM, and the following question arises naturally.

Question 2. *What can be said about* f *and* g, *if* f *and* g *share* 0 *CM in Theorem G?*

Theorem H. *(see [14]) Let* f, g *be two transcendental entire functions of finite order such that* f *and* g *share* 0 *CM. Suppose that* $P_0 \not\equiv 0$ *is a polynomial,* η *is a nonzero complex constant, and* n *is an integer such that* $deg(P_0) < n+1$. *Assume that* $P(f(z))f(z+\eta) - P_0$ *and* $P(g(z))g(z+\eta) - P_0$ *share* 0 *CM. If* $n > 2\Gamma_1 + 1$ *and* $P(f(z))f(z+\eta)$ *is a Mobius transformation of* $P(g(z))g(z+\eta)$, *or if* $n > 2\Gamma_2 + 1$, *then one of the following two cases hold:*
(i) $f = tg$, *where* $t^d = 1$;
(ii) $f = e^{\alpha}$, $g = te^{-\alpha}$, *where* P_0 *reduces to a nonzero constant* c, t *is a constant such that* $t^{n+1} = c^2$, *and* α *is a nonconstant polynomial.*

Regarding Theorems D, E and H, it is natural to ask the following question which is the motivation of the present paper.

Question 3. *Is it possible in some way to relax the nature of sharing in Theorems D, E and H?*

In this paper, our aim is to find out the possible answer to Question 3. We will prove three theorems which improves Theorems D, E and H by relaxing the nature of sharing. To state the main results, we need the following definition of weighted sharing which measures how close a shared value is to being shared CM or to being shared IM.

Definition 1.1 ([10]). *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f *where an* a*-point of multiplicity* m

is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f and g share the value a with weight k.

This definition implies that if f and g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m \leq k$ if and only if it is an a-point of g with multiplicity $m \leq k$, and z_0 is an a-point of f with multiplicity $m > k$ if and only if it is an a-point of g with multiplicity $n > k$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f and g share the value a with weight k. It is clear that if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also, note that f, g share the value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Remark 1.1. Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or infinity. Let $\overline{N}_{k)}^{E}(r, a; f, g)$ denote the reduced counting function of those a-points of f whose multiplicities are equal to that of the corresponding a-points of g, and both of their multiplicities are not greater than k. Also, let $\overline{N}_{(k}^{0}(r, a; f, g)$ denote the reduced counting function of those a-points of f which are a-points of g, and both of their multiplicities are not less than k. If

$$\overline{N}(r, a; f \mid\leq k) - \overline{N}_{k)}^{E}(r, a; f, g) = S(r, f),$$

$$\overline{N}(r, a; g \mid\leq k) - \overline{N}_{k)}^{E}(r, a; f, g) = S(r, g),$$

$$\overline{N}(r, a; f \mid\geq k+1) - \overline{N}_{(k+1}^{0}(r, a; f, g) = S(r, f),$$

$$\overline{N}(r, a; g \mid\geq k+1) - \overline{N}_{(k+1}^{0}(r, a; f, g) = S(r, g),$$

or if $k = 0$ and

$$\overline{N}(r, a; f) - \overline{N}_0(r, a; f, g) = S(r, f),$$

$$\overline{N}(r, a; g) - \overline{N}_0(r, a; f, g) = S(r, g),$$

then we say that f and g share "(a, k)".

Now we are ready to state our main results.

Theorem 1.1. *Let f and g be two distinct transcendental entire functions of finite order, and let P_0 ($\not\equiv 0$) be a polynomial. Suppose that η is a nonzero complex constant and $n \geq 4$ is an integer such that $2deg(P_0) < n+1$. Suppose that $f^n(z)f(z+\eta) - P_0(z)$ and $g^n(z)g(z+\eta) - P_0(z)$ share $(0, 2)$. If $n \geq 4$ and $f^n(z)f(z+\eta)/P_0(z)$ is a Mobius*

transformation of $g^n(z)g(z+\eta)/P_0(z)$, *or if* $n \geq 6$, *then one of the following two cases hold:*
(i) $f = tg$, *where* $t \neq 1$ *is a constant satisfying* $t^{n+1} = 1$;
(ii) $f = e^Q$ *and* $g = te^{-Q}$, *where* P_0 *reduces to a nonzero constant* c, t *is a constant such that* $t^{n+1} = c^2$, *and* Q *is a nonconstant polynomial.*

Theorem 1.2. *Let* f *and* g *be two transcendental entire functions of finite order, and let* $\alpha\,(\not\equiv 0, \infty)$ *be a meromorphic function such that* $\rho(\alpha) < \rho(f)$. *Suppose that* η *is a nonzero complex number, and* n *and* m *are positive integers such that* $n \geq m+6$. *If* $f^n(z)(f^m(z)-1)f(z+\eta)$ *and* $g^n(z)(g^m(z)-1)g(z+\eta)$ *share* $(\alpha, 2)$, *then* $f = tg$, *where* t *is a constant satisfying* $t^m = 1$.

Theorem 1.3. *Let* f *and* g *be two transcendental entire functions of finite order such that* f *and* g *share* 0 *CM, and let* $P_0(\not\equiv 0)$ *be a polynomial. Suppose that* η *is a nonzero complex constant and* n *is an integer such that* $deg(P_0) < n+1$. *Assume that* $P(f(z))f(z+\eta) - P_0$ *and* $P(g(z))g(z+\eta) - P_0$ *share* $(0, 2)$. *If* $n > 2\Gamma_1 + 1$ *and* $P(f(z))f(z+\eta)/P_0(z)$ *is a Mobius transformation of* $P(g(z))g(z+\eta)/P_0(z)$, *or if* $n > 2\Gamma_2 + 1$, *then one of the following two cases hold:*
(i) $f = tg$, *where* $t^d = 1$;
(ii) $f = e^\beta$, $g = te^{-\beta}$, *where* P_0 *reduces to a nonzero constant* c, t *is a constant such that* $t^{n+1} = c^2$, *and* β *is a nonconstant polynomial.*

Definition 1.2 ([11]). *For* $a \in \mathbb{C} \cup \{\infty\}$ *we define* $N(r, a; f \mid= 1)$ *to be the counting function of simple* a*-points of* f. *For a positive integer* p *we define* $N(r, a; f \mid\leq p)$ *to be the counting function of those* a*-points of* f *(counted with proper multiplicities) whose multiplicities are not greater than* p. *By* $\overline{N}(r, a; f \mid\leq p)$ *we denote the corresponding reduced counting function. In an analogous manner we define the functions* $N(r, a; f \mid\geq p)$ *and* $\overline{N}(r, a; f \mid\geq p)$.

Definition 1.3 ([10]). *Let* p *be a positive integer or infinity. We define* $N_p(r, a; f)$ *to be the counting function of* a*-points of* f, *where each* a*-point of multiplicity* m *is counted* m *times if* $m \leq p$ *and* p *times if* $m > p$. *Then define*

$$N_p(r, a; f) := \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq p).$$

Clearly, $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.4 ([1]). *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share* 1 *IM. Let* z_0 *be an* 1*-point of* f *and* g *with multiplicities* p *and* q, *respectively. Define* $\overline{N}_L(r,1;f)$ *to be the counting function of those 1-points of* f *and* g, *where* $p>q$, $N_E^{1)}(r,1;f)$ *to be the counting function of those 1-points of* f *and* g, *where* $p=q=1$, *and* $N_E^{(k}(r,1;f)$ $(k\geq 2$ *is an integer) to be the counting function of those 1-points of* f *and* g, *where* $p=q\geq k$, *and each point in these counting functions is counted only once. In the same manner we can define the functions* $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$ *and* $N_E^{(k}(r,1;g)$.

Definition 1.5 ([10]). *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share the value* a *IM. Define* $\overline{N}_*(r,a;f,g)$ *to be the reduced counting function of those* a*-points of* f *whose multiplicities differ from that of the corresponding* a*-points of* g. *Clearly,* $\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)$ *and* $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

2. Lemmas

In this section, we state some lemmas which will be needed in the sequel. We denote by H the following function:

$$H:=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right),$$

where F, G are nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 2.1 (see [23], Proof of Theorem 1.12). *Let* f *be a nonconstant meromorphic function in the complex plane, and let*

$$P(f)=a_nf^n(z)+a_{n-1}f^{n-1}(z)+\dots+a_1f(z)+a_0, \tag{2.1}$$

where a_0, a_1, ... , a_n *are constants and* $a_n\neq 0$. *Then* $m(r,P(f))=nm(r,f)+O(1)$.

Lemma 2.2 ([4]). *Let* f *be a meromorphic function of order* $\rho(f)<\infty$, *and let* $\eta\,(\neq 0)$ *be a complex number. Then for each* $\varepsilon>0$ *we have*

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta)}\right)=O\{r^{\rho(f)-1+\varepsilon}\}.$$

Lemma 2.3 ([4]). *Let* f *be a meromorphic function of order* $\rho(f)<\infty$, *and let* $\eta\,(\neq 0)$ *be a complex number. Then for each* $\varepsilon>0$ *we have*

$$T(r,f(z+\eta))=T(r,f(z))+O\{r^{\rho(f)-1+\varepsilon}\}+O\{\log r\}.$$

Lemma 2.4. *Let* f *be a transcendental entire function of order* $\rho(f)<\infty$, *and let* $\eta\,(\neq 0)$ *be a complex number. Suppose that* $F=P(f(z))f(z+\eta)$, *where* $P(f)$ *is as* *in (2.1). Then*

$$T(r,F)=(n+1)T(r,f)+O\{r^{\rho(f)-1+\varepsilon}\}+S(r,f).$$

Besides, we have $S(r,F)=S(r,f)$.

Proof. Noting that f is an entire function of finite order ρ, in view of Lemmas 2.1 and 2.2 and the standard Valiron-Mohon'ko theorem, we can write

$$\begin{aligned}&(n+1)T(r,f)=T(r,f(z)P(f(z)))+S(r,f)\\ =\ & m(r,f(z)P(f(z)))+S(r,f)\le m\left(r,\frac{f(z)P(f(z))}{F(z)}\right)+m(r,F(z))+S(r,f)\le\end{aligned}$$

$$(2.2)\quad m\left(r,\frac{f(z)}{f(z+\eta)}\right)+m(r,F(z))+S(r,f)\le T(r,F)+O\{r^{\rho(f)-1+\varepsilon}\}+S(r,f).$$

On the other hand, by Lemmas 2.1 and 2.3 and the fact that f is a transcendental entire function of finite order, we obtain

$$\begin{aligned}T(r,F)&\le T(r,P(f(z)))+T(r,f(z+\eta))+S(r,f)\\ &= nT(r,f)+T(r,f(z+\eta))+S(r,f)\\ &\le (n+1)T(r,f)+O\{r^{\rho(f)-1+\varepsilon}\}+S(r,f).\qquad (2.3)\end{aligned}$$

Now the result follows from (2.2) and (2.3). □

Lemma 2.5 ([14]). *Let f and g be two transcendental entire functions of finite order, $\eta\,(\neq 0)$ be a complex constant, $\alpha(z)$ be a small function of f and g, $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$ be a nonzero polynomial, where $a_0,\ a_1,\ \ldots,\ a_n\,(\neq 0)$ are complex constants, and let $n>\Gamma_1$ be an integer. If $P(f)f(z+\eta)$ and $P(g)g(z+\eta)$ share $\alpha(z)$ IM, then $\rho(f)=\rho(g)$.*

Lemma 2.6 (see [23], Lemma 7.1). *Let F and G be nonconstant meromorphic functions such that G is a Mobius transformation of F. Suppose that there exists a subset $I\subset R^+$ with linear measure $mesI=+\infty$ such that for $r\in I$ and $r\to\infty$*

$$\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)<(\lambda+o(1))T(r,F),$$

where $\lambda<1$. If there exists a point $z_0\in\mathbb{C}$ satisfying $F(z_0)=G(z_0)=1$, then either $F=G$ or $FG=1$.

Lemma 2.7 ([2]). *Let F and G be two nonconstant meromorphic functions sharing $(1,2)$, $(\infty,0)$ and $H\not\equiv 0$. Then the following assertions hold.*
(i) $T(r,F)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)-m(r,1;G)-N_E^{(3}(r,1;F)-\overline{N}_L(r,1;G)+S(r,F)+S(r,G);$

(ii) $T(r,G) \le N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) - m(r,1;F) - N_E^{(3}(r,1;G) - \overline{N}_L(r,1;F) + S(r,F) + S(r,G)$.

Lemma 2.8 ([24]). *Let F and G be two nonconstant meromorphic functions, and let $H \equiv 0$. If*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G)}{T(r)} < 1,$$

where $T(r) = \max\{T(r,F), T(r,G)\}$, $r \in I$ and I is a set with infinite linear measure, then either $F \equiv G$ or $FG \equiv 1$.

Lemma 2.9 ([3]). *Let f and g be two transcendental entire functions of finite order, and let $\eta\,(\neq 0)$ be a complex constant. Let n and m be positive integers, such that $n \ge m+5$ and*

$$f^n(z)(f^m(z)-1)f(z+\eta) \equiv g^n(z)(g^m(z)-1)g(z+\eta).$$

Then $f(z) \equiv tg(z)$, where t is a constant satisfying $t^m = 1$.

Though the authors of [3] claimed that the result of Lemma 2.9 is true for $n \ge m+6$, from the proof it can easily be viewed that in fact it is true for $n \ge m+5$.

3. Proof of theorems

Proof of Theorem 1.2. Let $F(z) = \frac{f^n(z)(f^m(z)-1)f(z+\eta)}{\alpha(z)}$ and $G(z) = \frac{g^n(z)(g^m(z)-1)g(z+\eta)}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share $(1,2)$. Noting that $\rho(\alpha) < \rho(f)$, from Lemma 2.4 we see that

$$T(r,F) = (n+m+1)T(r,f) + O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(\alpha)+\varepsilon}\}, \tag{3.1}$$

$$T(r,G) = (n+m+1)T(r,g) + O\{r^{\rho(g)-1+\varepsilon}\} + O\{r^{\rho(\alpha)+\varepsilon}\}. \tag{3.2}$$

From (3.1) and (3.2) we get

$$\rho(F) \le \max\{\rho(f), \rho(\alpha)\}, \quad \rho(f) \le \max\{\rho(F), \rho(\alpha)\}, \tag{3.3}$$

$$\rho(G) \le \max\{\rho(g), \rho(\alpha)\}, \quad \rho(g) \le \max\{\rho(G), \rho(\alpha)\}. \tag{3.4}$$

Using (3.3) and the fact that $\rho(\alpha) < \rho(f)$ we obtain

$$\rho(F) = \rho(f). \tag{3.5}$$

Now, using Nevanlinna's second fundamental theorem, we can write

$$\begin{aligned} T(r,F) &\le \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+S(r,f)\\ &\le \overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+\eta))+\overline{N}(r,1;f^{m}(z))\\ &\quad +\overline{N}(r,1;G)+O\{r^{\rho(\alpha)+\varepsilon}\}+S(r,f)\\ &\le (m+2)T(r,f)+T(r,G)+O\{r^{\rho(f)-1+\varepsilon}\}\\ &\quad +O\{r^{\rho(\alpha)+\varepsilon}\}+S(r,f). \end{aligned} \tag{3.6}$$

Similarly, we get

$$\begin{aligned} T(r,G) &\le (m+2)T(r,g)+T(r,F)+O\{r^{\rho(g)-1+\varepsilon}\}\\ &\quad +O\{r^{\rho(\alpha)+\varepsilon}\}+S(r,f). \end{aligned} \tag{3.7}$$

From (3.1), (3.5), (3.6) and the condition $\rho(\alpha)<\rho(f)<\infty$ we see that

$$\rho(F)\le\rho(G), \tag{3.8}$$

and from (3.4), (3.5), (3.8) and the condition $\rho(\alpha)<\rho(f)<\infty$ we obtain

$$\rho(G)=\rho(g). \tag{3.9}$$

Also, from (3.2), (3.5), (3.7) - (3.9) and the condition $\rho(\alpha)<\rho(f)<\infty$ we get

$$\rho(G)\le\rho(F). \tag{3.10}$$

Combining (3.5) and (3.8)-(3.10), we obtain

$$\rho(f)=\rho(g)=\rho(F)=\rho(G). \tag{3.11}$$

Suppose that $H\not\equiv 0$. Then using Lemmas 2.3 and 2.7 we can write

$$\begin{aligned} T(r,F)+T(r,G) &\le 2N_2(r,0;F)+2N_2(r,0;G)+2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\ &\quad +2\overline{N}_*(r,\infty;F,G)+S(r,F)+S(r,G)\\ &\le 4\overline{N}(r,0;f)+4\overline{N}(r,0;g)+2N(r,1;f^m)+2N(r,1;g^m)\\ &\quad +2N(r,0;f(z+\eta))+2N(r,0;g(z+\eta))+S(r,f)+S(r,g)\\ &\le (2m+6)\{T(r,f)+T(r,g)\}+O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})\\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{3.12}$$

Therefore, from (3.1), (3.2) and (3.12) we obtain

$$(n-m-5)\{T(r,f)+T(r,g)\}\le O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g),$$

yielding a contradiction with the assumption that $n \geq m+6$. Thus we must have $H \equiv 0$. Taking into account that

$$\begin{aligned}
&\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
&\leq \ \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,1;f^m)+\overline{N}(r,1;g^m)\\
&\quad +\overline{N}(r,0;f(z+\eta))+\overline{N}(r,0;g(z+\eta))+S(r,f)+S(r,g)\\
&\leq \ (m+2)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\
&\leq \ \frac{2m+4}{n+m+1}T(r),
\end{aligned}$$

where $T(r)=\max\{T(r,F),T(r,G)\}$, by Lemma 2.8, we deduce that either $F\equiv G$ or $FG\equiv 1$. Let $FG\equiv 1$. Then we have

$$f^n(z)(f^m(z)-1)f(z+\eta)g^n(z)(g^m(z)-1)g(z+\eta)\equiv \alpha^2,$$

implying that

$$\begin{aligned}
f^n(z)(f(z)-1)(f^{m-1}(z)+f^{m-2}(z)+...+1)f(z+\eta)g^n(z)(g(z)-1)\\
(g^{m-1}(z)+g^{m-2}(z)+...+1)g(z+\eta)=\alpha^2.
\end{aligned}$$

Noting that f and g are transcendental entire functions of finite order, it is easily seen from the above equality that $\overline{N}(r,0;f)=S(r,f)$, $\overline{N}(r,1;f)=S(r,f)$ and $\overline{N}(r,\infty;f)=S(r,f)$ for $r\in I$ and $r\to\infty$, where $I\subset(0,+\infty)$ is a subset of infinite linear measure. Thus, we obtain

$$T(r,f)\leq\overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f) \ = \ S(r,f),$$

for $r\in I$ and $r\to\infty$, which is meaningless. Thus, we must have $F\equiv G$, and hence

$$f^n(z)(f^m(z)-1)f(z+\eta)\equiv g^n(z)(g^m(z)-1)g(z+\eta).$$

Therefore by Lemma 2.9, it immediately follows that $f(z)\equiv tg(z)$, where t is a constant satisfying $t^m=1$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let $F_1=\frac{P(f(z))f(z+\eta)}{P_0(z)}$ *and* $G_1=\frac{P(g(z))g(z+\eta)}{P_0(z)}$. *Then* F_1 *and* G_1 *are two transcendental meromorphic functions sharing* $(1,2)$. *From Lemma 2.4 we get*

$$T(r,F_1)=(n+1)T(r,f)+O\{r^{\rho(f)-1+\varepsilon}\}+O\{\log r\}, \tag{3.13}$$

$$T(r,G_1)=(n+1)T(r,g)+O\{r^{\rho(g)-1+\varepsilon}\}+O\{\log r\}. \tag{3.14}$$

Since f and g are of finite order, it follows from (3.13) and (3.14) that F_1 and G_1 are also of finite order. Moreover, from Lemma 2.5 we deduce that

$$\rho(f) = \rho(g) = \rho(F_1) = \rho(G_1).$$

We now discuss the following two cases separately.

Case 1. Suppose that F_1 is a Mobius transformation of G_1. Then using the standard Valiron-Mohon'ko lemma we obtain $T(r, P(f)f(z+\eta)) = T(r, P(g)g(z+\eta)) + O\{\log r\}$. Then from (3.13) and (3.14) and the fact that f and g are transcendental entire functions of finite order we deduce

$$\frac{T(r,f)}{T(r,g)} \to 1, \quad \frac{T(r,F_1)}{T(r,f)} \to n+1 \quad \textit{as} \quad r \to \infty \textit{ and } r \in I.$$

From Lemma 2.3 and the condition that f and g are transcendental entire functions we have

$$\begin{aligned}
\overline{N}(r,0;F_1(z)) + \overline{N}(r,\infty;F_1(z)) &\leq \overline{N}(r,0;P(f(z))) + \overline{N}(r,0;f(z+\eta)) + O\{\log r\} \\
&\leq \Gamma_1 T(r,f(z)) + T(r,f(z+\eta)) + O\{\log r\} \\
&\leq (\Gamma_1+1)T(r,f(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + O\{\log r\},
\end{aligned}$$

as $r \to \infty$ and $r \in I$. Similarly, we get

$$\overline{N}(r,0;G_1(z)) + \overline{N}(r,\infty;G_1(z)) \leq (\Gamma_1+1)T(r,g(z)) + O\{r^{\rho(g)-1+\varepsilon}\} + O\{\log r\},$$

as $r \to \infty$ and $r \in I$. Thus

$$\begin{aligned}
&\overline{N}(r,0;F_1(z)) + \overline{N}(r,\infty;F_1(z)) + \overline{N}(r,0;G_1(z)) + \overline{N}(r,\infty;G_1(z)) \\
(3.15) \quad \leq\ &\frac{2(\Gamma_1+1)}{n+1} T(r,F_1)(1+o(1)),
\end{aligned}$$

as $r \to \infty$ and $r \in I$. In view of Nevanlinna's second fundamental theorem, we obtain

$$\begin{aligned}
T(r,F_1(z)) &\leq \overline{N}(r,0;F_1(z)) + \overline{N}(r,\infty;F_1(z)) + \overline{N}(r,1;F_1(z)) + O\{\log r\} \\
&\leq \overline{N}(r,0;P(f(z))) + \overline{N}(r,0;f(z+\eta)) + \overline{N}(r,1;F_1(z)) + O\{\log r\} \\
&\leq (\Gamma_1+1)T(r,f(z)) + \overline{N}(r,1;F_1(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + O\{\log r\},
\end{aligned}$$

which together with (3.13) gives $(n-\Gamma_1)T(r,f) \leq \overline{N}(r,0;F_1(z)) + S(r,f)$, as $r \to \infty$ and $r \in I$. From this and the fact that F_1 and G_1 share $(1,2)$ we conclude that there exists a point $z_0 \in \mathbb{C}$ such that $F_1(z_0) = G_1(z_0) = 1$. Hence from (3.15), Lemma 2.6 and the condition $n > 2\Gamma_1 + 1$ we infer that either $F_1G_1 = 1$ or $F_1 = G_1$. Now the conclusion of the theorem in this case follows from the proof of Case 1.1 and Case 1.2 of Theorem 5 from [14].

Case 2. Now we assume that $n > 2\Gamma_2 + 1$ and $H \not\equiv 0$. Then using Lemmas 2.3 and 2.7 we can write

$$\begin{aligned} & T(r,F_1)+T(r,G_1) \le 2N_2(r,0;F_1)+2N_2(r,0;G_1)+2\overline{N}(r,\infty;F_1)+2\overline{N}(r,\infty;G_1) \\ + & \ 2\overline{N}_*(r,\infty;F_1,G_1)+S(r,F_1)+S(r,G_1) \\ \le & \ 2N_2(r,0;P(f))+2N_2(r,0;P(g))+2N(r,0;f(z+\eta))+2N(r,0;g(z+\eta))+O\{\log r\} \\ \le & \ 2(\Gamma_2+1)\{T(r,f)+T(r,g)\}+O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g), \end{aligned}$$

which together with (3.13) and (3.14) gives

$$(n-2\Gamma_2-1)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

yielding a contradiction with the fact that $n > 2\Gamma_2 + 1$. Thus we must have $H \equiv 0$. Since $n > 2\Gamma_2 + 1 \ge 2\Gamma_1 + 1$, we obtain

$$\begin{aligned} & \overline{N}(r,0;F_1)+\overline{N}(r,0;G_1)+\overline{N}(r,\infty;F_1)+\overline{N}(r,\infty;G_1) \\ \le & \ \overline{N}(r,0;P(f))+\overline{N}(r,0;P(g))+\overline{N}(r,0;f(z+\eta))+\overline{N}(r,0;g(z+\eta))+O\{\log r\} \\ \le & \ (\Gamma_1+1)\{T(r,f)+T(r,g)\}+O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})+O\{\log r\} \\ \le & \ \frac{\Gamma_1+1}{n+1}\{T(r,F_1)+T(r,G_1)\}+O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g) \\ \le & \ \frac{2(\Gamma_1+1)}{n+1}T(r)+o\{T(r)\}, \end{aligned}$$

where $T(r) = \max\{T(r,F_1), T(r,G_1)\}$. Therefore, in view of Lemma 2.8, we can conclude that either $F_1 \equiv G_1$ or $F_1G_1 \equiv 1$. Now the result follows from Case 1. This completes the proof of Theorem 1.3.

Proof of Theorem 1.1. Let $F_2 = \frac{f^n(z)f(z+\eta)}{P_0(z)}$ and $G_2 = \frac{g^n(z)g(z+\eta)}{P_0(z)}$. Then F_2 and G_2 are two transcendental meromorphic functions that share $(1,2)$. Applying arguments similar to those used in the proof of Theorem 1.3, we conclude that in both cases either $F_2G_2 = 1$ or $F_2 = G_2$. Then the conclusion of the theorem follows from the proof of Subcase 1.1 and Subcase 1.2 of Theorem 1 from [15]. Here we omit the details. □

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