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# THE CASAZZA-TREMAIN CONJECTURE IN THE INFINITE DIMENSIONAL HILBERT SPACES

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Abstract. In this paper, the proof of the Casazza-Tremain Conjecture in the infinite dimensional Hilbert spaces is given.

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## 1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space with inner product $< ., . >$. A sequence of vectors $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $A$ and $B$ $(0 < A \leq B < \infty)$, such that for all $f \in \mathcal{H}$ we have

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} | < f, f_k > |^2 \leq B\|f\|^2. \tag{1.1}$$

The constants $A$ and $B$ are called the lower and upper frame bounds, respectively. The second inequality of the frame condition (1.1) is also known as the Bessel condition for $\{f_k\}_{k=1}^{\infty}$. If $A = B$, then $\{f_k\}_{k=1}^{\infty}$ is called a tight frame, and if $A = B = 1$, then $\{f_k\}_{k=1}^{\infty}$ is called a normalized tight frame or Parseval frame. A sequence $\{f_k\}_{k=1}^{\infty}$ in Hilbert space $\mathcal{H}$ is called a frame sequence in $\mathcal{H}$ if it is a frame for $\overline{span}\{f_k\}_{k=1}^{\infty}$.

A bounded linear operator $T$ defined by

$$T : \ell_2(\mathbb{N}) \to \mathcal{H}, \quad T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$$

is called the synthesis operator of $\{f_k\}_{k=1}^{\infty}$. Also, a bounded linear operator $S$ defined by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{k=1}^{\infty} < f, f_k > f_k$$

is called the frame operator of $\{f_k\}_{k=1}^{\infty}$. It is easy to show that $S = TT^*$, where $T^*$ is the adjoint operator of $T$.

Two frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are called dual frames for $\mathcal{H}$ if

$$f = \sum_{k=1}^{\infty} < f, f_k > g_k = \sum_{k=1}^{\infty} < f, g_k > f_k, \quad \text{for any} \quad f \in \mathcal{H}.$$

The frame $\{\tilde{f}_k\}_{k=1}^{\infty}$ defined by $\tilde{f}_k = S^{-1} f_k$ is a dual frame of frame $\{f_k\}_{k=1}^{\infty}$, and is called the canonical dual frame of $\{f_k\}_{k=1}^{\infty}$.

A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ae_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $A \in B(\mathcal{H})$ is an invertible operator. Every Riesz basis for $\mathcal{H}$ is a frame for $\mathcal{H}$. A sequence $\{f_k\}_{k=1}^{\infty}$ in the Hilbert space $\mathcal{H}$ is called a Riesz basic sequence in $\mathcal{H}$ if it is a Riesz basis for the Hilbert space $\overline{span}\{f_k\}_{k=1}^{\infty}$. For more information concerning frames we refer to [2, 3, 11].

In 1959, R. Kadison and I. Singer [4] introduced the problem of extensions of pure states.

**Kadison-Singer Problem.** *Does every pure state on the (Abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on $\ell_2$ have a unique extension to a (pure) state on $B(\ell_2)$, the von Neumann algebra of all bounded linear operators on the Hilbert space $\ell_2$?*

Recall that a state of von Neumann algebra $\mathcal{A}$ is a linear functional $\Lambda$ on $\mathcal{A}$ with $\Lambda(I) = 1$ and $\Lambda(T) \geq 0$ for all positive operators $T \in \mathcal{A}$. A pure state of $\mathcal{A}$ is an extreme point of the family of states of $\mathcal{A}$.

For $a, b \in \mathbb{R}$, consider the translation and modulation operators on $L_2(\mathbb{R})$, which are defined as $(T_a g)(x) = g(x-a)$, $(E_b g)(x) = e^{2\pi i b x} g(x)$, $\forall x \in \mathbb{R}$, respectively. A Gabor frame is a frame for $L_2(\mathbb{R})$ of the form $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$, where $a, b > 0$ and $g \in L_2(\mathbb{R})$ is a fixed function.

Studying the Gabor frames, H. Fiechtinger noted that all examples of Gabor frames can be written as a finite union of Riesz basis sequence and so he suggested the following conjecture:

**Fiechtinger Conjecture.** *Every bounded frame can be written as a finite union of Riesz basic sequences.*

This conjecture is equivalent to the Kadison-Singer problem (see [7]), which has been solved recently by Marcus, Spielman and Srivastava [5]. Hence the Fiechtinger conjecture is now a useful theorem in the frame theory.

Another conjecture that is equivalent to the Kadison-Singer problem is the Bourgain-Tzafriri conjecture.

**Bourgain-Tzafriri Conjecture.** *There is a universal constant $\eta > 0$ so that for* *every $\theta > 1$ there is a natural number $r = r(\theta)$ satisfying: for any natural number $n$, if $T : \mathbb{C}^n \to \mathbb{C}^n$ is a linear operator with $\|T\| \leq \theta$ and $\|Te_i\| = 1$ for all $i = 1, 2, ..., n$, then there is a partition $\{I_j\}_{j=1}^r$ of $\{1, 2, ..., n\}$ so that for all $j = 1, 2, ..., r$ and all choices of scalars $\{c_i\}_{i\in I_j}$ we have*

$$\|\sum_{i\in I_j} c_i Te_i\|^2 \geq \eta \sum_{i\in I_j} |c_i|^2.$$

In this paper, the proof of two conjectures (the Weaver conjecture $KS_2$ and the Casazza-Tremain conjecture) in the infinite dimensional Hilbert spaces is given.

## 2. The conjecture $KS_r$

The famous Kadison-Singer problem in $C^*$ algebra, posed on 1959, has been solved recently by Marcus, Spielman and Srivastava [5]. In fact, in [5] was proved the Weaver conjecture $KS_2$ in the finite dimensional Hilbert space $\mathbb{C}^d$, $d \in \mathbb{N}$, which implies the Kadison-Singer problem. Observe that the Kadison-Singer problem is also equivalent to the Paving conjecture (see [1, 10]).

**Paving Conjecture**: *For any $\epsilon > 0$, there is $r = r(\epsilon) \in \mathbb{N}$ such that for every zero diagonal operator $T$ on $\mathbb{C}^n$, $n \in \mathbb{N}$, there exists a partition $\{I_j\}_{j=1}^r$ of $\{1, 2, ..., n\}$ such that $\|P_{I_j} T P_{I_j}\| \leq \epsilon \|T\|$ for all $j \in \{1, 2, ..., n\}$, where $P_{I_j}$ is the orthogonal projection onto $\{e_i\}_{i\in I_j}$ and $\{e_i\}_{i=1}^n$ is the canonical orthonormal basis for $\mathbb{C}^n$.*

N. Weaver [6] using the equivalence of the Kadison-Singer problem and the Paving conjecture suggested another equivalent Conjecture $KS_r$.

**Conjecture** $KS_r$. *Let $r \in \mathbb{N}$. There exist universal constants $\eta \geq 2$ and $\theta > 0$ such that the following holds. Let $f_1, f_2, ..., f_m \in \mathbb{C}^n$ with $\|f_k\| \leq 1$ and suppose*

$$\sum_{k=1}^{m} |\langle f, f_k\rangle|^2 \leq \eta \quad \text{for all unit vectors } f \in \mathbb{C}^n.$$

*Then there exists a partition $\{I_j\}_{j=1}^r$ of $\{1, 2, ..., n\}$ such that*

$$\sum_{k\in I_j} |\langle f, f_k\rangle|^2 \leq \eta - \theta \quad \text{for all unit vectors } f \in \mathbb{C}^n \text{ and all } j \in \{1, 2, ..., r\}.$$

Note that the constants $\eta$ and $\theta > 0$ must be independent of $n$ and $m$.

In fact, N. Weaver has proved the following theorem.

**Theorem 2.1** ([6]). *The Kadison-Singer problem has a positive solution if and only if Conjecture $KS_r$ is true for some $r \geq 2$.*

Also, N. Weaver has indicated modifications in Conjecture $KS_r$, which do not alter its truth-value.

**Theorem 2.2** ([6]). *If either or both of the following modifications is made to Conjecture $KS_r$, the resulting conjecture is equivalent to Conjecture $KS_r$:*
*a) require $\theta = 1$;*
*b) assume $\sum_{k=1}^{m} | \langle f, f_k \rangle |^2 = \eta$ for all unit vectors $f \in \mathbb{C}^n$ instead of $\sum_{k=1}^{m} | \langle f, f_k \rangle |^2 \leq \eta$.*

In [5], A. Marcus, D. Spielman and N. Srivastava using the mixed characteristic polynomials and the above theorem have established the following theorem, and hence proved the Kadison-Singer problem.

**Theorem 2.3** ([5]). *There exist universal constants $\eta \geq 2$ and $\theta > 0$ such that the following holds. Let $f_1, f_2, ..., f_m \in \mathbb{C}^n$ with $\|f_k\| \leq 1$ and suppose*

$$\sum_{k=1}^{m} | \langle f, f_k \rangle |^2 = \eta \quad \text{for all unit vectors } f \in \mathbb{C}^n.$$

*Then there exist a partition $I_1, I_2$ of $\{1, 2, ..., n\}$ such that*

$$\sum_{k \in I_j} | \langle f, f_k \rangle |^2 \leq \eta - \theta \quad \text{for all unit vectors } f \in \mathbb{C}^n \text{ and all } j = 1, 2.$$

It is easy to show that Theorem 2.3 remains true in the case where $\mathbb{C}^n$ is replaced by any finite dimensional Hilbert space. To prove the Weaver Conjecture $KS_2$ in the infinite dimensional Hilbert spaces we need the following lemma (see [9], Proposition 2.1).

**Lemma 2.1.** *Fix $r \in \mathbb{N}$ and assume for every natural number $n$ we have a partition $\{I_i^n\}_{i=1}^r$ of $\{1, 2, ..., n\}$. Then there are natural numbers $\{n_1 < n_2 < ...\}$ such that $I_i^{n_j} \subset I_i^{n_k}$ for all $k \geq j$ and $i \in \{1, 2, ..., r\}$. Also, if $I_i := \{j | j \in I_i^{n_j}\} = \{j_1 < j_2 < ...\}$, then $\{I_i\}_{i=1}^r$ is a partition of $\mathbb{N}$ and $\{j_1, j_2, ... j_m\} \subset I_i^{n_{j_m}}$ for all $m \in \mathbb{N}$.*

The proof of Weaver Conjecture $KS_2$ in the infinite dimensional Hilbert spaces is given in the next theorem.

**Theorem 2.4.** *There exist universal constants $\eta \geq 2$ and $\theta > 0$ such that the following holds. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in an infinite dimensional Hilbert space $\mathcal{H}$ with $\|f_k\| \leq 1$ and suppose*

$$\sum_{k=1}^{\infty} | \langle f, f_k \rangle |^2 = \eta \quad \text{for all unit vectors } f \in \mathcal{H}.$$

*Then there exists a partition $I_1, I_2$ of $\mathbb{N}$ such that*

$$\text{(2.1)} \qquad \sum_{k\in I_i} |\langle f, f_k\rangle|^2 \le \eta - \theta \text{ for all unit vectors } f \in \mathcal{H} i = 1, 2.$$

**Proof.** Let $\eta \geq 2$ and $\theta > 0$ be as in Theorem 2.3, and let (2.1) holds. We fix a unit vector $f \in \mathcal{H}$ and for any $n \in \mathbb{N}$ consider the space $\mathcal{H}_n := span\{f_1, f_2, ..., f_n, f\}$. Then for all $n \in \mathbb{N}$ and for all $g \in \mathcal{H}_n$ with $\|g\| = 1$, we have

$$\sum_{k=1}^{n} |\langle g, f_k\rangle|^2 \le \sum_{k=1}^{\infty} |\langle g, f_k\rangle|^2 = \eta.$$

By Theorems 2.2 and 2.3, for any $n \in \mathbb{N}$ there exists a partition $I_1^n, I_2^n$ of $\{1, 2, ..., n\}$ such that

$$\sum_{k\in I_i^n} |\langle g, f_k\rangle|^2 \le \eta - \theta \text{ for all unit vectors } g \in \mathcal{H}_n \text{ and } i \in \{1, 2\}.$$

Next, by Lemma 2.1, there exist natural numbers $n_1 < n_2 < ...$ such that $I_i^{n_j} \subset I_i^{n_k}$ for all $k \geq j$ and $i \in \{1, 2\}$. Also, if $I_i := \{j | j \in I_i^{n_j}\} = \{j_1 < j_2 < ...\}$, then $\{I_i\}_{i=1}^2$ is a partition of $\mathbb{N}$ and $\{j_1, j_2, ... j_m\} \subset I_i^{n_{j_m}}$ for all $m \in \mathbb{N}$.

Thus, for all $m \in \mathbb{N}$ we have

$$\sum_{k=1}^{m} |\langle f, f_{j_k}\rangle|^2 \le \sum_{k\in I_i^{n_{j_m}}} |\langle f, f_{j_k}\rangle|^2 \le \eta - \theta,$$

and hence we get (2.2). Theorem 2.4 is proved. □

P. Casazza and J. Tremain suggested the following conjecture (see [7, Conjecture 8.2]).

**Casazza-Tremain Conjecture.** *There exists an $\varepsilon > 0$ so that for large $K$, for all $n$ and all equal norm Parseval frames $\{f_i\}_{i=1}^{Kn}$ for $\ell_2^n$, there is a $J \subset \{1, 2, ..., Kn\}$ so that both $\{f_i\}_{i\in J}$ and $\{f_i\}_{i\in J^c}$ have lower frame bounds which are greater than $\varepsilon$.*

The proof of the Casazza-Tremain Conjecture in the finite dimensional Hilbert spaces can be found in [8]. The following theorem contains a proof of the Casazza-Tremain Conjecture in the infinite dimensional Hilbert spaces.

**Theorem 2.5.** *Let the constants $\eta$ and $\theta$ be as in Theorem 2.4. Then every $\eta$-tight frame in the infinite dimensional Hilbert space $\mathcal{H}$ with normalized elements can be partitioned into two frames with frame bounds $\theta$ and $\eta$.*

**Proof.** Let $\{f_k\}_{k=1}^{\infty}$ be a $\eta$-tight frame in the infinite dimensional Hilbert space $\mathcal{H}$ with $\|f_k\| \le 1$. Then by Theorem 2.4, there exists a partition $\{I_i\}_{i=1}^2$ of $\mathbb{N}$ such that

(2.2) holds, and

$$\eta = \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 = \sum_{k\in I_1} |\langle f, f_k\rangle|^2 + \sum_{k\in I_2} |\langle f, f_k\rangle|^2 \leq \sum_{k\in I_1} |\langle f, f_k\rangle|^2 + \eta - \theta,$$

and hence $\sum_{k\in I_1} |\langle f, f_k\rangle|^2 \geq \theta$. Therefore

$$\theta \leq \sum_{k\in I_1} |\langle f, f_k\rangle|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 = \eta.$$

Similar arguments can be used to show that

$$\theta \leq \sum_{k\in I_2} |\langle f, f_k\rangle|^2 \leq \eta.$$

Theorem 2.5 is proved. □

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