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FURTHER RESULTS ON UNIQUENESS OF DERIVATIVES OF MEROMORPHIC FUNCTIONS SHARING THREE SETS

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Abstract. [1] In this paper we prove some uniqueness theorems concerning the derivatives of meromorphic functions when they share three sets. The obtained results improve some recent existing results.

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1. Introduction, definitions and results

In this paper by meromorphic functions we always will mean meromorphic functions in the complex plane, and will use the standard notation of value distribution theory (see [7]):

$$T(r, f), \;\; m(r, f), \;\; N(r, \infty; f), \;\; \overline{N}(r, \infty; f), \ldots .$$

By letter E we will denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. We denote by $T(r)$ the maximum of $T(r, f^{(k)})$ and $T(r, g^{(k)})$, and by $S(r)$ any quantity satisfying the relation $S(r) = o(T(r))$ as $r \longrightarrow \infty, r \notin E$.

If for some $a \in \mathbb{C} \cup \{\infty\}$, the functions f and g have the same sets of a-points with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). If the multiplicities are not taken into account, then we say that f and g share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$. Denote $E_f(S) := \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count

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the multiplicities, then the set $\bigcup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$, then we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, then we say that f and g share the set S IM. Evidently, if S contains only one element, these definitions coincide with the usual definitions of CM and IM shared values, respectively.

In 1926, R. Nevanlinna showed that a meromorphic function on the complex plane $\mathbb{C}$ is uniquely determined by the pre-images, ignoring multiplicities, of 5 distinct values (including infinity). A few years latter, he showed that when multiplicities are taken into consideration, then 4 points are enough. More precisely, Nevanlinna proved that if two meromorphic functions share four distinct values CM, then either they coincide or one of them is the bilinear transformation of the other.

These two results are the starting point of uniqueness theory. The research became more interesting, although sophisticated, when F. Gross and C. C. Yang transferred the study of uniqueness theory to a more general setting, namely considering sets of distinct elements instead of values. For instance, they proved that if f and g are two non-constant entire functions and S_1, S_2 and S_3 are three distinct finite sets such that $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2, 3$, then $f \equiv g$.

The following question was asked in [19].

Question A. *Can one find three finite sets S_j $(j = 1, 2, 3)$ such that any two non-constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2, 3$ must be identical ?*

Question A may be considered as an inception of a new horizon in the uniqueness of meromorphic functions concerning three set sharing problem and so far the quest for affirmative answer to Question A under weaker hypothesis has made a great stride (see, e.g., [1], [2], [5] – [7], [14], [16], [18] – [21], [22]).

Unfortunately the derivative counterparts of the results obtained in the above cited papers are scanty in number. In 2003, in the direction of Question A concerning the uniqueness of derivatives of meromorphic functions, Qiu and Fang obtained the following result.

Theorem A ([18]). *Let $S_1 = \{z : z^n - z^{n-1} - 1 = 0\}$, $S_2 = \{\infty\}$ and $S_3 = \{0\}$, and let $n \geq 3$ and $k > 0$ be integers. Let f and g be two non-constant meromorphic functions such that $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 3$ and $E_f(S_2) = E_g(S_2)$, then $f^{(k)} \equiv g^{(k)}$.*

In 2004, Yi and Lin [22] proved the following theorem.

Theorem B ([22]). *Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}$, $S_2 = \{\infty\}$ and $S_3 = \{0\}$, where a and b are nonzero constants such that $z^n+az^{n-1}+b=0$ has no repeated roots, and let $n \geq 3$ and $k > 0$ be integers. Let f and g be two non-constant meromorphic functions such that $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2, 3$, then $f^{(k)} \equiv g^{(k)}$.*

The following examples show that in Theorems A and B the condition $a \neq 0$ is necessary.

Example 1.1 ([4]). *Let $f(z) = e^z$ and $g(z) = (-1)^k e^{-z}$, and let $S_1 = \{z : z^3-1 = 0\}$, $S_2 = \{\infty\}$, $S_3 = \{0\}$. Since $f^{(k)} - \omega^l = g^{(k)} - \omega^{3-l}$, where $\omega = cos\frac{2\pi}{3} + isin\frac{2\pi}{3}$, $0 \leq l \leq 2$, clearly we have $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2, 3$, while $f^{(k)} \not\equiv g^{(k)}$.*

The following examples establish the sharpness of the lower bound of n in Theorems A and B.

Example 1.2 ([4]). *Let $f(z) = \sqrt{\alpha+\beta}\,\sqrt{\alpha\beta}\,e^z$ and $g(z) = (-1)^k\sqrt{\alpha+\beta}\,\sqrt{\alpha\beta}\,e^{-z}$, and let $S_1 = \{\alpha+\beta, \alpha\beta\}$, $S_2 = \{\infty\}$, $S_3 = \{0\}$ with $\alpha+\beta = -a$ and $\alpha\beta = b$, where a, b are nonzero complex numbers. Clearly we have $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2, 3$, while $f^{(k)} \not\equiv g^{(k)}$.*

Example 1.3. *Let $f = \alpha\sqrt{\beta}e^z$ and $g = (-1)^k\beta\sqrt{\alpha}e^{-z}$, where α and β are two non zero complex numbers such that $\sqrt{\frac{\alpha}{\beta}} \neq -1$. Let $S_1 = \{\beta\sqrt{\alpha}, \alpha\sqrt{\beta}\}$, $S_2 = \{\infty\}$ and $S_3 = \{0\}$. Clearly we have $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2, 3$, while $f^{(k)} \not\equiv g^{(k)}$.*

Example 1.4. *Let $f = \sqrt{2}e^z$ and $g = (-1)^k\sqrt{2}e^{-z}$. Let $S_1 = \{1+i, 1-i\}$, $S_2 = \{\infty\}$ and $S_3 = \{0\}$. Clearly we have $E_{f^{(k)}}(S_j) = E_{g^{(k)}}(S_j)$ for $j = 1, 2, 3$, while $f^{(k)} \not\equiv g^{(k)}$.*

The above examples assure the fact that in Theorems A and B, the cardinality of the set S_1 cannot be further reduced. Rather, on the basis of these examples, one may try to relax the nature of sharing the range sets. For the purpose of relaxation of the nature of sharing the sets the notion of weighted sharing of values and sets, which appeared in [12, 13], has become an effective tool.

Definition 1.1 ([12, 13]). *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.*

It follows from Definition 1.1 that if f and g share a value a with weight k, then a point z_0 is an a-point of f with multiplicity $m \leq k$ if and only if it is an a-point of

g with multiplicity $m \le k$, and z_0 is an a-point of f with multiplicity $m > k$ if and only if it is an a-point of g with multiplicity $n > k$, where m is not necessarily equal to n.

We will write "f, g share (a, k)" to mean that f and g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \le p < k$. Also, we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Definition 1.2 ([12]). *Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$, and let k be a nonnegative integer or ∞. We define $E_f(S, k) := \bigcup_{a\in S} E_k(a; f)$. It is clear that $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.*

In 2009, Banerjee and Bhattacharjee [3] used the concept of weighted sharing of sets to improve Theorems A and B.

Theorem C ([3]). *Let S_i, $i = 1 - 3$ be as in Theorem B and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 4) = E_{g^{(k)}}(S_1, 4)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_{f^{(k)}}(S_3, 7) = E_{g^{(k)}}(S_3, 7)$, then $f^{(k)} \equiv g^{(k)}$.*

Theorem D ([3]). *Let S_i, $i = 1 - 3$ be as in Theorem B and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 5) = E_{g^{(k)}}(S_1, 5)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_{f^{(k)}}(S_3, 1) = E_{g^{(k)}}(S_3, 1)$, then $f^{(k)} \equiv g^{(k)}$.*

Theorem E ([3]). *Let S_i, $i = 1, 2, 3$ be as in Theorem B and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 6) = E_{g^{(k)}}(S_1, 6)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_{f^{(k)}}(S_3, 0) = E_{g^{(k)}}(S_3, 0)$, then $f^{(k)} \equiv g^{(k)}$.*

In 2011, Banerjee and Bhattacharjee [4] further improved the above results, by proving the following theorems.

Theorem F ([4]). *Let S_i, $i = 1 - 3$ be as in Theorem B and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1, 5) = E_{g^{(k)}}(S_1, 5)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$ and $E_{f^{(k)}}(S_3, 0) = E_{g^{(k)}}(S_3, 0)$, then $f^{(k)} \equiv g^{(k)}$.*

Theorem G ([4]). *Let S_i, $i=1,2,3$ be as in Theorem B and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1,4) = E_{g^{(k)}}(S_1,4)$, $E_f(S_2,\infty) = E_g(S_2,\infty)$ and $E_{f^{(k)}}(S_3,1) = E_{g^{(k)}}(S_3,1)$, then $f^{(k)} \equiv g^{(k)}$.*

Theorem H ([4]). *Let S_i, $i=1,2,3$ be as in Theorem B and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1,5) = E_{g^{(k)}}(S_1,5)$, $E_f(S_2,9) = E_g(S_2,9)$ and $E_{f^{(k)}}(S_3,\infty) = E_{g^{(k)}}(S_3,\infty)$, then $f^{(k)} \equiv g^{(k)}$.*

In the present paper we significantly reduce the weight of the range sets in all the above theorems. The following theorem is the main result of this paper.

Theorem 1.1. *Let S_i, $i=1,2,3$ be as in Theorem B and k be a positive integer. If f and g are two non-constant meromorphic functions such that $E_{f^{(k)}}(S_1,k_1) = E_{g^{(k)}}(S_1,k_1)$, $E_f(S_2,k_2) = E_g(S_2,k_2)$ and $E_{f^{(k)}}(S_3,k_3) = E_{g^{(k)}}(S_3,k_3)$, where $k_1 \geq 4$, $k_2 \geq 0$, $k_3 \geq 0$ are integers satisfying*

$$2k_1k_2k_3 > k_1 + k_2 + 2k_3 + k - 2kk_1k_3 - k_1k_2 - kk_1 + 3,$$

then $f^{(k)} \equiv g^{(k)}$.

Remark 1.1. Note that Theorem 1.1 holds for $k_1 = 4$, $k_2 = 2$ and $k_3 = 0$, and so it improves Theorems A-H.

Remark 1.2. Examples 1.2-1.4 assure the fact that in Theorem 1.1, $n \geq 3$ is the best possible.

Throughout the paper we will use the standard definitions and notation of the value distribution theory available in [10]. Below we recall some notation and definitions which are used in this paper.

Definition 1.3 ([11]). *For $a \in \mathbb{C} \cup \{\infty\}$ and a meromorphic function f, we denote by $N(r,a;f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f \mid\leq m)$ (resp. $N(r,a;f \mid\geq m)$) the counting function of those a-points of f whose multiplicities are not greater (resp. less) than m, where each a-point is counted according to its multiplicity. The functions $\overline{N}(r,a;f \mid\leq m)$ and $(\overline{N}(r,a;f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also, the functions $N(r,a;f \mid< m)$, $N(r,a;f \mid> m)$, $\overline{N}(r,a;f \mid< m)$ and $\overline{N}(r,a;f \mid> m)$ are defined analogously.*

Definition 1.4. *We denote by $\overline{N}(r,a;f\mid=k)$ the reduced counting function of those a-points of a function f whose multiplicities are exactly k, where $k\geq 2$ is an integer.*

Definition 1.5 ([2]). *Let f and g be two non-constant meromorphic functions such that f and g share (a,k), where $a\in\mathbb{C}\cup\{\infty\}$. Let z_0 be an a-point of f with multiplicity p, and an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g, for which $p>q$ and each a-point is counted only once. In the same way we can define the function $\overline{N}_L(r,a;g)$.*

Definition 1.6 ([13]). *We denote $N_2(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)$.*

Definition 1.7 ([12, 13]). *Let f and g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly, we have $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.*

Definition 1.8 ([15]). *Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.*

Definition 1.9 ([15]). *Let $a,b_1,b_2,\ldots,b_q\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid g\neq b_1,b_2,\ldots,b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i=1,2,\ldots,q$.*

Definition 1.10. *Let f and g be two non-constant meromorphic functions such that $E_f(S,k)=E_g(S,k)$, and let a and b be any two elements of S. We denote by $\overline{N}_*(r,a;f|g=b)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding b-points of g. Clearly, we have $\overline{N}_*(r,a;f|g=b)=\overline{N}_*(r,b;g|f=a)$. Also, if $a=b$, then $\overline{N}_*(r,a;f|g=b)=\overline{N}_*(r,a;f,g)$.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Let F and G be two non-constant meromorphic functions defined as follows:

$$F=\frac{\left(f^{(k)}\right)^{n-1}\left(f^{(k)}+a\right)}{-b},\qquad G=\frac{\left(g^{(k)}\right)^{n-1}\left(g^{(k)}+a\right)}{-b}, \tag{2.1}$$

where $n \geq 2$ and $k > 0$ are integers. Define the following functions:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

$$\Phi_1 = \frac{F'}{F-1} - \frac{G'}{G-1},$$

$$\Phi_2 = \frac{(f^{(k)})'}{f^{(k)}} - \frac{(g^{(k)})'}{g^{(k)}}$$

and

$$\Phi_3 = \left(\frac{(f^{(k)})'}{f^{(k)} - \omega_i} - \frac{(f^{(k)})'}{f^{(k)}}\right) - \left(\frac{(g^{(k)})'}{g^{(k)} - \omega_j} - \frac{(g^{(k)})'}{g^{(k)}}\right),$$

where ω_i and ω_j are any two roots of the equation $z^n + az^{n-1} + b = 0$.

Lemma 2.1 ([13], Lemma 1). *Let F, G share $(1,1)$ and $H \not\equiv 0$. Then*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *Let S_1, S_2 and S_3 be as in Theorem 1.1, and let F and G be given by (2.1). If for two non-constant meromorphic functions f and g we have $E_{f^{(k)}}(S_1,0) = E_{g^{(k)}}(S_1,0)$, $E_{f^{(k)}}(S_2,0) = E_{g^{(k)}}(S_2,0)$, $E_f(S_3,0) = E_g(S_3,0)$ and $H \not\equiv 0$, then*

$$\begin{aligned} N(r,H) \leq\ & \overline{N}_*(r,0;f^{(k)},g^{(k)}) + \overline{N}_*(r,1;F,G) + \overline{N}(r,-a\frac{n-1}{n};f^{(k)}) + \overline{N}(r,-a\frac{n-1}{n};g^{(k)}) \\ & + \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;(f^{(k)})') + \overline{N}_0(r,0;(g^{(k)})'), \end{aligned}$$

where $\overline{N}_0(r,0;(f^{(k)})')$ is the reduced counting function of those zeros of $(f^{(k)})'$ which are not the zeros of $f^{(k)}(F-1)$ and $\overline{N}_0(r,0;(g^{(k)})')$ is defined similarly.

Proof. Since $E_{f^{(k)}}(S_1,0) = E_{g^{(k)}}(S_1,0)$, it follows that F, G share $(1,0)$. From (2.1) we have

$$F' = [nf^{(k)} + (n-1)a](f^{(k)})^{n-2}(f^{(k)})'/(-b)$$

and

$$G' = [ng^{(k)} + (n-1)a](g^{(k)})^{n-2}(g^{(k)})'/(-b).$$

We can easily verify that the possible poles of H can occur at:

(i) those zeros of $f^{(k)}$ and $g^{(k)}$ whose multiplicities are different from the multiplicities of the corresponding zeros of $g^{(k)}$ and $f^{(k)}$, respectively,

(ii) the zeros of $nf^{(k)} + a(n-1)$ and $ng^{(k)} + a(n-1)$,

(iii) those poles of f and g whose multiplicities are different from the multiplicities of the corresponding poles of g and f, respectively,

(iv) the 1-points of F and G with different multiplicities,

(v) the zeros of $(f^{(k)})'$ which are not zeros of $f^{(k)}(F-1)$,

(vi) the zeros of $(g^{(k)})'$ which are not zeros of $g^{(k)}(G-1)$.

Lemma 2.2 is proved. □

Lemma 2.3 ([17]). *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.4 ([4]). *Let F and G be given by (2.1). If $f^{(k)}$, $g^{(k)}$ share $(0,0)$ and 0 is not a Picard exceptional value of $f^{(k)}$ and $g^{(k)}$, then $\Phi_1 \equiv 0$ implies $F \equiv G$.*

Lemma 2.5 ([4]). *Let F and G be given by (2.1), $n \geq 3$ be an integer and $\Phi_1 \not\equiv 0$. If F, G share $(1, k_1)$; f, g share (∞, k_2), and $f^{(k)}$, $g^{(k)}$ share $(0, k_3)$, where $0 \leq k_3 < \infty$, then*

$$[(n-1)k_3 + n - 2]\, \overline{N}(r, 0; f^{(k)} \mid\geq k_3 + 1) \leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + S(r).$$

Lemma 2.6. *Let f and g be two non-constant meromorphic functions, F and G be given by (2.1), $n \geq 3$ be an integer and $\Phi_2 \not\equiv 0$. If F, G share $(1, k_1)$; $f^{(k)}$, $g^{(k)}$ share $(0, k_3)$, and f, g share (∞, k_2), where $1 \leq k_1 \leq \infty$, then*

$$k_1 \overline{N}(r, 1; F \mid\geq k_1 + 1) \leq \overline{N}_*(r, 0; f^{(k)}, g^{(k)}) + \overline{N}_*(r, \infty; f, g) + S(r, f^{(k)}) + S(r, g^{(k)}).$$

Proof. Note that

$$k_1 \overline{N}(r, 1; F \mid\geq k_1 + 1) \leq N(r, 0; \Phi_2) \leq N(r, \Phi_2) + S(r, f^{(k)}) + S(r, g^{(k)}) \leq$$
$$\overline{N}_*(r, 0; f^{(k)}, g^{(k)}) + \overline{N}_*(r, \infty; f, g) + S(r, f^{(k)}) + S(r, g^{(k)}),$$

and the result follows. □

Lemma 2.7. *Let f and g be two non-constant meromorphic functions, F and G be given by (2.1), $n \geq 3$ be an integer, and let $\Phi_1 \not\equiv 0$, $\Phi_2 \not\equiv 0$. If F, G share $(1, k_1)$, where $k_1 \geq 2$; $f^{(k)}$, $g^{(k)}$ share $(0, k_3)$, and f, g share (∞, k_2), $0 \leq k_2 \leq \infty$, then*

$$\overline{N}(r, 0; f^{(k)} \mid\geq k_3 + 1) \leq \frac{k_1 + 1}{k_1[(n-1)k_3 + (n-2)] - 1} \overline{N}_*(r, \infty; f, g) + S(r, f^{(k)}) + S(r, g^{(k)}),$$

A similar result also holds for $g^{(k)}$.

Proof. Using Lemmas 2.5 and 2.6, and noting that $\overline{N}_*(r,0;f^{(k)},g^{(k)}) \leq \overline{N}(r,0;f^{(k)}| \geq k_3+1)$, we can write

$$\begin{aligned} & [(n-1)k_3+(n-2)]\overline{N}(r,0;f^{(k)}| \geq k_3+1) \\ \leq\ & \overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;f,g)+S(r,f^{(k)})+S(r,g^{(k)}) \\ \leq\ & \frac{1}{k_1}\overline{N}(r,0;f^{(k)}| \geq k_3+1)+\frac{k_1+1}{k_1}\overline{N}_*(r,\infty;f,g) \\ & +S(r,f^{(k)})+S(r,g^{(k)}), \end{aligned}$$

and the result follows. Lemma 2.7 is proved. □

Lemma 2.8. *Let f and g be two non-constant meromorphic functions. Suppose f, g share $(\infty,0)$ and ∞ is not a Picard exceptional value of f and g. Then $\Phi_3 \equiv 0$ implies $f^{(k)} \equiv \frac{\omega_i}{\omega_j}g^{(k)}$.*

Proof. Suppose $\Phi_3 \equiv 0$. Then by integration we obtain

$$1-\frac{\omega_i}{f^{(k)}} \equiv A(1-\frac{\omega_j}{g^{(k)}}),$$

where $A \neq 0$. Since f, g share $(\infty,0)$ it follows that $A=1$, and hence $f^{(k)} \equiv \frac{\omega_i}{\omega_j}g^{(k)}$.

Lemma 2.9. *Let f and g be two non-constant meromorphic functions and $\Phi_3 \not\equiv 0$, and let F and G be given by (2.1). If $f^{(k)}$, $g^{(k)}$ share $(0,k_3)$; f, g share (∞,k_2), where $1 \leq k_2 \leq \infty$, and $E_{f^{(k)}}(S_1,k_1)=E_{g^{(k)}}(S_1,k_1)$, where $1 \leq k_1 \leq \infty$ and the set S_1 is as in Theorem 1.1, then*

$$\begin{aligned} & (k_2+k)\ \overline{N}(r,\infty;f\ | \geq k_2+1) \\ \leq\ & \overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)+\overline{N}_*(r,1;F,G)+S(r). \end{aligned}$$

A similar result also holds for $g^{(k)}$.

Proof. If ∞ is an e.v.P. (Picard exceptional value) of $f^{(k)}$ and $g^{(k)}$, then the result follows immediately.
Next, suppose ∞ is not e.v.P. of $f^{(k)}$ and $g^{(k)}$. Since $E_{f^{(k)}}(S_1,k_1)=E_{g^{(k)}}(S_1,k_1)$, it

follows that $\overline{N}_*(r,\omega_i;f^{(k)}|g^{(k)}=\omega_j)\leq\overline{N}_*(r,1;F,G)$. Hence we can write

$$\begin{aligned}
&(k_2+k)\ \overline{N}(r,\infty;f|\geq k_2+1)\\
=\ &(k_2+k)\ \overline{N}(r,\infty;g|\geq k_2+1)\\
\leq\ &N(r,0;\Phi_3)\\
\leq\ &N(r,\Phi_3)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\ &\overline{N}_*(r,0;f^{(k)},g^{(k)})+\overline{N}_*(r,\omega_i;f^{(k)}|g^{(k)}=\omega_j)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\ &\overline{N}_*(r,0;f^{(k)},g^{(k)})+\overline{N}_*(r,1;F,G)+S(r),
\end{aligned}$$

and the result follows. Lemma 2.9 is proved. □

Lemma 2.10. *Let f and g be two non-constant meromorphic functions, $\Phi_2\not\equiv 0$, $\Phi_3\not\equiv 0$, and let F and G be given by (2.1). If $f^{(k)}$, $g^{(k)}$ share $(0,k_3)$; f, g share (∞,k_2), where $0\leq k_2<\infty$, and F, G share $(1,k_1)$, where $k_1>1$, then*

$$\overline{N}(r,\infty;f\ |\geq k_2+1)\frac{k_1+1}{k_1(k_2+k)-1}\overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)+S(r).$$

A similar result also holds for $g^{(k)}$ also.

Proof. Using Lemmas 2.6 and 2.9 and noting that $\overline{N}_*(r,\infty;f,g)\leq\overline{N}(r,\infty;f|\geq k_2+1)$, we can write

$$\begin{aligned}
(k_2+k)\overline{N}(r,\infty;f|\geq k_2+1)\ &\leq\ \overline{N}_*(r,1;F,G)+\overline{N}_*(r,0;f^{(k)},g^{(k)})+S(r)\\
&\leq\ \frac{1}{k_1}\overline{N}(r,\infty;f|\geq k_2+1)+\frac{k_1+1}{k_1}\overline{N}_*(r,0;f^{(k)},g^{(k)})\\
&\quad +S(r,f^{(k)})+S(r,g^{(k)}),
\end{aligned}$$

and the result follows. Lemma 2.10 is proved. □

Lemma 2.11 ([4]). *Let F and G be given by (2.1) and $H\not\equiv 0$. If $f^{(k)}$, $g^{(k)}$ share $(0,k_3)$; f, g share (∞,k_2), where $0\leq k_2<\infty$, and F, G share $(1,k_1)$, where $1\leq k_1\leq\infty$, then*

$$\begin{aligned}
&\{(nk_2+nk+n)-1\}\ \overline{N}(r,\infty;f\ |\geq k_2+1)\\
\leq\ &\overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)+\overline{N}\left(r,0;f^{(k)}+a\right)+\overline{N}\left(r,0;g^{(k)}+a\right)+\overline{N}_*(r,1;F,G)+S(r).
\end{aligned}$$

A similar result also holds for g.

Lemma 2.12. *Let f and g be two non-constant meromorphic functions. Also, let F and G be given by (2.1), $n\geq 3$ be an integer, and $\Phi_1\not\equiv 0$, $\Phi_2\not\equiv 0$ and $\Phi_3\not\equiv 0$. If F,*

G share $(1,k_1)$; $f^{(k)}$, $g^{(k)}$ share $(0,k_3)$, and f, g share (∞,k_2), where $k_1>1$, $k_2\geq 0$ and $k_3\geq 0$ are integers satisfying

$$2k_1k_2k_3>k_1+k_2+2k_3+k-2kk_1k_3-k_1k_2-kk_1+3,$$

then

$$\overline{N}(r,1;F|\geq k_1+1)+\overline{N}(r,\infty;f|\geq k_2+1)+\overline{N}(r,0;f^{(k)}|\geq k_3+1)=S(r).$$

Proof. Since $\Phi_1\not\equiv 0$ from Lemma 2.5 we get

$$(2k_3+1)\,\overline{N}(r,0;f^{(k)}|\geq k_3+1)\quad\leq\quad\overline{N}(r,1;F\,|\geq k_1+1)+\overline{N}(r,\infty;f|\geq k_2+1)+S(r).$$

Next, since $\Phi_2\not\equiv 0$ and $\Phi_3\not\equiv 0$, we can apply Lemmas 2.6 and 2.9 to obtain

$$k_1\,\overline{N}(r,1;F|\geq k_1+1)\quad\leq\quad\overline{N}(r,0;f^{(k)}\,|\geq k_3+1)+\overline{N}(r,\infty;f|\geq k_2+1)+S(r),$$

and

$$(k_2+k)\ \overline{N}(r,\infty;f|\geq k_2+1)\quad\leq\quad\overline{N}(r,1;F\,|\geq k_1+1)+\overline{N}(r,0;f^{(k)}|\geq k_3+1)+S(r).$$

Using the above inequalities and arguments similar to those applied in the proof of Lemma 2.6 from [20], we can complete the proof the lemma. We omit the details.
□

Lemma 2.13 ([13]). *Let $N(r,0;f^{(k)}\mid f\neq 0)$ be the counting function of those zeros of $f^{(k)}$ which are not zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity. Then*

$$N(r,0;f^{(k)}\mid f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f\,|<k)+k\overline{N}(r,0;f\,|\geq k)+S(r,f).$$

Lemma 2.14. *Let F and G be given by (2.1), F, G share $(1,k_1)$, $2\leq k_1\leq\infty$, and let $\Phi_1\not\equiv 0$ and $n\geq 3$. Also, let $f^{(k)}$, $g^{(k)}$ share $(0,k_3)$ and f, g share (∞,∞). Then*

$$\overline{N}(r,0;f^{(k)})\leq\frac{1}{k_1(n-2)-1}\ \overline{N}(r,\infty;f)+S(r,f^{(k)}).$$

Proof. Using Lemmas 2.3 and 2.13 we can write

$$\overline{N}_*(r,1;F,G)\leq\overline{N}(r,1;F\,|\geq k_1+1)\leq\frac{1}{k_1}\ \left(N(r,1:F)-\overline{N}(r,1;F)\right)$$
$$\leq\ \frac{1}{k_1}\,[\sum_{j=1}^{n}\left(N(r,\omega_j;f^{(k)})-\overline{N}(r,\omega_j;f^{(k)})\right)]\leq\frac{1}{k_1}\ \left(N(r,0;(f^{(k)})'\mid f^{(k)}\neq 0)\right)$$
$$\leq\ \frac{1}{k_1}\left[\overline{N}(r,0;f^{(k)})+\overline{N}(r,\infty;f)\right]+S(r,f^{(k)}),$$

where $\omega_1,\omega_2\ldots\omega_n$ are the distinct roots of equation $z^n+az^{n-1}+b=0$. The rest of the proof follows from Lemma 2.5 with $k_3=0$. □

Lemma 2.15. *Let F and G be given by (2.1), F, G share $(1, k_1)$, $2 \leq k_1 \leq \infty$, and let $\Phi_1 \not\equiv 0$ and $n \geq 3$. Also, let $f^{(k)}$, $g^{(k)}$ share $(0, k_3)$ and f, g share (∞, ∞), where $0 \leq k_3 \leq \infty$. Then*

$$\overline{N}_L(r, 1; F) \leq \frac{k_1(n-2)}{(k_1+1)[k_1(n-2)-1]}\ \overline{N}(r, \infty; f) + S(r, f^{(k)}).$$

A similar result also holds for G.

Proof. Using Lemmas 2.3 and 2.13 we can write

$$\begin{aligned}\overline{N}_L(r, 1; F) &\leq \overline{N}(r, 1; F \mid\geq k_1+2) \leq \frac{1}{k_1+1}\ (N(r, 1 : F) - \overline{N}(r, 1; F)) \\ &\leq \frac{1}{k_1+1}\left[\overline{N}(r, 0; f^{(k)}) + \overline{N}(r, \infty; f)\right] + S(r, f^{(k)}).\end{aligned}$$

Now using Lemma 2.14 the proof of the lemma can easily be completed. We omit the details. □

Lemma 2.16 ([1]). *Let f and g be two non-constant meromorphic functions sharing $(1, k_1)$, where $2 \leq k_1 \leq \infty$. Then*

$$\begin{aligned}&\overline{N}(r, 1; f| = 2) + 2\,\overline{N}(r, 1; f| = 3) + \ldots + (k_1 - 1)\,\overline{N}(r, 1; f| = k_1) + k_1\,\overline{N}_L(r, 1; f) \\ &\quad + (k_1+1)\,\overline{N}_L(r, 1; g) + k_1\,\overline{N}_E^{(k_1+1}(r, 1; g) \leq N(r, 1; g) - \overline{N}(r, 1; g).\end{aligned}$$

Lemma 2.17. *Let F and G be given by (2.1) and they share $(1, k_1)$. If $f^{(k)}$, $g^{(k)}$ share $(0, k_3)$ and f, g share (∞, k_2), where $2 \leq k_1 \leq \infty$, and $H \not\equiv 0$. Then*

$$\begin{aligned}nT(r, f^{(k)}) \leq\ &\overline{N}(r, \infty; f) + \overline{N}(r, -a\frac{n-1}{n}; f^{(k)}) + \overline{N}(r, \infty; g) + \overline{N}(r, -a\frac{n-1}{n}; g^{(k)}) \\ &+ \overline{N}(r, 0; f^{(k)}) + \overline{N}(r, 0; g^{(k)}) + \overline{N}_*(r, 0; f^{(k)}, g^{(k)}) + \overline{N}_*(r, \infty; f, g) \\ &- (k_1-1)\overline{N}_*(r, 1; F, G) + \overline{N}_L(r, 1; F) + S(r, f^{(k)}) + S(r, g^{(k)}).\end{aligned}$$

A similar result also holds for $g^{(k)}$.

Proof. Using Lemmas 2.13 and 2.16 we can write

$$\overline{N}_0(r, 0; (g^{(k)})') + \overline{N}(r, 1; F \mid\geq 2) + \overline{N}_*(r, 1; F, G) \tag{2.2}$$

$$\begin{aligned}&\leq \overline{N}_0(r, 0; (g^{(k)})') + \overline{N}(r, 1; F| = 2) + \overline{N}(r, 1; F| = 3) + \ldots + \overline{N}(r, 1; F| = k_1) \\ &\quad + \overline{N}_E^{(k_1+1}(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_*(r, 1; F, G) \\ &\leq \overline{N}_0(r, 0; (g^{(k)})') - \overline{N}(r, 1; F| = 3) - \ldots - (k_1-2)\overline{N}(r, 1; F| = k_1) - (k_1-1)\overline{N}_L(r, 1; F) \\ &\quad - k_1\overline{N}_L(r, 1; G) - (k_1-1)\overline{N}_E^{(k_1+1}(r, 1; F) + N(r, 1; G) - \overline{N}(r, 1; G) + \overline{N}_*(r, 1; F, G) \\ &\leq \overline{N}_0(r, 0; (g^{(k)})') + N(r, 1; G) - \overline{N}(r, 1; G) - (k_1-2)\overline{N}_L(r, 1; F) - (k_1-1)\overline{N}_L(r, 1; G) \\ &\leq \overline{N}_0(r, 0; (g^{(k)})') + N(r, 1; G) - \overline{N}(r, 1; G) - (k_1-2)\overline{N}_L(r, 1; F) - (k_1-1)\overline{N}_L(r, 1; G)\end{aligned}$$

$$\leq N(r,0;(g^{(k)})' \mid g^{(k)} \neq 0) - (k_1-2)\overline{N}_L(r,1;F) - (k_1-1)\overline{N}_L(r,1;G)$$

$$\leq \overline{N}(r,0;g^{(k)}) + \overline{N}(r,\infty;g) - (k_1-2)\overline{N}_L(r,1;F) - (k_1-1)\overline{N}_L(r,1;G) =$$

$$\overline{N}(r,0;g^{(k)}) + \overline{N}(r,\infty;g) - (k_1-1)\overline{N}_*(r,1;F,G) + \overline{N}_L(r,1;F),$$

where $\overline{N}_0(r,0;(g^{(k)})')$ has the same meaning as in Lemma 2.2. Hence using (2.2), Lemmas 2.1, 2.2 and 2.3, in view of second fundamental theorem, we obtain

$$n\,T(r,f^{(k)}) \leq \overline{N}(r,0;f^{(k)}) + \overline{N}(r,\infty;f) + N(r,1;F \mid= 1) + \tag{2.3}$$

$$+\overline{N}(r,1;F \mid\geq 2) - N_0(r,0;(f^{(k)})') + S(r,f^{(k)}) \leq \overline{N}(r,0;f^{(k)}) + \overline{N}(r,\infty;f)$$

$$+\overline{N}(r,-a\frac{n-1}{n};f^{(k)}) + \overline{N}(r,-a\frac{n-1}{n};g^{(k)}) + \overline{N}_*(r,0;f^{(k)},g^{(k)}) + \overline{N}_*(r,\infty;f,g) +$$

$$+\overline{N}_*(r,1;F,G) + \overline{N}(r,1;F \mid \geq 2) + \overline{N}_0(r,0;(g^{(k)})') + S(r,f^{(k)}) + S(r,g^{(k)})$$

$$\leq \overline{N}(r,\infty;f) + \overline{N}(r,-a\frac{n-1}{n};f^{(k)}) + \overline{N}(r,\infty;g) + \overline{N}(r,-a\frac{n-1}{n};g^{(k)}) + \overline{N}(r,0;f^{(k)})$$

$$+\overline{N}(r,0;g^{(k)}) + \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,0;f^{(k)},g^{(k)}) - (k_1-1)\overline{N}_*(r,1;F,G) +$$

$$+\overline{N}_L(r,1;F) + S(r,f^{(k)}) + S(r,g^{(k)}).$$

This proves the lemma. □

Lemma 2.18 ([4]). *Let F and G be given by (2.1) and they share $(1,k_1)$, and let $n \geq 3$. If $f^{(k)}$, $g^{(k)}$ share $(0,0)$ and f, g share (∞,k_2), and $H \equiv 0$. Then $f^{(k)} \equiv g^{(k)}$.*

3. Proofs of the theorem

Proof of Theorem 1.1 Let F and G be given by (2.1). Then F, G share $(1,k_1)$ and $(\infty;k_2)$. We consider the following cases.

Case 1. Let $H \not\equiv 0$. Clearly $F \not\equiv G$ and so $f^{(k)} \not\equiv g^{(k)}$.

Subcase 1.1: Let $\Phi_2 \not\equiv 0$.

Subcase 1.1.1: Suppose $\Phi_3 \not\equiv 0$.

Suppose first that 0 is not an e.v.P. of $f^{(k)}$ and $g^{(k)}$. Then by Lemma 2.4 we get $\Phi_1 \not\equiv 0$. Since $f^{(k)}$, $g^{(k)}$ share $(0,k_3)$ it follows that $\overline{N}_*(r,0;f^{(k)},g^{(k)}) \leq \overline{N}(r,0;f^{(k)})$. Hence, successively applying Lemmas 2.17 and 2.7 for $k_3=0$, Lemma 2.10 for $k_2=0$

and Lemma 2.12, we can write

$$
\begin{aligned}
&(3.1)\; nT(r,f^{(k)})\\
\leq\;& \overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f^{(k)})+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g^{(k)})\\
&+2\,\overline{N}(r,0;f^{(k)})+\overline{N}_*(r,0;f^{(k)},g^{(k)})+\overline{N}_*(r,\infty;f,g)-(k_1-1)\overline{N}_*(r,1;F,G)\\
&+\overline{N}_L(r,1;F)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\;& \overline{N}(r,-a\frac{n-1}{n};f^{(k)})+\overline{N}(r,-a\frac{n-1}{n};g^{(k)})+3\;\overline{N}(r,0;f^{(k)})+2\overline{N}(r,\infty;f)\\
&+\overline{N}_*(r,\infty;f,g)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\;& \overline{N}(r,-a\frac{n-1}{n};f^{(k)})+\overline{N}(r,-a\frac{n-1}{n};g^{(k)})+\frac{3k_1+3}{(n-2)k_1-1}\;\overline{N}(r,\infty;f\mid\geq k_2+1)\\
&+\overline{N}(r,\infty;f\mid\geq k_2+1)+\frac{2k_1+2}{k_1k-1}\;\overline{N}(r,0;f^{(k)}\mid\geq k_3+1)+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\;& T(r,f^{(k)})+T(r,g^{(k)})+S(r,f^{(k)})+S(r,g^{(k)})\\
\leq\;& 2\;T(r)+S(r).
\end{aligned}
$$

Next, suppose 0 is an e.v.P. of $f^{(k)}$ and $g^{(k)}$. Then $\overline{N}(r,0;f^{(k)})=S(r,f^{(k)})$. Assuming that $\Phi_1\not\equiv 0$, we can apply Lemma 2.10 for $k_2=0$ to get $\overline{N}(r,\infty;f)=S(r)$. Hence $\overline{N}_*(r,\infty;f,g)=S(r)$, showing that (3.1) holds.

Now assume that $\Phi_1\equiv 0$. Then $(F-1)\equiv d(G-1)$, where $d\neq 0,1$. Since f, g share (∞,k_2), it follows that f, g share (∞,∞) which implies $\overline{N}_*(r,\infty;f,g)=S(r)$. Also, by Lemma 2.10 for $k_2=0$ we have $\overline{N}(r,\infty;f)=S(r)$. Therefore, in this case also (3.1) holds.

Arguments similar to those applied above can be used to obtain

$$(3.2)\qquad nT(r,g^{(k)})\;\leq\;2\;T(r)+S(r).$$

Combining (3.1) and (3.2) we get

$$(3.3)\qquad (n-2)\,T(r)\leq S(r),$$

which leads to a contradiction for $n\geq 3$.

Subcase 1.1.2: Suppose $\Phi_3\equiv 0$. Then by integration we obtain

$$1-\frac{\omega_i}{f^{(k)}}\equiv A(1-\frac{\omega_j}{g^{(k)}}),$$

where $A\neq 0$. If $A=1$ then $f^{(k)}=\frac{\omega_i}{\omega_j}g^{(k)}$, which contradicts $\Phi_2\not\equiv 0$. So $A\neq 0,1$. Since f, g share (∞,k_3), it follows that $N(r,\infty;f)=S(r,f^{(k)})$ and $N(r,\infty;g)=S(r,g^{(k)})$. Now proceeding as in Subcase 1.1.1, we can arrive at a contradiction.

Subcase 1.2: Let $\Phi_2\equiv 0$.

By integration we have $f^{(k)} \equiv cg^{(k)}$, where $c \neq 0, 1$. Since $f^{(k)}$, $g^{(k)}$ share $(0, k_3)$ and f, g share (∞, k_2), it follows that $\overline{N}_*(r, 0; f^{(k)}, g^{(k)}) = 0$ and $\overline{N}_*(r, \infty; f, g) = 0$.

Subcase 1.2.1 Suppose $\Phi_3 \not\equiv 0$.

If 0 is not an e.v.P. of $f^{(k)}$ and $g^{(k)}$, then by Lemma 2.4 we get $\Phi_1 \not\equiv 0$. Now consecutively applying Lemmas 2.17, 2.14 and 2.9 for $k_2 = 0$, and Lemma 2.15, we can write

$$(3.4) \qquad nT(r, f^{(k)}) \leq \overline{N}(r, \infty; f) + \overline{N}(r, -a\frac{n-1}{n}; f^{(k)}) + \overline{N}(r, \infty; g) +$$

$$+\overline{N}(r, -a\frac{n-1}{n}; g^{(k)}) + 2\overline{N}(r, 0; f^{(k)}) + \overline{N}_*(r, 0; f^{(k)}, g^{(k)}) + \overline{N}_*(r, \infty; f, g) -$$

$$-(k_1 - 1)\ \overline{N}_*(r, 1; F, G) + \overline{N}_L(r, 1; F) + S(r, f^{(k)}) + S(r, g^{(k)}) \leq \overline{N}(r, -a\frac{n-1}{n}; f^{(k)}) +$$

$$\overline{N}(r, -a\frac{n-1}{n}; g^{(k)}) + 2\ \overline{N}(r, \infty; f) + \frac{2}{k_1(n-2)-1}\ \overline{N}(r, \infty; f) - (k_1 - 1)\ \overline{N}_*(r, 1; F, G) +$$

$$+\overline{N}_L(r, 1; F) + S(r, f^{(k)}) + S(r, g^{(k)}) \leq \overline{N}(r, -a\frac{n-1}{n}; f^{(k)}) + \overline{N}(r, -a\frac{n-1}{n}; g^{(k)}) +$$

$$+3\ \overline{N}(r, \infty; f) - (k_1 - 1)\overline{N}_*(r, 1; F, G) + \overline{N}_L(r, 1; F) + S(r, f^{(k)}) + S(r, g^{(k)}) \leq 2\,T(r) +$$

$$+\frac{3}{k}\ \overline{N}_*(r, 1; F, G) - (k_1 - 1)\overline{N}_*(r, 1; F, G) + \overline{N}_L(r, 1; F) + S(r, f^{(k)}) + S(r, g^{(k)}) \leq$$

$$\leq 2\,T(r) + \frac{k_1(n-2)}{(k_1+1)[k_1(n-2)-1]}\ \overline{N}(r, \infty; g) + S(r, f^{(k)}) + S(r, g^{(k)}) \leq$$

$$\leq \left(2 + \frac{3k_1(n-2)}{(k_1+1)(k+1)[k_1(n-2)-1]}\right) T(r) + S(r).$$

Therefore

$$(3.5) \qquad \left(n - 2 - \frac{3k_1(n-2)}{(k_1+1)(k+1)[k_1(n-2)-1]}\right) T(r) \leq S(r).$$

Since $n \geq 3$, the inequality (3.5) leads to a contradiction.

In the case where 0 is an e.v.P. of $f^{(k)}$ and $g^{(k)}$, we can apply Lemma 2.9 for $k_2 = 0$, to get $\overline{N}(r, \infty; f) = \frac{1}{k}\overline{N}_*(r, 1; F, G)$. Hence, proceeding as above in this case also we arrive at a contradiction.

Subcase 1.2.2: Suppose $\Phi_3 \equiv 0$.

Suppose ∞ is not an e.v.P. of f and g. Since $f^{(k)}$, $g^{(k)}$ share $(0, k_3)$ and f, g share (∞, k_2), it follows from Lemma 2.8 that $\overline{N}_*(r, 0; f^{(k)}, g^{(k)}) = 0$ and $\overline{N}_*(r, \infty; f, g) = 0$.

Assuming that 0 is not an e.v.P of $f^{(k)}$ and $g^{(k)}$, by Lemma 2.4 we get $\Phi_1 \not\equiv 0$. Now, consecutively using Lemmas 2.17 and 2.5 for $k_3 = 0$, and Lemma 2.11 for $k_2 = 0$, we can write

$$(3.6) \qquad nT(r, f^{(k)}) \leq \overline{N}(r, \infty; f) + \overline{N}(r, -a\frac{n-1}{n}; f^{(k)}) + \overline{N}(r, \infty; g) +$$

$$\overline{N}(r,-a\frac{n-1}{n};g^{(k)})+2\overline{N}(r,0;f^{(k)})+\overline{N}_*(r,0;f^{(k)},g^{(k)})+\overline{N}_*(r,\infty;f,g)-$$
$$-(k_1-1)\ \overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)+S(r,f^{(k)})+S(r,g^{(k)})\leq\overline{N}(r,-a\frac{n-1}{n};f^{(k)})+$$
$$+\overline{N}(r,-a\frac{n-1}{n};g^{(k)})+2\ \overline{N}(r,\infty;f)+2\ \overline{N}_*(r,1;F,G)-(k_1-1)\ \overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)+$$
$$+S(r,f^{(k)})+S(r,g^{(k)})\leq\overline{N}(r,-a\frac{n-1}{n};f^{(k)})+\overline{N}(r,-a\frac{n-1}{n};g^{(k)})-$$
$$-(k_1-3)\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)+\frac{2}{nk+n-1}\{\overline{N}(r,0;f^{(k)}+a)+\overline{N}(r,0;g^{(k)}+a)+$$
$$+\overline{N}_*(r,1;F,G)\}+S(r,f^{(k)})+S(r,g^{(k)})\leq 2\,T(r)+\frac{4}{nk+n-1}\,T(r)+\frac{2}{5}\,\overline{N}_L(r,1;F)+$$
$$+S(r,f^{(k)})+S(r,g^{(k)})\leq\left(2+\frac{4}{nk+n-1}+\frac{2k_1(n-2)}{5(k_1+1)[k_1(n-2)-1]}\right)T(r)+S(r).$$

Therefore

$$\left(n-2-\frac{4}{nk+n-1}-\frac{2k_1(n-2)}{5(k_1+1)[k_1(n-2)-1]}\right)T(r)\leq S(r). \tag{3.7}$$

Since $n\geq 3$, the inequality (3.7) leads to a contradiction.

If 0 is an e.v.P. of $f^{(k)}$ and $g^{(k)}$, then with the help of Lemmas 2.17 and 2.11 for $k_2=0$ and the above arguments, we arrive at a contradiction.

If ∞ is an e.v.P. of f and g, then proceeding as in Subcase 1.2.1, we can arrive at a contradiction.

Case 2. Let $H\equiv 0$. In this case the assertion of the theorem follows from Lemma 2.18. Theorem 1.1 is proved.

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