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FIXED POINTS OF MIXED MONOTONE OPERATORS FOR EXISTENCE AND UNIQUENESS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems by using new fixed point results of mixed monotone operators on cones.

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1. Introduction

In recent years, boundary value problems for nonlinear fractional differential equations with a variety of boundary conditions have been investigated by many researchers. Fractional differential equations appear naturally in various fields of science and engineering, and thus constitute an important field of research (see [1 – 3]). As a matter of fact, fractional derivatives provide a powerful tool for the description of memory and hereditary properties of various materials and processes. A significant feature of a fractional order differential operator, in contrast to its counterpart in classical calculus, is its nonlocal behavior, meaning that the future state of a dynamical system or process based on the fractional differential operator depends on its current state as well its past states. In other words, differential equations of arbitrary order are capable of describing memory and hereditary properties of certain important materials and processes. This aspect of fractional calculus has contributed towards the growing popularity of the subject. Mixed monotone operators were introduced by Guo and Lakshmikantham in [4]. Their study has wide applications in the applied sciences such as engineering, biological chemistry technology, nuclear physics and in mathematics (see [6 – 8]). Various existence and uniqueness theorems of fixed points for mixed monotone operators have been obtained by a number of authors (see [9] – [12]). Bhaskar and Lakshmikantham, [9], established some coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces and discussed a question of existence and uniqueness of a solution for a periodic boundary value problem. Recently Y. Sang, [13], proved some new existence and uniqueness theorems of a fixed point of mixed monotone operators with perturbations.

In this paper, by applying Sang's results, we obtain some new results on the existence and uniqueness of positive solutions for some nonlinear fractional differential equations via given boundary value problems.
We first introduce some notations, definitions and known results to be used in the paper.

Definition 1.1 ([1, 2]). *For a continuous function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined by*

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $n-1 < \alpha < [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 1.2 ([1, 2]). *The Riemann-Liouville fractional derivative of order α for a continuous function f is defined by*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n-1}} ds, \qquad n = [\alpha]+1,$$

where the right-hand side is defined pointwise on $(0,\infty)$.

Definition 1.3 ([1, 2]). *Let $[a,b]$ be an interval in $\mathbb{R}$ and $\alpha > 0$. The Riemann-Liouville fractional order integral of a function $f \in L^1([a,b],\mathbb{R})$ is deined by*

$$I_a^{\alpha} f(t) = \frac{1}{\gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

whenever the integral exists.

Let $(E, \|\cdot\|)$ be a Banach space which is partially ordered by a cone $P \subseteq E$, that is, $x \le y$ if and only if $y - x \in P$. If $x \neq y$, then we denote $x < y$ or $x > y$. Also, the zero element of E we denote by θ. Recall that a non-empty closed convex set $P \subset E$ is called a cone if it satisfies the conditions: (i) $x \in P$, $\lambda \ge 0 \Longrightarrow \lambda x \in P$, (ii) $x \in P$, $-x \in P \Longrightarrow x = \theta$. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \le x \le y$ implies $\| x \| \le N \| y \|$. Also, we define the order interval $[x_1, x_2] = \{x \in E | x_1 \le x \le x_2\}$ for all $x_1, x_2 \in E$. We say that an operator $A : E \to E$ is increasing whenever $x \le y$ implies $Ax \le Ay$.

Definition 1.4 ([4, 5]). *Let $D \subset E$. An operator $A : D \times D \to D$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, that is, $u_i, v_i \in D\,(i = 1,2)$, $u_1 \le u_2, v_1 \ge v_2$ implies $A(u_1, v_1) \le A(u_2, v_2)$.*

An element $x^* \in D$ is called a fixed point of A if it satisfies $A(x^*, x^*) = x^*$. For $h > \theta$ we define $P_h = \{x \in E| \quad \exists \lambda, \mu > 0;\ \lambda h \leq x \leq \mu h\}$.

In this paper, using the existence and uniqueness results for the solution of the following operator equation

$$A(x, x) + Bx = x, \tag{1.1}$$

where A is a mixed monotone operator, B is sublinear and E is a real ordered Banach space, obtained in [13] by the partial ordering theory and monotone iterative technique, we study a question of existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems.

Theorem 1.1 ([13]). *Let P be a normal cone in E. $A : P \times P \to P$ be a mixed monotone operator, and let $B : E \to E$ be sublinear. Assume that for all $a < t < b$, there exist two positive-valued functions $\tau(t)$ and $\varphi(t, x, y)$ defined on an interval (a, b) such that:*
(H_1) $\tau : (a, b) \to (0, 1)$ is surjection;
(H_2) $\varphi(t, x, y) > \tau(t)$ for all $t \in (a, b)$, $x, y \in P$;
(H_3) $A(\tau(t)x, \frac{1}{\tau(t)}y) \geq \varphi(t, x, y)A(x, y)$ for all $t \in (a, b)$, $x, y \in P$;
(H_4) $(I - B)^{-1} : E \to E$ exists and is an increasing operator.
Furthermore, for any $t \in (a, b)$ the function $\phi(t, x, y)$ is nonincreasing in x for fixed y, and nondecreasing in y for fixed x. In addition, suppose that there exist $h \in P - \{\theta\}$ and $t_0 \in (a, b)$ such that
$\tau(t_0)h \leq (I - B)^{-1}A(h, h) \leq \frac{\varphi(t_0, \frac{1}{\tau(t_0)}, \tau(t_0)h)}{\tau(t_0)}h.$
Then the following assertions hold:
(i) there are $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0$ and

$$u_0 \leq (I - B)^{-1}A(u_0, v_0) \leq (I - B)^{-1}A(v_0, u_0) \leq v_0;$$

(ii) the equation (1.1) has a unique solution x^* in $[u_0, v_0]$;
(iii) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = (I - B)^{-1}A(x_{n-1}, y_{n-1}), \quad y_n = (I - B)^{-1}A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $\| x_n - x^* \| \to 0$ and $\| y_n - x^* \| \to 0$ as $n \to \infty$.

2. Main results

We study the existence and uniqueness of a solution of a fractional differential equation on a partially ordered Banach space with two types of boundary conditions and two types of fractional derivatives. We first study the existence and uniqueness of a positive solution for the following fractional differential equation:

$$\frac{D^\alpha}{Dt}u(t) = f(t, u(t), u(t)), \qquad t \in [0, 1],\ 3 < \alpha \leq 4, \tag{2.1}$$

subject to conditions

$$u(0) = u'(0) = u(1) = u'(1) = 0, \tag{2.2}$$

where D^α is the Riemann-Liouville fractional derivative of order α.

Consider the Banach space of continuous functions on $[0,1]$ with sup norm and set $P = \{y \in C[0,1] : \min_{t\in[0,1]} y(t) \geq 0\}$. Then P is a normal cone. The next two lemmas were proved in [14].

Lemma 2.1 ([14]). *Given $y \in C[0,1]$ and a number α such that $3 < \alpha \leq 4$. Then the unique solution of the following fractional differential equation boundary value problem*

$$\begin{aligned} &\frac{D^\alpha}{Dt} u(t) = f(t, y(t)), \qquad t \in [0,1],\ 3 < \alpha \leq 4, \\ &u(0) = u'(0) = u(1) = u'(1) = 0, \end{aligned} \tag{2.3}$$

is given by

$$u(t) = \int_0^1 G(t,s) f(s, y(s)) ds,$$

where

$$G(t,s) = \begin{cases} \frac{(t-1)^{\alpha-1} + (1-s)^{\alpha-2} t^{\alpha-2}[(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\alpha-2} t^{\alpha-2}[(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.4}$$

If $f(t, u(t)) = 1$, then the unique solution of (2.3) is given by
$u_0(t) = \int_0^1 G(t,s) ds = \frac{1}{\Gamma(\alpha+1)} t^{\alpha-2}(1-t)^2$.

Lemma 2.2 ([14]). *The Green's function $G(t,s)$ has the following properties:*
(1) $G(t,s) > 0$ and $G(t,s)$ is continuous for $t, s \in [0,1]$;
(2) $\frac{(\alpha-2)h(t)k(s)}{\Gamma(\alpha)} \leq G(t,s) \leq \frac{M_0 k(s)}{\Gamma(\alpha)}$,
where $M_0 = \max\{\alpha - 1, (\alpha-2)^2\}$, $h(t) = t^{\alpha-2}(1-t)^2$, $k(s) = s^2(1-s)^{\alpha-2}$.

Now we are ready to state and prove our first main result.

Theorem 2.1. *Let $f(t, u(t), v(t)) \in C([0,1] \times [0,\infty) \times [0,\infty))$ be an increasing in u and decreasing in v function. Assume that for all $a < t < b$ there exist two positive-valued functions $\tau(t)$ and $\varphi(t,u,v)$ defined on an interval (a,b) such that:*
(H_1) $\tau : (a,b) \to (0,1)$ is surjection;
(H_2) $\varphi(t,u,v) > \tau(t)$ for all $t \in (a,b)$, $u, v \in P$;
(H_3) $\int_0^1 G(t,s) f(s, \tau(t)u(s), \frac{1}{\tau(t)} v(s)) \geq \varphi(t,u,v) \int_0^1 G(t,s) f(s, u(s), v(s)) ds$.
Furthermore, for any $t \in (a,b)$ the function $\varphi(t,u,v)$ is nonincreasing in u for fixed v, and nondecreasing in v for fixed u. In addition, suppose that there exist $h \in P - \{\theta\}$

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$\tau(t_0)h \leq (I - B)^{-1}A(h, h) \leq \frac{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h)}{\tau(t_0)}h.$
Then the following assertions hold:
(i) there are $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0$ and

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is given by

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If $f(t, u(t)) = 1$, then the unique solution of (2.3) is given by
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(2) $\qquad \frac{(\alpha-2)h(t)k(s)}{\Gamma(\alpha)} \leq G(t,s) \leq \frac{M_0 k(s)}{\Gamma(\alpha)}$,
where $M_0 = \max\{\alpha - 1, (\alpha-2)^2\}$, $\quad h(t) = t^{\alpha-2}(1-t)^2$, $\quad k(s) = s^2(1-s)^{\alpha-2}$.

Now we are ready to state and prove our first main result.

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(H_1) $\tau : (a,b) \to (0,1)$ is surjection;
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(H_3) $\int_0^1 G(t,s) f(s, \tau(t)u(s), \frac{1}{\tau(t)}v(s)) \geq \varphi(t,u,v) \int_0^1 G(t,s) f(s, u(s), v(s)) ds$.
Furthermore, for any $t \in (a,b)$ the function $\varphi(t,u,v)$ is nonincreasing in u for fixed v, and nondecreasing in v for fixed u. In addition, suppose that there exist $h \in P - \{\theta\}$

and $t_0 \in (a,b)$ such that

$$\tau(t_0)h \leq \int_0^1 G(t,s)f(s,h(s),h(s))ds \leq \frac{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h)}{\tau(t_0)}h. \tag{2.5}$$

Then the following assertions hold:

(i) there are $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 \leq v_0$, and
$u_0 \leq \int_0^1 G(t,s)f(s,u_0(s),v_0(s))ds \leq \int_0^1 G(t,s)f(s,v_0(s),u_0(s))ds \leq v_0$,

(ii) the problem (2.1), (2.2) has a unique solution x^* in $[u_0, v_0]$,

(iii) for any initial values $u_0, v_0 \in P_h$ and $n = 1, 2, \ldots$, constructing successively the sequences

$$u_n = \int_0^1 G(t,s)f(s,u_{n-1}(s),v_{n-1}(s))ds, \quad v_n = \int_0^1 G(t,s)f(s,v_{n-1}(s),u_{n-1}(s))ds,$$

we have $\| u_n - u^* \| \to 0$ and $\| v_n - v^* \| \to 0$ as $n \to \infty$.

Proof. By Lemma 2.1, the problem is equivalent to equation $u(t) = \int_0^1 G(t,s)f(s,y(s))ds$, where $G(t,s)$ is defined by (2.4). Define the operator $A : P \times P \to E$ as follows: $A(u(t), v(t)) = \int_0^1 G(t,s)f(s,u(s),v(s))ds$, and observe that u is a solution for the problem if and only if $u = A(u,u)$.

Next, it is easy to see that the operator A is increasing in u and decreasing in v on P. Hence, under the assumptions of the theorem we have $A(\tau(t)u, \frac{1}{\tau(t)}v) \geq \varphi(t,u,v)A(u,v)$ for all $t \in (a,b)$, $u, v \in P$ and $\tau(t_0)h \leq A(h,h) \leq \frac{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h)}{\tau(t_0)}h$. Thus, the operator A satisfies all the conditions of Theorem 1.1, and hence A has a unique positive solution (u^*, u^*) such that $A(u^*, u^*) = u^*$, $u^* \in [u_0, v_0]$. □

Example 2.1. Consider the following periodic boundary value problem:

$$D^{\frac{7}{2}}u(t) = f(t,u(t),u(t)) = g(t) + u(t) + \frac{1}{\sqrt{u(t)}}, \qquad t \in [0,1], \tag{2.6}$$

$$u(0) = u'(0) = u(1) = u'(1) = 0,$$

where $g(t)$ is continuous on $[0,1]$ with $388.625 \leq g(t) \leq 63728$.

For every $\lambda \in (0,1)$ and $u, v \in P$ we have
$\int_0^1 G(t,s)[g(s) + \lambda u(s) + \frac{1}{\sqrt{\frac{1}{\lambda}v(s)}}]ds = \lambda \int_0^1 G(t,s)[\frac{g(s)}{\lambda} + u(s) + \frac{1}{\sqrt{\lambda v(s)}}]ds$
$\leq \lambda \frac{\frac{g(s)}{\lambda} + u(s) + \frac{1}{\sqrt{\lambda v(s)}}}{g(s) + u(s) + \frac{1}{\sqrt{v(s)}}} \int_0^1 G(t,s)[g(s) + u(s) + \frac{1}{\sqrt{v(s)}}]ds.$
We note that

$$\lambda < \varphi(\lambda, u, v) = \lambda \frac{\frac{g(s)}{\lambda} + u(s) + \frac{1}{\sqrt{\lambda v(s)}}}{g(s) + u(s) + \frac{1}{\sqrt{v(s)}}} \int_0^1 G(t,s)[g(s) + u(s) + \frac{1}{\sqrt{v(s)}}]ds < 1.$$

By means of some calculations, we can conclude that for any $\lambda \in (0,1)$ the function φ is nonincreasing in u for fixed v and nondecreasing in v for fixed u.

So, it is enough to verify that the condition (2.6) of Theorem 2.1 is satisfied. Putting $u = v = h = 1$, and taking into account that $M_1 = \min_{t\in[0,1]} \int_0^1 G(t,s)ds = 0.00001$ and $M_2 = \max_{t\in[0,1]} \int_0^1 G(t,s)ds = 0.004$, we can easily get
$2^{-8} \leq 0.00001 \times 390.625 \leq \int_0^1 G(t,s)[g(s) + u(s) + \frac{1}{\sqrt{v(s)}}]ds$
$\leq 0.004 \times 63730 \leq \frac{2^8(g(s)+2)}{g(s)+2^8+2^4} = \frac{s(2^{-8},2^8,2^{-8})}{2^{-8}}, \quad s \in [0,1]$,
implying that the condition (2.5) of Theorem 2.1 holds. Therefore, we can apply this theorem to conclude that the problem in the example has a unique solution.

Now, we study the existence and uniqueness of a positive solution for the following fractional differential equation:

$$ {}^cD^\alpha y(t) = h(t), \qquad t \in [0,T], \quad T \geq 1. \tag{2.7}$$

subject to

$$ y(0) + \int_0^T y(s)ds = y(T). \tag{2.8}$$

Lemma 2.3 ([15]). *Let $0 < \alpha \leq 1$ and let $h \in C([0,T],\mathbb{R})$ be a given function. Then the boundary value problem (2.7), (2.8) has a unique solution given by*

$$y(t) = \int_0^T G(t,s)h(s)ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \frac{-(T-s)^\alpha + \alpha T(t-s)^{\alpha-1}}{T\Gamma(\alpha+1)} + \frac{(T-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \leq s < t, \\ \frac{-(T-s)^\alpha}{T\Gamma(\alpha+1)} + \frac{(T-s)^{\alpha-1}}{T\Gamma(\alpha)}, & t \leq s < T. \end{cases}$$

By using arguments similar to those applied in the proof of Theorem 2.1, it can easily be verified that Theorem 2.1 remains true for Green function defined in Lemma 2.3.

Example 2.2. Consider the following boundary value problem

$$ {}^cD^{\frac{1}{2}}u(t) = f(t,u(t),u(t)) = g(t)u(t)^{\frac{1}{2}} + u(t)^{\frac{-1}{3}}, \qquad t \in [0,1],$$

$$u(0) + \int_0^1 u(s)ds = u(1),$$

where $g(t)$ is continuous on $[0,1]$ with $0.6378 \leq g(t) \leq 2.89967$.

For every $\lambda \in (0,1)$ and $u, v \in P$ we have

$$\int_0^1 G(t,s)[g(s)((\lambda u(s))^{\frac{1}{2}} + (\frac{1}{\lambda}v(s))^{\frac{-1}{3}})]ds \geq \lambda \int_0^1 G(t,s)[\frac{g(s)}{\lambda} + \tag{2.9}$$

$$+u(s) + \frac{1}{\sqrt{v(s)}}]ds \geq \lambda \int_0^1 G(t,s)[(g(s)(\frac{u(s)}{\lambda})^{\frac{1}{2}} + (\lambda v(s))^{\frac{-1}{3}})]ds$$

$$\geq \lambda \frac{g(s)(\frac{u(s)}{\lambda})^{\frac{1}{2}} + (\lambda v(s))^{\frac{-1}{3}}}{g(s)u(s)^{\frac{1}{2}} + v(s)^{\frac{-1}{3}}} \int_0^1 G(t,s)[g(s)u(s)^{\frac{1}{2}} + v(s)^{\frac{-1}{3}}]ds.$$

Note that

$$\lambda < \varphi(\lambda, u, v) = \lambda \frac{g(s)(\frac{u(s)}{\lambda})^{\frac{1}{2}} + (\lambda v(s))^{\frac{-1}{3}}}{g(s)u(s)^{\frac{1}{2}} + v(s)^{\frac{-1}{3}}} < 1.$$

By means of some calculations, we can conclude that for any $\lambda \in (0, 1)$ the function φ is nonincreasing in u for fixed v and nondecreasing in v for fixed u.

So, it is enough to verify that the condition (2.5) of Theorem 2.1 is satisfied. Putting $u = v = h = 1$, and taking into account that $M_1 = \min_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{3}$, and $M_2 = \max_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{80}{51}$, we can easily get $2^{-6} \leq \frac{1}{3} \times 1.6378 \leq \int_0^1 G(t,s)[g(s)u(s)^{\frac{1}{2}} + u(s)^{\frac{-1}{3}}]ds \leq \frac{80}{51} \times 3.8967 \leq \frac{16g(s)+8}{2g(s)+1} = \frac{\varphi(2^{-6}, 2^{6}, 2^{-6})}{2^{-6}}$, $s \in [0, 1]$, implying that the condition (2.5) of Theorem 2.1 holds. Therefore, we can apply this theorem to conclude that the problem in the example has a unique solution.

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