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BRUCK CONJECTURE FOR A LINEAR DIFFERENTIAL POLYNOMIAL

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Abstract. In the paper we study the Brück Conjecture for a linear differential polynomial.

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1. Introduction, definitions and results

Let f, g be nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f, g share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicities, and we say that f, g share the value a IM (ignoring multiplicities) if f, g have the same a-points but the multiplicities are not taken into account.

The monograph [7] is a good source of standard notations and definitions of the value distribution theory. We now introduce some notation and a definition.

Definition 1.1. *Given a meromorphic function f, a number $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer k,*

(i) $N_{(k}(r, a; f)$ $(\overline{N}_{(k}(r, a; f))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than k;*

(ii) $N_{k)}(r, a; f)$ $(\overline{N}_{k)}(r, a; f))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than k;*

Definition 1.2. *A meromorphic function $a = a(z)$ is called a small function of a meromorphic function f if $T(r, a) = S(r, f)$.*

In [5], R. Brück considered the uniqueness problem of an entire function when it shares a single value CM with its first derivative, and proposed the following conjecture, which inspired a number of people to work on the topic.

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Brück Conjecture: *Let f be a nonconstant entire function satisfying $\nu(f) < \infty$, and let $\nu(f)$ be not a positive integer, where $\nu(f)$ is the hyper-order of f. If f and f' share one finite value a CM, then $f' - a = c(f - a)$ for some constant $c \neq 0$.*

R. Brück [5] himself proved the following result.

Theorem A ([5]). *Let f be a nonconstant entire function. If f and f' share the value 1 CM and $N(r, 0; f') = S(r, f)$, then $f - 1 = c(f' - 1)$, where c is a nonzero constant.*

Considering entire functions of finite order, L. Z. Yang [9] proved the following theorem.

Theorem B ([9]). *Let f be a nonconstant entire function of finite order, and let $a(\neq 0)$ be a finite constant. If f and $f^{(k)}$ share the value a CM, then $f - a = c(f^{(k)} - a)$, where c is a nonzero constant and $k \geq 1$ is an integer.*

In 2005, A. H. H. Al-khaladi [2] extended Theorem A to the class of meromorphic functions and proved the following result.

Theorem C ([2]). *Let f be a nonconstant meromorphic function satisfying $N(r, 0; f') = S(r, f)$. If f and f' share the value 1 CM, then $f - 1 = c(f' - 1)$ for some nonzero constant c.*

Also, in [2] were considered the following examples, showing that the value sharing cannot be relaxed from CM to IM, and the condition $N(r, 0; f') = S(r, f)$ is essential.

Example 1.1. Let $f = 1 + \tan z$. Then $f' - 1 = (f - 1)^2$ and $N(r, 0; f') \equiv 0$. Clearly f and f' share the value 1 IM but the conclusion of Theorem C does not hold.

Example 1.2. Let $f = \frac{z}{1 + e^{-z}}$. Then f and f' share the value 1 CM and $N(r, 0; f') \neq S(r, f)$. It is easy to verify that $f' - 1 = \frac{1}{1 + e^{z}}(f - 1)$.

A. H. H. Al-khaladi [1] also observed by the following example that in Theorem A the shared value cannot be replaced by a shared small function.

Example 1.3. Let $f = 1 + e^{e^z}$ and $a = \frac{1}{1 - e^{-z}}$. Then a is a small function of f and $f - a$, $f' - a$ share the value 0 CM and $N(r, 0; f') \equiv 0$. Also, we see that $f - a = \frac{1}{e^z}(f' - a)$.

Considering the sharing of small functions, A. H. H. Al-khaladi [1] proved the following result.

Theorem D ([1]). *Let f be a nonconstant entire function satisfying $N(r,0;f') = S(r,f)$, and let $a\,(\not\equiv 0,\infty)$ be a meromorphic small function of f. If $f-a$ and $f'-a$ share the value 0 CM, then $f-a = \left(1+\frac{c}{a}\right)(f'-a)$, where $1+\frac{c}{a} = e^{\beta}$, c is a constant and β is an entire function.*

For higher order derivatives, A. H. H. Al-khaladi [3] proved the following theorem.

Theorem E ([3]). *Let f be a nonconstant entire function satisfying $\overline{N}(r,0;f^{(k)}) = S(r,f)$ $(k>1)$, and let $a\,(\not\equiv 0,\infty)$ be a meromorphic small function of f. If $f-a$ and $f^{(k)}-a$ share the value 0 CM, then $f-a = \left(1+\frac{P_{k-1}}{a}\right)(f^{(k)}-a)$, where P_{k-1} is a polynomial of degree at most $k-1$ and $1+\frac{P_{k-1}}{a}\not\equiv 0$.*

Recently A. H. H. Al-khaladi [4] extended Theorem E to meromorphic functions. A natural extension of a derivative is a linear differential polynomial. For a transcendental meromorphic function f we denote by $L = L(f^{(k)})$ a linear differential polynomial of the form

$$L = L(f^{(k)}) = a_0 f^{(k)} + a_1 f^{(k+1)} + \cdots + a_p f^{(k+p)}, \tag{1.1}$$

where $a_0, a_1, \ldots, a_p\,(\neq 0)$ are constants, and $k\,(\geq 1)$ and $p(\geq 0)$ are integers such that $p=0$ if $k=1$ and $0\leq p\leq k-2$ if $k\geq 2$.

In the present paper we consider the problem of sharing a small function by a meromorphic function and a linear differential polynomial in conformity with Brück conjecture. The following theorem is the main result of the paper.

Theorem 1.1. *Let f be a transcendental meromorphic function and let the differential polynomial $L = L(f^{(k)})$, given by (1.1), be nonconstant. Suppose that $f-a$ and $L-a$ share 0 CM, where $a\,(\not\equiv 0,\infty)$ is a small function of f. If $N(r,0;f^{(k)}) = S(r,f)$, then*

$$f-a = \left(1+\frac{P_{k-1}}{a}\right)(L-a),$$

where P_{k-1} is a polynomial of degree at most $k-1$ and $1+\frac{P_{k-1}}{a}\not\equiv 0$.

The following example shows that the condition $N(r,0;f^{(k)}) = S(r,f)$ is essential in Theorem 1.1.

Example 1.4. Let P be a nonconstant polynomial, and let $f = \dfrac{Pe^z}{1+e^z}$. Then $f' = \dfrac{e^z(P + P' + P'e^z)}{(1+e^z)^2}$, and hence $N(r, 0; f') \neq S(r, f)$. Also, $f - P'$ and $f' - P'$ share 0 CM but $f' - P' = \dfrac{1}{1+e^z}(f - P')$, where $T(r, P') = S(r, f)$.

2. Lemmas

In this section we present some necessary lemmas to be used in the proof of Theorem 1.1.

Lemma 2.1. *Let f be a nonconstant meromorphic function and let $L = L(f^{(k)})$, given by (1.1), be nonconstant. If $f - a$ and $L - a$ share 0 CM, where $a = a(z)(\not\equiv 0, \infty)$ is a small function of f, then one of the following assertions holds:*

(i) $f - a = \left(1 + \frac{P_{k-1}}{a}\right)(L - a)$, *where P_{k-1} is a polynomial of degree at most $k-1$ and $1 + \dfrac{P_{k-1}}{a} \not\equiv 0$.*

(ii) $T(r, f^{(k)}) \leq (k + p + 1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(k)}) + N(r, 0; f^{(k)}) + S(r, f)$.

Proof. Let $h = \dfrac{f - a}{L - a}$. Then h is an entire function and the poles of f are precisely the zeros of h. Now differentiating

$$f - a = hL - ah \tag{2.1}$$

k-times we get

$$f^{(k)} - a^{(k)} = (hL)^{(k)} - (ha)^{(k)}. \tag{2.2}$$

We now consider the following cases.

Case I. Let $a^{(k)} \not\equiv 0$. We put

$$W = \frac{(hL)^{(k)}}{hf^{(k)}} - \frac{(ha)^{(k)}}{ha^{(k)}}. \tag{2.3}$$

Since $W' = \dfrac{(hL)^{(k)}}{hL} \cdot \dfrac{L}{f^{(k)}} - \dfrac{(ha)^{(k)}}{ha} \cdot \dfrac{a}{a^{(k)}}$, we have $m(r, W) = S(r, f)$.

We first suppose that $W \not\equiv 0$. Let z_0 be a zero of $f^{(k)} - a^{(k)}$ and $a^{(k)}(z_0) \neq 0, \infty$. Then from (2.2) we see that z_0 is a zero of $(hL)^{(k)} - (ha)^{(k)}$. Hence $W(z_0) = 0$ and we have

$$\begin{aligned} \overline{N}(r, 0; f^{(k)} - a^{(k)}) &\leq N(r, 0; W) + S(r, f) \\ &\leq T(r, W) + S(r, f) \\ &= N(r, W) + S(r, f). \end{aligned} \tag{2.4}$$

Also

$$N(r,W) \le (k+p)\overline{N}(r,\infty;f) + N(r,0;f^{(k)}) + S(r,f). \tag{2.5}$$

By Nevanlinna's three small functions theorem (see [7], p. 47), and formulas (2.4) and (2.5), we get

$$T(r,f^{(k)}) \le (k+p+1)\overline{N}(r,\infty;f) + \overline{N}(r,0;f^{(k)}) + N(r,0;f^{(k)}) + S(r,f),$$

which is (ii).

Now let $W \equiv 0$. Then from (2.2) and (2.3) we get

$$(f^{(k)} - a^{(k)})a^{(k)} \equiv (ha)^{(k)}(f^{(k)} - a^{(k)}).$$

Since $f^{(k)} - a^{(k)} \not\equiv 0$, we obtain $(ha)^{(k)} \equiv a^{(k)}$. Integrating the last equality k-times we get $ha = a + P_{k-1}(z)$, where $P_{k-1}(z)$ is a polynomial of degree at most $k-1$. So $h = 1 + \dfrac{P_{k-1}}{a}$ and hence $f - a = \left(1 + \dfrac{P_{k-1}}{a}\right)(L-a)$, which is (i).

Case II. Let $a^{(k)} \equiv 0$. Then a is a polynomial of degree at most $k-1$. From (2.2) we get $f^{(k)} = (hL)^{(k)} - (ah)^{(k)}$, and hence

$$\frac{1}{h} = \frac{(hL)^{(k)}}{hf^{(k)}} - \frac{(ah)^{(k)}}{hf^{(k)}}. \tag{2.6}$$

Putting $F = f^{(k)}$, $G = \dfrac{(hL)^{(k)}}{hf^{(k)}}$ and $b = \dfrac{(ah)^{(k)}}{h}$, from (2.6) we get

$$\frac{1}{h} = G - \frac{b}{F}. \tag{2.7}$$

Differentiating (2.7) we obtain

$$-\frac{1}{h}\cdot\frac{h'}{h} = G' - \frac{b'}{F} + \frac{b}{F}\cdot\frac{F'}{F}. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\frac{A}{F} = G' + G\cdot\frac{h'}{h}, \tag{2.9}$$

where $A = b\cdot\dfrac{h'}{h} + b' - b\cdot\dfrac{F'}{F}$.

We first suppose that $G \equiv 0$. Then by integration we get $hL = Q_{k-1}$, where $Q_{k-1} = Q_{k-1}(z)$ is a polynomial of degree at most $k-1$. Putting $h = \dfrac{f-a}{L-a}$ we get

$$(f-a)L = (L-a)Q_{k-1}. \tag{2.10}$$

Since a is a polynomial, from (2.10) we see that f is an entire function. Hence h is an entire function having no zeros. We put $h = e^{\alpha}$, where α is an entire function, and so $f = a + h(L-a) = a + Q_{k-1} - ae^{\alpha}$ and $L = Q_{k-1}e^{-\alpha}$. It follows from the definition of L that $L = R(a', \alpha')e^{\alpha}$, where $R(a', \alpha')$ is a differential polynomial in a' and α'. Hence

$$R(a', \alpha')e^{2\alpha} = Q_{k-1}. \tag{2.11}$$

From (2.11) we see that $T(r, e^{\alpha}) = S(r, e^{\alpha})$, yielding a contradiction. Therefore $G \not\equiv 0$.

If h is a constant, say c, then $f - a = c(L - a)$, which is (i).

Now we suppose that h is nonconstant and $b \equiv 0$. Then by integration we get $ah = P_{k-1}$, where $P_{k-1} = P_{k-1}(z)$ is a polynomial of degree at most $k - 1$.

Since h is an entire function and a is a polynomial of degree at most $k - 1$, the equality $h = \dfrac{P_{k-1}}{a}$ implies that a is a factor of P_{k-1}, and hence

$$h = Q^{*}_{k-t}, \tag{2.12}$$

where $Q^{*}_{k-t} = Q^{*}_{k-t}(z)$ is a polynomial of degree at most $k - t$ $(t \geq 1)$.

If z_0 is a pole of f, then z_0 is a zero of h with multiplicity $k + p$, which is impossible by (2.12). So, f is an entire function, and hence h is an entire function having no zeros. Therefore from (2.12) we see that h is a constant, which is impossible.

Now we suppose that $b \not\equiv 0$. Let $A = 0$, then from (2.9) we get $\dfrac{G'}{G} + \dfrac{h'}{h} \equiv 0$. By integration we obtain $Gh = K$ and hence

$$(hL)^{(k)} = Kf^{(k)}, \tag{2.13}$$

where K is a nonzero constant.

Again, $\dfrac{A}{b} = \dfrac{h'}{h} + \dfrac{b'}{b} - \dfrac{F'}{F} = 0$ implies by integration $hb = MF$, and so

$$(ah)^{(k)} = Mf^{(k)}, \tag{2.14}$$

where M is a nonzero constant.

Since a is a polynomial and h is an entire function, we see from (2.14) that f is an entire function. So, h is an entire function having no zeros and we can put $h = e^{\alpha}$, where α is an entire function.

Integrating (2.13) k-times we get

$$hL = Kf + P^{*}_{k-1}, \tag{2.15}$$

where $P^{*}_{k-1} = P^{*}_{k-1}(z)$ is a polynomial of degree at most $k - 1$.

Since $hL = f - a + ah$, from (2.15) we get

$$(1 - K)f = a(1 - e^{\alpha}) + P^{*}_{k-1}. \tag{2.16}$$

If $K = 1$, from (2.16) we see that $e^\alpha = 1 + \frac{P^*_{k-1}}{a}$, which is impossible. Hence $K \neq 1$. Now from (2.16) we get

$$f = \frac{ae^\alpha}{K-1} - \frac{a + P^*_{k-1}}{K-1}. \tag{2.17}$$

Therefore from (1.1) we have

$$L = R(\alpha')e^\alpha, \tag{2.18}$$

where $R(\alpha')(\not\equiv 0)$ is a differential polynomial in α' with polynomial coefficients.

From (2.15) we obtain

$$L = \frac{Ka}{K-1} - \frac{Ka + P^*_{k-1}}{K-1}e^{-\alpha}. \tag{2.19}$$

It follows from (2.18) and (2.19) that

$$R(\alpha')e^{2\alpha} = \frac{Ka}{K-1}e^\alpha - \frac{Ka + P^*_{k-1}}{K-1}.$$

This implies $T(r, e^\alpha) = S(r, e^\alpha)$, yielding a contradiction. Therefore $A \not\equiv 0$.

Now observe that $A = b\left(\frac{h'}{h} + \frac{b'}{b} - \frac{F'}{F}\right)$ implies $m(r, A) = S(r, f)$. Also, the poles of A are contributed by: (i) the poles of $b = \frac{(ah)^{(k)}}{h}$, (ii) the poles of $\frac{h'}{h}$ and (iii) the poles of $\frac{F'}{F} = \frac{f^{(k+1)}}{f^{(k)}}$. Since h is entire and the zeros of h are precisely the poles of f, and each zero of h is of multiplicity $k + p$, we get

$$N(r, A) \leq (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(k)}) + S(r, f).$$

Therefore

$$T(r, A) \leq (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(k)}) + S(r, f). \tag{2.20}$$

From (2.9) and (2.20) we get

$$\begin{aligned} m(r, \frac{1}{F}) &\leq m(r, \frac{1}{A}) + m(r, G' + G\frac{h'}{h}) \leq T(r, A) + S(r, f) \\ &\leq (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(k)}) + S(r, f). \end{aligned}$$

So, by the first fundamental theorem, we obtain

$$T(r, f^{(k)}) \leq (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(k)}) + N(r, 0; f^{(k)}) + S(r, f),$$

which implies (ii). This completes the proof of Lemma 2.1. □

Lemma 2.2 ([2]). *Let k be a positive integer and f be a meromorphic function such that $f^{(k)}$ is not constant. Then either $\left(f^{(k+1)}\right)^{k+1} = c\left(f^{(k)} - \lambda\right)^{k+2}$ for some nonzero constant c or*

$$kN_{1)}(r,\infty;f) \le \overline{N}_{(2}(r,\infty;f) + N_{1)}(r,\lambda;f^{(k)}) + \overline{N}(r,0;f^{(k+1)}) + S(r,f),$$

where λ is a constant.

Lemma 2.3 ([10], p.39). *Let f be a nonconstant meromorphic function in the complex plane and let k be a positive integer. Then*

$$N(r,0;f^{(k)}) \le N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.4 ([8]). *Given a transcendental meromorphic function f and a constant $K > 1$. Then there exists a set $M(K)$ whose upper logarithmic density is at most*

$$\delta(K) = \min\{(2e^{K-1} - 1)^{-1}, (1 + e(K-1)\exp(e(1-K)))\}$$

such that for every positive integer k,

$$\limsup_{r\to\infty, r\notin M(K)} \frac{T(r,f)}{T(r,f^{(k)})} \le 3eK.$$

Lemma 2.5. *Let f be a transcendental meromorphic function such that $N(r,0;f^{(1)}) = S(r,f)$. If $f-a$ and $a_1 f^{(1)} - a$ share 0 CM, where $a = a(z)(\not\equiv 0,\infty)$ is a small function of f and a_1 is a nonzero constant, then*

$$N_{1)}(r,0;f^{(2)}) \le \overline{N}_{(2}(r,\infty;f) + S(r,f).$$

Proof. If $a+a' \equiv 0$, then using the method of [4] (pp. 349 - 351), we get $N_{1)}(r,0;f^{(2)}) = S(r,f)$, and the result follows. If $a + a' \not\equiv 0$, then again using the method of [4] (pp. 351 - 354), we get $N_{1)}(r,\infty;f) = S(r,f)$. Now by Lemma 2.3 we obtain

$$\begin{aligned} N(r,0;f^{(2)}) &\le N(r,0;f^{(1)}) + \overline{N}(r,\infty;f) + S(r,f) \\ &= \overline{N}_{(2}(r,\infty;f) + S(r,f). \end{aligned}$$

Since $N_{1)}(r,0;f^{(2)}) \le N(r,0;f^{(2)})$, the lemma is proved. □

Lemma 2.6 ([6]). *Let f be a transcendental meromorphic function and k be a positive integer. Then*

$$k\overline{N}(r,\infty;f) \le N(r,0;f^{(k)}) + (1+\varepsilon)N(r,\infty;f) + S(r,f),$$

where ε is any fixed positive number.

3. Proof of Theorem 1.1

Proof. First we verify that

$$\left(f^{(k+1)}\right)^{k+1} \neq c\left(f^{(k)}\right)^{k+2}, \tag{3.1}$$

where $c \neq 0$ is a constant. Indeed, if (3.1) does not hold, then we get

$$\left(\frac{f^{(k+1)}}{f^{(k)}}\right)^{k+1} = cf^{(k)}. \tag{3.2}$$

Differentiating (3.2) and then using (3.2) we obtain

$$\left(\frac{f^{(k+1)}}{f^{(k)}}\right)^{-2}\left(\frac{f^{(k+1)}}{f^{(k)}}\right)' = \frac{1}{k+1}.$$

Integrating twice we get

$$f^{(k)} = \frac{1}{\{Cz + D(k+1)\}^{k+1}},$$

where $C \neq 0$ and D are constants. This is impossible because f is transcendental.

Let $k \geq 2$. We suppose that

$$T(r, f^{(k)}) \leq (k+p+1)\overline{N}(r,\infty; f) + \overline{N}(r,0; f^{(k)}) + N(r,0; f^{(k)}) + S(r,f).$$

Since $N(r,0; f^{(k)}) = S(r,f)$, we get from above

$$T(r, f^{(k)}) \leq (k+p+1)\overline{N}(r,\infty; f) + S(r,f). \tag{3.3}$$

Also, from Lemma 2.6 we obtain for $0 < \varepsilon < \dfrac{k}{p+1} - 1$,

$$k\overline{N}(r,\infty; f) \leq (1+\varepsilon)N(r,\infty; f) + S(r,f).$$

Hence from (3.3) we obtain

$$m(r, f^{(k)}) + N(r,\infty; f) \leq \frac{p+1}{k}(1+\varepsilon)N(r,\infty; f) + S(r,f)$$

and so $m(r, f^{(k)}) + N(r,\infty; f) = S(r,f)$. Therefore

$$T(r, f^{(k)}) = S(r,f). \tag{3.4}$$

Let $M(K)$ be defined as in Lemma 2.4. By (3.4) we can choose a sequence $r_n \to \infty$ such that $r_n \notin M(K)$ and $\lim\limits_{n\to\infty} \dfrac{T(r_n, f^{(k)})}{T(r_n, f)} = 0$. This contradicts Lemma 2.4.

Next, let $k = 1$. We suppose

$$T(r, f^{(1)}) \leq 2\overline{N}(r,\infty; f) + \overline{N}(r,0; f^{(1)}) + N(r,0; f^{(1)}) + S(r,f).$$

Since $N(r,0; f^{(1)}) = S(r,f)$, we obtain

$$m(r, f^{(1)}) + N(r,\infty; f) \leq \overline{N}(r,\infty; f) + S(r,f)$$

and so

$$(3.5)\qquad m(r, f^{(1)}) + N_{(2}(r, \infty; f) = S(r, f).$$

By the second fundamental theorem we get in view of (3.5)

$$T(r, f^{(1)}) \le N(r, 1; f^{(1)}) + N(r, 0; f^{(1)}) + \overline{N}(r, \infty; f) - N(r, 0; f^{(2)}) + S(r, f)$$

and so

$$(3.6)\qquad m(r, 1; f^{(1)}) + N(r, 0; f^{(2)}) \le N_{1)}(r, \infty; f) + S(r, f).$$

Now by Lemma 2.2 and (3.5) we get for $\lambda = 0$

$$(3.7)\qquad N_{1)}(r, \infty; f) \le \overline{N}(r, 0; f^{(2)}) + S(r, f).$$

From (3.6) and (3.7) we get

$$(3.8)\qquad N_{(2}(r, 0; f^{(2)}) = S(r, f).$$

By (3.5), (3.8) and Lemma 2.5 we obtain

$$(3.9)\qquad N(r, 0; f^{(2)}) = S(r, f).$$

Hence by (3.5), (3.7) and (3.9) we get $N(r, \infty; f) = S(r, f)$, and so by (3.5) we have $T(r, f^{(1)}) = S(r, f)$, which is (3.4) for $k = 1$. Similarly using Lemma 2.4 we arrive at a contradiction. Therefore by Lemma 2.1 we obtain

$$f - a = \left(1 + \frac{P_{k-1}}{a}\right)(L - a).$$

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