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ON L^p-INTEGRABILITY OF A SPECIAL DOUBLE SINE SERIES FORMED BY ITS BLOCKS

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Abstract. In this paper we deal with a special double sine trigonometric series formed by its blocks. This type of trigonometric series is of particular interest since its blocks always are bounded, that is, under some additional assumptions the sum-function of such series always exists. We give some conditions under which such sum-function is integrable of power $p \in \{2, 3, \dots\}$, as well as is integrable with some natural weight.

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1. Introduction

Let $\Lambda_1 = \{n_i\}$ and $\Lambda_2 = \{r_j\}$ be two strictly increasing sequences of natural numbers $1 = n_1 < n_2 < n_3 < \cdots$ and $1 = r_1 < r_2 < r_3 < \cdots$ satisfying the conditions:

$$\sum_{i=1}^{\infty} \frac{1}{n_i} < +\infty \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{r_j} < +\infty.$$

Considering the special double sine series

$$\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{\sin kx \sin \ell y}{k\ell},$$

we form the following series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left| \sum_{k=n_i}^{n_{i+1}-1} \sum_{\ell=r_j}^{r_{j+1}-1} \frac{\sin kx \sin \ell y}{k\ell} \right|. \tag{1.1}$$

According to the well-known estimate

$$\left| \sum_{k=v}^{V} \frac{\sin kx}{k} \right| \le \frac{\pi}{vx}, \quad v \le V \le \infty, \quad 0 < x \le \pi, \tag{1.2}$$

the series (1.1) converges for all (x, y) and its sum $G_{\Lambda_1,\Lambda_2}(x, y)$ is a continuous function on $(0, \pi] \times (0, \pi]$. This fact is of particular interest and therefore this is the main reason why we have formed the series (1.1).

In the one-dimensional case such series has been considered by Telyakovskii [1] and Trigub [3]. In particular, Telyakovskii [2] has considered the question: when the sum-function $g_{\Lambda_1}(x)$ of the series

$$\sum_{k=1}^{\infty}\left|\sum_{k=n_i}^{n_{i+1}-1}\frac{\sin kx}{k}\right|$$

belongs to the spaces $L^p[0,\pi]$ for $p=2,3,\dots$?

Specifically, in [2] was proved the following theorem.

Theorem 1.1. *For any natural* $p=2,3,\dots$ *the function* $g_{\Lambda_1}(x)$ *belongs to the space* $L^p[0,\pi]$ *if the series* $\sum_{i=1}^{\infty}\frac{1}{n_i}m_i^{1-\frac{1}{p}}$ *is convergent, where* $m_i=\min(n_i,n_{i+1}-n_i+1)$.

In the same paper was considered the problem of integrability of the function $g_{\Lambda_1}(x)$ with weight $x^{-\gamma}$ under natural condition $0<\gamma<1$. Among others, the following result was proved in [2].

Theorem 1.2. *If for* $\gamma\in(0,1)$ *the series*

$$\sum_{i=1}^{\infty}\frac{1}{n_i}m_i^{\gamma}$$

is convergent, then the integral $\int_0^{\pi}\frac{1}{x^{\gamma}}g_{\Lambda_1}(x)dx$ *converges.*

Note that questions pertaining to trigonometric series formed by their blocks were considered in [4] - [6], and still receive considerable attention. The main aim of this paper is to extend the above results to two-dimensional case. In order to do this we will use the technique developed in [2], the estimate (1.2) and the following inequality (see [2], page 818):

$$u_i(x):=\left|\sum_{k=n_i}^{n_{i+1}-1}\frac{\sin kx}{k}\right|\le\frac{A}{n_i}\min\left(\frac{1}{x},m_i\right),\quad 0<x\le\pi. \tag{1.3}$$

where A is an absolute constant. Here and in the sequel we write $G_{\Lambda_1,\Lambda_2}\in L^p$, $p\ge1$, if the integral $\int_0^{\pi}\int_0^{\pi}|G_{\Lambda_1,\Lambda_2}(x,y)|^p dxdy$ is finite.

2. The main results

In this section we state and prove the main results of the paper. We first prove the following result.

Theorem 2.1. *For any natural $p = 2, 3, \dots$ the function G_{Λ_1,Λ_2} belongs to the space $L^p([0,\pi]\times[0,\pi])$ if the series*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(m_i s_j)^{1-\frac{1}{p}}}{n_i r_j}$$

is convergent, where $m_i = \min(n_i, n_{i+1} - n_i + 1)$ and $s_j = \min(r_j, r_{j+1} - r_j + 1)$.

Proof. For arbitrary natural numbers M and N we have

$$\int_0^\pi\int_0^\pi\left(\sum_{i=1}^{M}\sum_{j=1}^{N}u_i(x)u_j(y)\right)^p dxdy$$
$$=\int_0^\pi\int_0^\pi\sum_{i_1=1}^{M}u_{i_1}(x)\cdots\sum_{i_p=1}^{M}u_{i_p}(x)\sum_{j_1=1}^{N}u_{j_1}(y)\cdots\sum_{j_p=1}^{N}u_{j_p}(y)dxdy$$

$$=\sum_{i_1=1}^{M}\cdots\sum_{i_p=1}^{M}\sum_{j_1=1}^{N}\cdots\sum_{j_p=1}^{N}\int_0^\pi\int_0^\pi u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy. \tag{2.1}$$

Next, we split the square $[0,\pi]\times[0,\pi]$ into the rectangles $[0,\alpha]\times[0,\beta]$, $[0,\alpha]\times[\pi,\beta]$, $[\alpha,\pi]\times[0,\beta]$ and $[\alpha,\pi]\times[\beta,\pi]$, where α and β will be determined later in an appropriate way. Using the estimates (1.3) we can write

$$\int_0^\alpha\int_0^\beta u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy$$

$$\le A^{2p}\int_0^\alpha\int_0^\beta\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}dxdy = A^{2p}\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\alpha\beta, \tag{2.2}$$

$$\int_0^\alpha\int_\beta^\pi u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy \tag{2.3}$$
$$\le A^{2p}\int_0^\alpha\int_\beta^\pi\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{1}{r_{j_1}\cdots r_{j_p}}\frac{dxdy}{y^p} < \frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{A^{2p}}{r_{j_1}\cdots r_{j_p}}\frac{\alpha\beta^{1-p}}{p-1},$$

$$\int_\alpha^\pi\int_0^\beta u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy \tag{2.4}$$
$$\le A^{2p}\int_\alpha^\pi\int_0^\beta\frac{1}{n_{i_1}\cdots n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{dxdy}{x^p} < \frac{A^{2p}}{n_{i_1}\cdots n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{\alpha^{1-p}\beta}{p-1},$$

$$\int_\alpha^\pi\int_\beta^\pi u_{i_1}(x)\cdots u_{i_p}(x)u_{j_1}(y)\cdots u_{j_p}(y)dxdy \tag{2.5}$$
$$\le A^{2p}\int_\alpha^\pi\int_\beta^\pi\frac{1}{n_{i_1}\cdots n_{i_p}}\frac{1}{r_{j_1}\cdots r_{j_p}}\frac{dxdy}{(xy)^p} < \frac{A^{2p}}{n_{i_1}\cdots n_{i_p}r_{j_1}\cdots r_{j_p}}\frac{(\alpha\beta)^{1-p}}{(p-1)^2}$$

Inserting the estimates (2.2)-(2.5) into (2.1), we obtain

$$\int_0^\pi\int_0^\pi\left(\sum_{i=1}^{M}\sum_{j=1}^{N}u_i(x)u_j(y)\right)^p dxdy < A^{2p}\sum_{i_1=1}^{M}\cdots\sum_{i_p=1}^{M}\sum_{j_1=1}^{N}$$

$$\cdots\sum_{j_p=1}^{N}\left(\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\alpha\beta+\frac{m_{i_1}}{n_{i_1}}\cdots\frac{m_{i_p}}{n_{i_p}}\frac{1}{r_{j_1}\cdots r_{j_p}}\frac{\alpha\beta^{1-p}}{p-1}\right.$$

$$\left.+\frac{1}{n_{i_1}\cdots n_{i_p}}\frac{s_{j_1}}{r_{j_1}}\cdots\frac{s_{j_p}}{r_{j_p}}\frac{\alpha^{1-p}\beta}{p-1}+\frac{1}{n_{i_1}\cdots n_{i_p}r_{j_1}\cdots r_{j_p}}\frac{(\alpha\beta)^{1-p}}{(p-1)^2}\right). \tag{2.6}$$

Whence, choosing in (2.6) $\alpha=(m_{i_1}\cdots m_{i_p})^{-\frac{1}{p}}$ and $\beta=(s_{j_1}\cdots s_{j_p})^{-\frac{1}{p}}$, we find that

$$\int_0^\pi\int_0^\pi\left(\sum_{i=1}^{M}\sum_{j=1}^{N}u_i(x)u_j(y)\right)^p dx\,dy < 4A^{2p}\sum_{i_1=1}^{M}\cdots\sum_{i_p=1}^{M}\sum_{j_1=1}^{N}\cdots$$

$$\sum_{j_p=1}^{N}\frac{(m_{i_1}\cdots m_{i_p}s_{j_1}\cdots s_{j_p})^{1-\frac{1}{p}}}{n_{i_1}\cdots n_{i_p}r_{j_1}\cdots r_{j_p}} < 4A^{2p}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(m_is_j)^{1-\frac{1}{p}}}{n_ir_j}\right)^p.$$

Consequently, since the last series converges by assumption, the integrals

$$\int_0^\pi\int_0^\pi\left(\sum_{i=1}^{M}\sum_{j=1}^{N}u_i(x)u_j(y)\right)^p dx\,dy$$

are bounded by a quantity that is independent of M, N. Therefore, based on the double version of the Levi's theorem, we conclude that the function G_{Λ_1,Λ_2} belongs to the space $L^p([0,\pi]\times[0,\pi])$. □

The next result gives an answer to the following question: under what conditions the function G_{Λ_1,Λ_2} belongs to the space $L^p([0,\pi]\times[0,\pi])$ with weight $x^{-\gamma_1}y^{-\gamma_2}$, $\gamma_1,\gamma_2\in(0,1)$?

Theorem 2.2. *If for $\gamma_1,\gamma_2\in(0,1)$, the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{n_ir_j}m_i^{\gamma_1}s_j^{\gamma_2}$ is convergent, then the following integral converges*

$$\int_0^\pi\int_0^\pi\frac{G_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy.$$

Proof. Based on the uniform convergence of the series (1.1) we have

$$\int_0^\pi\int_0^\pi\frac{G_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^\pi\int_0^\pi\frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy.$$

Splitting the square $[0,\pi]\times[0,\pi]$ into the rectangles $[0,\alpha_i]\times[0,\beta_j]$, $[0,\alpha_i]\times[\pi,\beta_j]$, $[\alpha_i,\pi]\times[0,\beta_j]$ and $[\alpha_i,\pi]\times[\beta_j,\pi]$, where α_i and β_j are determined by

$$\alpha_i=\frac{1}{m_i}\quad\text{and}\quad\beta_j=\frac{1}{s_j}, \tag{2.7}$$

we find that

$$\int_0^{\alpha_i}\int_0^{\beta_j} \frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy \le A^2 \int_0^{\alpha_i}\int_0^{\beta_j} \frac{1}{x^{\gamma_1}y^{\gamma_2}}\frac{m_i s_j}{n_i r_j}dxdy = \frac{A^2}{(1-\gamma_1)(1-\gamma_2)}\frac{m_i s_j}{n_i r_j}\alpha_i^{1-\gamma_1}\beta_j^{1-\gamma_2},$$

$$\int_{\alpha_i}^{\pi}\int_0^{\beta_j} \frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy \le A^2 \int_{\alpha_i}^{\pi}\int_0^{\beta_j} \frac{1}{x^{\gamma_1}y^{\gamma_2}}\frac{s_j}{n_i x r_j}dxdy < \frac{A^2}{\gamma_1(1-\gamma_2)}\frac{s_j}{n_i r_j}\alpha_i^{-\gamma_1}\beta_j^{1-\gamma_2},$$

$$\int_0^{\alpha_i}\int_{\beta_j}^{\pi} \frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy \le A^2 \int_0^{\alpha_i}\int_{\beta_j}^{\pi} \frac{1}{x^{\gamma_1}y^{\gamma_2}}\frac{m_i}{n_i r_j y}dxdy < \frac{A^2}{(1-\gamma_1)\gamma_2}\frac{m_i}{n_i r_j}\alpha_i^{1-\gamma_1}\beta_j^{-\gamma_2},$$

$$\int_{\alpha_i}^{\pi}\int_{\beta_j}^{\pi} \frac{u_i(x)u_j(y)}{x^{\gamma_1}y^{\gamma_2}}dxdy \le A^2 \int_{\alpha_i}^{\pi}\int_{\beta_j}^{\pi} \frac{1}{x^{\gamma_1}y^{\gamma_2}}\frac{1}{n_i r_j xy}dxdy < \frac{A^2}{\gamma_1\gamma_2}\frac{1}{n_i r_j}\alpha_i^{-\gamma_1}\beta_j^{-\gamma_2}.$$

Finally, using (2.7) and the latest estimates, we obtain

$$\int_0^{\pi}\int_0^{\pi} \frac{G_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy < \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\Big(\frac{A^2}{(1-\gamma_1)(1-\gamma_2)}\frac{m_i s_j}{n_i r_j}\alpha_i^{1-\gamma_1}\beta_j^{1-\gamma_2}$$
$$+\frac{A^2}{\gamma_1(1-\gamma_2)}\frac{s_j}{n_i r_j}\alpha^{-\gamma_1}\beta^{1-\gamma_2}+\frac{A^2}{(1-\gamma_1)\gamma_2}\frac{m_i}{n_i r_j}\alpha^{1-\gamma_1}\beta^{-\gamma_2}$$
$$+\frac{A^2}{\gamma_1\gamma_2}\frac{1}{n_i r_j}\alpha^{-\gamma_1}\beta^{-\gamma_2}\Big) = C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{n_i r_j}m_i^{\gamma_1}s_j^{\gamma_2} < +\infty.$$

where $C = A^2 \cdot \max\left\{\frac{1}{(1-\gamma_1)(1-\gamma_2)}, \frac{1}{\gamma_1(1-\gamma_2)}, \frac{1}{(1-\gamma_1)\gamma_2}, \frac{1}{\gamma_1\gamma_2}\right\}$. □

The next statement supplements Theorem 2.1, and gives conditions under which the integral

$$\int_0^{\pi}\int_0^{\pi} \frac{G^p_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy$$

is convergent for $\gamma_1, \gamma_2 \in (0,1)$ and $p = 2, 3, \dots$.

Theorem 2.3. *If $p = 2, 3, \dots$ and $\gamma_1, \gamma_2 \in (1-p, 1)$, then the integral*

$$\int_0^{\pi}\int_0^{\pi} \frac{G^p_{\Lambda_1,\Lambda_2}(x,y)}{x^{\gamma_1}y^{\gamma_2}}dxdy$$

is convergent provided that the series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{n_i r_j}m_i^{1-\frac{1}{p}(1-\gamma_1)}s_j^{1-\frac{1}{p}(1-\gamma_2)}$$

is convergent.

Proof. Using a similar technique as in the proof Theorem 2.1 we have

$$\int_0^\pi \int_0^\pi \frac{1}{x^{\gamma_1} y^{\gamma_2}} \left(\sum_{i=1}^M \sum_{j=1}^N u_i(x) u_j(y) \right)^p dx\,dy$$

$$(2.8) \quad = \sum_{i_1=1}^M \cdots \sum_{i_p=1}^M \sum_{j_1=1}^N \cdots \sum_{j_p=1}^N \int_0^\pi \int_0^\pi \frac{1}{x^{\gamma_1} y^{\gamma_2}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy,$$

for all $p = 2, 3, \ldots$ and natural numbers M, N.

Again we split the square $[0, \pi] \times [0, \pi]$ into the rectangles $[0, \alpha] \times [0, \beta]$, $[0, \alpha] \times [\pi, \beta]$, $[\alpha, \pi] \times [0, \beta]$ and $[\alpha, \pi] \times [\beta, \pi]$, where α and β are determined as in Theorem 2.1.

Using the estimates (1.3) and taking into account that $\gamma_1 \wedge \gamma_2 \in (1 - p, 1)$, we can write

$$\int_0^\alpha \int_0^\beta \frac{1}{x^{\gamma_1} y^{\gamma_2}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy$$

$$\leq A^{2p} \int_0^\alpha \int_0^\beta \frac{1}{x^{\gamma_1} y^{\gamma_2}} \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} dx dy$$

$$(2.9) \quad = A^{2p} \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} \frac{\alpha^{1-\gamma_1} \beta^{1-\gamma_2}}{(1-\gamma_1)(1-\gamma_2)},$$

$$\int_0^\alpha \int_\beta^\pi \frac{1}{x^{\gamma_1} y^{\gamma_2}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy$$

$$\leq A^{2p} \int_0^\alpha \int_\beta^\pi \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{1}{r_{j_1} \cdots r_{j_p}} \frac{dx dy}{x^{\gamma_1} y^{\gamma_2 + p}}$$

$$(2.10) \quad < \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{A^{2p}}{r_{j_1} \cdots r_{j_p}} \frac{\alpha^{1-\gamma_1} \beta^{1-\gamma_2-p}}{(1-\gamma_1)(\gamma_2 + p - 1)},$$

$$\int_\alpha^\pi \int_0^\beta \frac{1}{x^{\gamma_1} y^{\gamma_2}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy$$

$$\leq A^{2p} \int_\alpha^\pi \int_0^\beta \frac{1}{n_{i_1} \cdots n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} \frac{dx dy}{x^{\gamma_1 + p} y^{\gamma_2}}$$

$$(2.11) \quad < \frac{A^{2p}}{n_{i_1} \cdots n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} \frac{\alpha^{1-\gamma_1-p} \beta^{1-\gamma_2}}{(\gamma_1 + p - 1)(1-\gamma_2)},$$

and

$$\int_\alpha^\pi \int_\beta^\pi \frac{1}{x^{\gamma_1} y^{\gamma_2}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy$$

$$\leq A^{2p} \int_\alpha^\pi \int_\beta^\pi \frac{1}{n_{i_1} \cdots n_{i_p}} \frac{1}{r_{j_1} \cdots r_{j_p}} \frac{dx dy}{x^{\gamma_1 + p} y^{\gamma_2 + p}}$$

$$< \frac{A^{2p}}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}} \frac{\alpha^{1-\gamma_1-p} \beta^{1-\gamma_2-p}}{(\gamma_1 + p - 1)(\gamma_2 + p - 1)}. \tag{2.12}$$

The above estimates along with

$$\alpha = \frac{1}{(m_{i_1} \cdots m_{i_p})^{\frac{1}{p}}} \quad \text{and} \quad \beta = \frac{1}{(s_{j_1} \cdots s_{j_p})^{\frac{1}{p}}}$$

imply

$$\int_0^\pi \int_0^\pi \frac{1}{x^{\gamma_1} y^{\gamma_2}} \left(\sum_{i=1}^M \sum_{j=1}^N u_i(x) u_j(y) \right)^p dx dy$$

$$< A(p, \gamma_1, \gamma_2) \sum_{i_1=1}^M \cdots \sum_{i_p=1}^M \sum_{j_1=1}^N \cdots \sum_{j_p=1}^N \frac{(m_{i_1} \cdots m_{i_p})^{1-\frac{1}{p}(1-\gamma_1)} (s_{j_1} \cdots s_{j_p})^{1-\frac{1}{p}(1-\gamma_2)}}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}},$$

where $A(p, \gamma_1, \gamma_2)$ is a constant that depends only on p, γ_1, and γ_2.

Hence,

$$\int_0^\pi \int_0^\pi \frac{1}{x^{\gamma_1} y^{\gamma_2}} \left(\sum_{i=1}^M \sum_{j=1}^N u_i(x) u_j(y) \right)^p dx dy$$

$$< A(p, \gamma_1, \gamma_2) \left(\sum_{i=1}^\infty \sum_{j=1}^\infty \frac{m_i^{1-\frac{1}{p}(1-\gamma_1)} s_j^{1-\frac{1}{p}(1-\gamma_2)}}{n_i r_j} \right)^p.$$

Finally, the use of the double version of the Levi's theorem implies the statement of the theorem. □

It is clear that the conditions $\gamma_1, \gamma_2 > 1 - p$ in Theorem 2.3 are essential, therefore in the next theorem we examine the boundary case $\gamma_1, \gamma_2 = 1 - p$.

Theorem 2.4. *If $p = 2, 3, \ldots$ and $\gamma_1, \gamma_2 = 1 - p$, then the integral*

$$\int_0^\pi \int_0^\pi \frac{G^p_{\Lambda_1, \Lambda_2}(x, y)}{(xy)^{1-p}} dx dy$$

is convergent provided that the series

$$\sum_{i=1}^\infty \sum_{j=1}^\infty \frac{1}{n_i r_j} (\log m_i)(\log s_j)$$

is convergent.

Proof. Observe first that in this boundary case the equality (2.8) reduces to the following:

$$\int_0^\pi \int_0^\pi \frac{1}{(xy)^{1-p}} \left(\sum_{i=1}^{M} \sum_{j=1}^{N} u_i(x) u_j(y) \right)^p dx dy \tag{2.13}$$

$$= \sum_{i_1=1}^{M} \cdots \sum_{i_p=1}^{M} \sum_{j_1=1}^{N} \cdots \sum_{j_p=1}^{N} \int_0^\pi \int_0^\pi \frac{1}{(xy)^{1-p}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy.$$

Also, for $\gamma_1, \gamma_2 = 1-p$ the estimates (2.9)-(2.12) take the following forms:

$$\int_0^\alpha \int_0^\beta \frac{1}{(xy)^{1-p}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy$$
$$\leq A^{2p} \int_0^\alpha \int_0^\beta \frac{1}{(xy)^{1-p}} \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} dx dy = A^{2p} \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} \frac{(\alpha\beta)^p}{p^2},$$

$$\int_0^\alpha \int_\beta^\pi \frac{1}{(xy)^{1-p}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy$$
$$\leq A^{2p} \int_0^\alpha \int_\beta^\pi \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{1}{r_{j_1} \cdots r_{j_p}} \frac{dx dy}{(xy)^{1-p}} < \frac{m_{i_1}}{n_{i_1}} \cdots \frac{m_{i_p}}{n_{i_p}} \frac{A^{2p}}{r_{j_1} \cdots r_{j_p}} \frac{\alpha^p \log \frac{\pi}{\beta}}{p},$$

$$\int_\alpha^\pi \int_0^\beta \frac{1}{(xy)^{1-p}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy$$
$$\leq A^{2p} \int_\alpha^\pi \int_0^\beta \frac{1}{n_{i_1} \cdots n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} \frac{dx dy}{(xy)^{1-p}} < \frac{A^{2p}}{n_{i_1} \cdots n_{i_p}} \frac{s_{j_1}}{r_{j_1}} \cdots \frac{s_{j_p}}{r_{j_p}} \frac{\beta^p \log \frac{\pi}{\alpha}}{p},$$

$$\int_\alpha^\pi \int_\beta^\pi \frac{1}{(xy)^{1-p}} u_{i_1}(x) \cdots u_{i_p}(x) u_{j_1}(y) \cdots u_{j_p}(y) dx dy$$
$$\leq A^{2p} \int_\alpha^\pi \int_\beta^\pi \frac{1}{n_{i_1} \cdots n_{i_p}} \frac{1}{r_{j_1} \cdots r_{j_p}} \frac{dx dy}{(xy)^{1-p}} < \frac{A^{2p}}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}} \log \frac{\pi}{\alpha} \log \frac{\pi}{\beta},$$

respectively.

Next, specifying $\alpha = (m_{i_1} \cdots m_{i_p})^{-\frac{1}{p}}$ and $\beta = (s_{j_1} \cdots s_{j_p})^{-\frac{1}{p}}$ we obviously have

$$\log \frac{\pi}{\alpha} = \log \pi + \frac{1}{p} \log(m_{i_1} \cdots m_{i_p}) \quad \text{and} \quad \log \frac{\pi}{\beta} = \log \pi + \frac{1}{p} \log(s_{j_1} \cdots s_{j_p}).$$

Using these equalities, the above estimates and the equality (2.13)), we obtain

$$\int_0^\pi \int_0^\pi \frac{1}{(xy)^{1-p}} \left(\sum_{i=1}^{M} \sum_{j=1}^{N} u_i(x) u_j(y) \right)^p dxdy$$

$$< \sum_{i_1=1}^{M} \cdots \sum_{i_p=1}^{M} \sum_{j_1=1}^{N} \cdots \sum_{j_p=1}^{N} \frac{A^{2p}}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}} \left\{ \left(\frac{1}{p} + \log\pi \right)^2 \right.$$

$$\left. + \left(\frac{1}{p^2} + \log\pi \right) \left[\sum_{\nu=1}^{p} \log(m_{i_\nu}) + \sum_{\mu=1}^{p} \log(s_{j_\mu}) \right] + \frac{1}{p^2} \sum_{\nu=1}^{p} \sum_{\mu=1}^{p} \log(m_{i_\nu}) \log(s_{j_\mu}) \right\}.$$

Therefore, we have

$$\int_0^\pi \int_0^\pi \frac{G^p_{\Lambda_1,\Lambda_2}(x,y)}{(xy)^{1-p}} dxdy$$

$$\leq K A^{2p} \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \sum_{j_1=1}^{\infty} \cdots \sum_{j_p=1}^{\infty} \frac{1}{n_{i_1} \cdots n_{i_p} r_{j_1} \cdots r_{j_p}} \sum_{\nu=1}^{p} \sum_{\mu=1}^{p} \log(m_{i_\nu}) \log(s_{j_\mu})$$

$$\leq K A^{2p} \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{n_i r_j} \right)^{p-1} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\log(m_i) \log(s_j)}{n_i r_j},$$

where K is an absolute positive constant. The proof is completed. □

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