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HYPERSURFACES OF A FINSLER SPACE WITH A SPECIAL (α, β)-METRIC

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Abstract. In the present paper we study the Finslerian hypersurfaces of a Finsler space with a special (α, β) metric, and examine the hypersurfaces of this special metric as a hyperplane of first, second and third kinds.

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1. Introduction

We consider an n-dimensional Finsler space $F^n = (M^n, L)$, that is, a pair consisting of an n-dimensional differentiable manifold M^n equipped with a Fundamental function L. The concept of an (α, β) metric, denoted by $L(\alpha, \beta)$, was introduced by M. Matsumoto [5], and later on has been studied by many authors (see [1 - 5, 8 - 9] and references therein). Well-known examples of (α, β) metrics are the Rander's metric $(\alpha+\beta)$, the Kropina metric $\frac{\alpha^2}{\beta}$ and the generalized Kropina metric $\frac{\alpha^{m+1}}{\beta^m}(m \neq 0, -1)$. Recall that a Finsler metric $L(x, y)$ is called an (α, β) metric if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is an 1−form on M^n.

We consider a special Finsler Space $F^n = \{M^n, L(\alpha, \beta)\}$ with the metric $L(\alpha, \beta)$ given by

$$L(\alpha, \beta) = \alpha + \beta + \frac{\alpha^2}{(\alpha - \beta)}. \tag{1.1}$$

Differentiating equation (2.1) partially with respect to α and β, we get

$$L_\alpha = \frac{2\alpha^2 + \beta^2 - 4\alpha\beta}{(\alpha-\beta)^2}, \quad L_\beta = \frac{2\alpha^2 + \beta^2 - 2\alpha\beta}{(\alpha-\beta)^2},$$

$$L_{\alpha\alpha} = \frac{2\beta^2}{(\alpha-\beta)^3}, \quad L_{\beta\beta} = \frac{2\alpha^2}{(\alpha-\beta)^3}, \quad L_{\alpha\beta} = \frac{-2\alpha\beta}{(\alpha-\beta)^3},$$

where

$$L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}, \quad L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}, \quad L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}.$$

In the Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$ the normalized element of the support $l_i = \dot\partial_i L$ and the angular metric tensor h_{ij} are given by the following formulas (see [5]):

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i,$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j,$$

where $Y_i = a_{ij} y^j$. For the fundamental function (2.1) the constants p, q_0, q_{-1} and q_{-2} in the last equation are given by the following formulas:

(1.2) $$p = L L_\alpha \alpha^{-1} = \frac{4\alpha^4 - \beta^4 - 8\alpha^3\beta + 4\alpha\beta^3}{\alpha(\alpha - \beta)^3},$$

$$q_0 = L L_{\beta\beta} = \frac{4\alpha^4 - 2\alpha^2\beta^2}{(\alpha - \beta)^4}, \quad q_{-1} = L L_{\alpha\beta} \alpha^{-1} = \frac{2\beta^3 - 4\alpha^2\beta}{(\alpha - \beta)^4},$$

$$q_{-2} = L \alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \frac{-4\alpha^5 - 2\alpha^2\beta^3 + 8\alpha^4\beta + \alpha\beta^4 - \beta^5}{\alpha^3(\alpha - \beta)^4}.$$

The fundamental metric tensor $g_{ij} = \frac{1}{2}\dot\partial_i \dot\partial_j L^2$ for $L = L(\alpha, \beta)$ is given by the following formula (see [4, 5]):

(1.3) $$g_{ij} = p a_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j,$$

where

(1.4) $$p_0 = q_0 + L_\beta^2 = \frac{8\alpha^4 + \beta^4 + 6\alpha^2\beta^2 - 8\alpha^3\beta - 4\alpha\beta^3}{(\alpha - \beta)^4},$$

$$p_{-1} = q_{-1} + L^{-1} p L_\beta = \frac{2\alpha\beta^3 - 4\alpha^3\beta + (2\alpha^2 + \beta^2 - 2\alpha\beta)^2}{\alpha(\alpha - \beta)^4},$$

$$p_{-2} = q_{-2} + p^2 L^{-2} = \frac{2\beta^4 + 8\alpha^2\beta^2 - 6\alpha\beta^3 + \frac{\beta^5}{\alpha}}{\alpha^2(\alpha - \beta)^4}.$$

The reciprocal tensor g^{ij} of g_{ij} is given by the following formula (see [4, 5]):

(1.5) $$g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j,$$

where $b^i = a^{ij} b_j$, $b^2 = a_{ij} b^i b^j$, and

(1.6) $$s_0 = \frac{1}{\tau p}\{p p_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\},$$

$$s_{-1} = \frac{1}{\tau p}\{p p_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\},$$

$$s_{-2} = \frac{1}{\tau p}\{p p_{-2} + (p_0 p_{-2} - p_{-1}^2) b^2\},$$

$$\tau = p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2).$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by formula (see [10]):

$$2pC_{ijk} = p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k, \tag{1.7}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \quad m_i = b_i - \alpha^{-2}\beta Y_i. \tag{1.8}$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\{^i_{jk}\}$ be the component of the Christoffel symbol of the associated Riemannian space R^n, and let ∇_k be the covariant derivative with respect to x^k relative to this Christoffel symbol. Define

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji}, \tag{1.9}$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^i_{jk})$ be the Cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^i_{jk}\}$ of the special Finsler space F^n is given by

$$\begin{aligned} D^i_{jk} = {} & B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ & - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + \\ & C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji}, \\ B_{ij} = \frac{1}{2}\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}, \quad B^k_i = g^{kj} B_{ji}, \\ A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \\ \lambda^m = B^m E_{00} + 2B_0 F^m_0, \quad B_0 = B_i y^i, \end{aligned} \tag{1.11}$$

and '0' denotes the contraction with y^i except for the quantities p_0, q_0 and s_0.

2. Induced Cartan connection

Let F^{n-1} be a hypersurface of F^n given by the equation $x^i = x^i(u^\alpha)$, where $\alpha = 1, 2, 3...(n-1)$. The element of the support y^i of F^n is taken to be tangential to F^{n-1}, that is, it is given by formula (see [6]):

$$y^i = B^i_\alpha(u) v^\alpha. \tag{2.1}$$

The metric tensor $g_{\alpha\beta}$ and the hv-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \quad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma,$$

and at each point (u^α) of F^{n-1}, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}\{x(u,v),y(u,v)\}B^i_\alpha N^j = 0, \quad g_{ij}\{x(u,v),y(u,v)\}N^iN^j = 1.$$

The angular metric tensor $h_{\alpha\beta}$ of the hypersurface is determined by formulas:

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \quad h_{ij}B^i_\alpha N^j = 0, \quad h_{ij}N^iN^j = 1. \tag{2.2}$$

The inverse (B^α_i, N_i) of (B^i_α, N^i) is given by

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^\alpha_i N^i = 0, \quad B^i_\alpha N_i = 0,$$
$$N_i = g_{ij}N^j, \quad B^k_i = g^{kj}B_{ji}, \quad B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j.$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^{*i}_{jk})$ is given by formulas (see [6]):

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma,$$
$$G^\alpha_\beta = B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta), \quad C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma,$$

where

$$M_{\beta\gamma} = N_iC^i_{jk}B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}, \quad H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta),$$

and

$$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial v^\gamma}, \quad B^i_{0\beta} = B^i_{\alpha\beta}v^\alpha.$$

The quantities $M_{\beta\gamma}$ and H_β are called the second fundamental v-tensor and the normal curvature vector, respectively (see [6]). The second fundamental h-tensor $H_{\beta\gamma}$ is defined as follows (see [6]):

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{2.3}$$

where

$$M_\beta = N_iC^i_{jk}B^j_\beta N^k. \tag{2.4}$$

The relative $h-$ and $v-$covariant derivatives of the projection factor B^i_α with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \quad B^i_\alpha|_\beta = M_{\alpha\beta}N^i.$$

It easily follows form equation (3.3) that $H_{\beta\gamma}$ generally is not symmetric and satisfies the equation

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{2.5}$$

implying that

$$H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{2.6}$$

The following lemmas, due to Matsumoto [6], will be used in Section 4

Lemma 2.1. *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.*

Lemma 2.2. *A hypersurface $F^{(n-1)}$ is a hyperplane of the first kind with respect to the connection CΓ if and only if $H_\alpha = 0$.*

Lemma 2.3. *A hypersurface $F^{(n-1)}$ is a hyperplane of the second kind with respect to the connection CΓ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 2.4. *A hypersurface $F^{(n-1)}$ is a hyperplane of the third kind with respect to the connection CΓ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.*

3. A hypersurface $F^{(n-1)}(c)$ of a special Finsler space

Let us consider a Finsler space with the metric $L = \alpha + \beta + \frac{\alpha^2}{\alpha-\beta}$, where the vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now we consider a hypersurface $F^{(n-1)}(c)$ given by the equation $b(x) = c$, where c is a constant (see [10]). From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$ we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0,$$

$$\frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0,$$

$$b_i B^i_\alpha = 0,$$

showing that $b_i(x)$ is a covariant component of a normal vector field of the hypersurface $F^{n-1}(c)$. Further, we have

(3.1) $$b_i B^i_\alpha = 0 \quad \text{and} \quad b_i y^i = 0, \quad \text{that is,} \quad \beta = 0,$$

and the induced matric $L(u, v)$ of $F^{n-1}(c)$ is given by

(3.2) $$L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta,$$

which is a Riemannian metric.

Taking $\beta = 0$ in the equations (2.2), (2.3) and (2.5) we get

(3.3) $$p = 4, \quad q_0 = 4, \quad q_{-1} = 0, \quad q_{-2} = -4\alpha^{-2},$$

$$p_0 = 8, \quad p_{-1} = 4\alpha^{-1}, \quad p_{-2} = 0, \quad \tau = 16(1 + b^2),$$

$$s_0 = \frac{1}{4(1+b^2)}, \quad s_{-1} = \frac{1}{4\alpha(1+b^2)}, \quad s_{-2} = \frac{-b^2}{4\alpha^2(1+b^2)}.$$

From (2.4) we get

(3.4) $$g^{ij} = \frac{1}{4}a^{ij} - \frac{1}{4(1+b^2)}b^i b^j - \frac{1}{4\alpha(1+b^2)}(b^i y^j + b^j y^i) + \frac{b^2}{4\alpha^2(1+b^2)}y^i y^j.$$

Thus, from (4.1) and (4.4), along $F^{n-1}(c)$ we obtain

$$g^{ij}b_ib_j = \frac{b^2}{4(1+b^2)}.$$

Therefore we have

$$b_i(x(u)) = \sqrt{\frac{b^2}{4(1+b^2)}}N_i, \quad b^2 = a^{ij}b_ib_j. \tag{3.5}$$

where b is the length of the vector b^i.

Next, from (4.4) and (4.5) we get

$$b^i = a^{ij}b_j = \sqrt{\frac{4b^2(1+b^2)}{\{1+b^2(1-\alpha^2)\}^2}}N^i + \frac{\alpha b^2 y^i}{1+b^2(1-\alpha^2)}. \tag{3.6}$$

Thus, we have the following result.

Theorem 3.1. *In a special Finsler hypersurface $F^{(n-1)}(c)$, the induced Riemannian metric is given by (4.2) and the scalar function $b(x)$ is given by (4.5) and (4.6).*

Now, observe that the angular metric tensor h_{ij} and the metric tensor g_{ij} of F^n are given by formulas:

$$h_{ij} = 4a_{ij} + 4b_ib_j - \frac{4}{\alpha^2}Y_iY_j \quad \textit{and} \quad g_{ij} = 4a_{ij} + 8b_ib_j + \frac{4}{\alpha}(b_iY_j + b_jY_i. \tag{3.7}$$

From equations (4.1), (4.7) and (3.2) it follows that if $h^{(a)}_{\alpha\beta}$ denotes the angular metric tensor of the Riemannian $a_{ij}(x)$, then along $F^{n-1}_{(c)}$ we have $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. Thus, along $F^{n-1}_{(c)}$ we have $\frac{\partial\alpha}{\partial\beta} = \frac{2\beta}{\alpha}$, and hence from equation (2.6) we get

$$Y_1 = \frac{4\beta}{\alpha}, \qquad m_i = b_i.$$

Therefore, in the special Finsler hypersurface $F^{(n-1)}_{(c)}$, the hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{2\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + \frac{6}{\alpha}b_ib_jb_k. \tag{3.8}$$

Next, it follows from (3.2), (3.3), (3.5), (4.1) and (4.8) that

$$M_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\frac{b^2}{4(1+b^2)}}h_{\alpha\beta} \quad \textit{and} \quad M_\alpha = 0. \tag{3.9}$$

Therefore, it follows from equation (3.6) that $H_{\alpha\beta}$ is symmetric. Thus, we have the following result.

Theorem 3.2. *The second fundamental v-tensor of the special Finsler hypersurface $F^{(n-1)}_{(c)}$ is given by (4.9) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now, from (4.1) we have $b_iB^i_\alpha = 0$, and hence

$$b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0.$$

Therefore, using the equality $b_{i|\beta} = b_{i|j}B^j_\beta + b_i|_j N^j H_\beta$, from (3.5) we obtain

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta}N^i = 0. \tag{3.10}$$

Since $b_i|_j = -b_h C^h_{ij}$, we get $b_{i|j}B^i_\alpha N^j = 0$. Therefore, taking into account that $b_{i|j}$ is symmetric, from equation (4.10) we have

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0. \tag{3.11}$$

Next, contracting (4.11) with v^β and using (3.1), we get

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_\alpha + b_{i|j}B^i_\alpha y^j = 0. \tag{3.12}$$

Again contracting by v^α the equation (4.12) and using (3.1), we have

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_0 + b_{i|j}y^i y^j = 0. \tag{3.13}$$

It follows from Lemmas 3.1 and 3.2 that the hypersuface $F^{(n-1)}_{(c)}$ is a hyperplane of first kind if and only if $H_0 = 0$. Thus, in view of (4.13), it is obvious that $F^{n-1}_{(c)}$ is a hyperplane of first kind if and only if $b_{i|j}y^i y^j = 0$. On the other hand, $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n is defined on y^i, but $b_{ij} = \nabla_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^i_{jk}\}$ constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i.

Below we consider the difference $b_{i|j} - b_{ij}$, where $b_{ij} = \nabla_j b_i$. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^i_{jk}\}$ is given by (2.10), and since b_i is a gradient vector, then from (2.9) we have $E_{ij} = b_{ij}$, $F_{ij} = 0$ *and* $F^i_j = 0$. Thus, (2.10) reduces to the following

$$\begin{aligned} D^i_{jk} = {} & B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - \\ & C^i_{km}A^m_j + C_{jkm}A^m_s g^{is} + \lambda^s(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} & B_i = 8b_i + 4\alpha^{-1}Y_i, \quad B^i = (\frac{1}{1+b^2})b^i + \frac{1}{\alpha(1+b^2)}y^i, \\ & \lambda^m = B^m b_{00}, \quad B_{ij} = \frac{2}{\alpha}(a_{ij} - \frac{Y_i Y_j}{\alpha^2}) + \frac{12}{\alpha}b_i b_j, \\ & B^i_j = \frac{1}{2\alpha}(\delta^i_j - \alpha^{-1}y^i Y_j) + \frac{5}{2\alpha(1+b^2)}b^i b_j - \\ & \frac{(1+6b^2)}{2\alpha^2(1+b^2)}b_j Y^i, \quad A^m_k = B^m_k b_{00} + B^m b_{k0}. \end{aligned} \tag{3.15}$$

In view of (4.3) and (4.4), the relation in (2.11) becomes to by virtue of (4.15) we have $B^i_0 = 0$, $B_{i0} = 0$ which leads $A^m_0 = B^m b_{00}$.

Now contracting (4.14) by y^k we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}.$$

Again contracting the above equation with respect to y^j we obtain

$$D^i_{00} = B^i b_{00} = \{(\tfrac{1}{1+b^2})b^i + \tfrac{1}{\alpha(1+b^2)} y^i\} b_{00}.$$

In view of (4.1), along $F^{(n-1)}_{(c)}$ we get

$$(3.16) \qquad b_i D^i_{j0} = \frac{b^2}{(1+b^2)} b_{j0} + \frac{(1+6b^2)}{2\alpha(1+b^2)} b_j b_{00} + \frac{1}{(1+b^2)} b_i b^m C^i_{jm} b_{00}.$$

Now we contract (4.16) by y^j to obtain

$$(3.17) \qquad b_i D^i_{00} = \frac{1}{(1+b^2)} b_{00}.$$

From (3.3). (4.5), (4.6), (4.9) and $M_\alpha = 0$ we obtain

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$$

Thus. the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and the equations (4.16), (4.17) give

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \tfrac{1}{1+b^2} b_{00}.$$

Consequently. the equations (4.12) and (4.13) can be written as follows:

$$(3.18) \qquad \sqrt{\frac{b^2}{4(1+b^2)}} H_\alpha + \frac{1}{1+b^2} b_{i0} B^i_\alpha = 0,$$

$$\sqrt{\frac{b^2}{4(1+b^2)}} H_0 + \frac{1}{1+b^2} b_{00} = 0.$$

Thus, the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact that $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Therefore. we can write

$$(3.19) \qquad 2b_{ij} = b_i c_j + b_j c_i.$$

Now from (4.1) and (4.19) we get

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} B^i_\alpha y^j = 0.$$

It follows from (4.18) that $H_\alpha = 0$, and hence in view of (4.15) and (4.19) we get $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{2}{\alpha} h_{\alpha\beta}$.

Next, we use the equations (3.3), (4.4) (4.6), (4.9) and (4.14) to obtain

$$(3.20) \qquad b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^2 (4+3b^2)}{16\alpha(1+b^2)^2} h_{\alpha\beta}.$$

Thus, the equation (4.11) reduces to the following

$$\sqrt{\frac{b^2}{4(1+b^2)}}H_{\alpha\beta} + \frac{b^2(4+3b^2)}{16\alpha(1+b^2)^2}h_{\alpha\beta} = 0. \tag{3.21}$$

and hence the hypersurface $F^{n-1}_{(c)}$ is umbilic. Thus, we have the following result.

Theorem 3.3. *A necessary and sufficient condition for $F^{(n-1)}_{(c)}$ to be a hyperplane of first kind is (4.19). In this case the second fundamental tensor of $F^{n-1}_{(c)}$ is proportional to its angular metric tensor.*

Now, taking into account that by Lemma 3.3, $F^{(n-1)}_{(c)}$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$, from (4.20) we get $c_0 = c_i(x)y^i = 0$. Therefore, there exists a function $\psi(x)$ such that $c_i(x) = \psi(x)b_i(x)$, and, in view of (4.19), we get $2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$. The last equation can also be written as follows $b_{ij} = \psi(x)b_ib_j$. Thus, we have the following result.

Theorem 3.4. *A necessary and sufficient condition for a hypersurface $F^{(n-1)}_{(c)}$ to be a hyperplane of second kind is (4.21).*

Putting together Lemma 3.4 and formula (4.9), we conclude that $F^{n-1}_{(c)}$ is not a hyperplane of third kind. Thus, we have the following result.

Theorem 3.5. *The hypersurface $F^{(n-1)}_{(c)}$ is not a hyperplane of the third kind.*

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