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THE PARTICLE STRUCTURE OF THE QUANTUM MECHANICAL BOSE AND FERMI GAS

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Abstract. In the framework of von Neumann's description of measurements of discrete quantum observable we establish a one-to-one correspondence between symmetric statistical operators W of quantum mechanical systems and classical point processes κ_W, thereby giving a particle picture of indistinguishable quantum particles. This holds true under irreducibility assumptions if we fix the underlying complete orthonormal system. The method of the Campbell measure is developed for such statistical operators: it is shown that the Campbell measure of a statistical operator W coincides with the Campbell measure of the corresponding point process κ_W. Moreover, again under irreducibility assumptions, a symmetric statistical operator is completely determined by its Campbell measure. The method of the Campbell measure then is used to characterize Bose-Einstein and Fermi-Dirac statistical operators. This is an elementary introduction into the work of Fichtner and Freudenberg [10, 11] combined with the quantum mechanical investigations of [2] and the corresponding point process approach of [30]. It is based on the classical work of von Neumann [22], Segal, Cook and Chaiken [28, 8, 7] as well as Moyal [18].

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1. Introduction

We consider quantum statistical states and ask for a precise particle picture of them. Under irreducibility assumptions we develop a one-to-one correspondence between symmetric statistical operators W of finite quantum mechanical systems and point processes κ_W, thereby giving a particle picture of indistinguishable quantum particles. This is done by developing a disintegration theory for such statistical operators in complete analogy to the decomposition of classical into conditional probabilities.

We also need the *method of the Campbell measure*, which is well known for point processes, and which is developed here for statistical operators. (This is inspired by the work of Fichtner, see for instance [12], and Liebscher [16].) We show that the Campbell measure of a symmetric statistical operator W coincides with the usual

Campbell measure of its law κ_W, moreover, under irreducibility assumptions, W is then completely determined by its Campbell measure.

We then present the point processes which correspond to the quantum statistical operators of Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac in the case of a fixed number of particles. Surprisingly, only the point process belonging to the Maxwell-Boltzmann statistical operator is really known and has been considered in probability theory until now.

We then extend our considerations to systems with a random number of particles and therefore work on Fock spaces. In this framework the Poisson point process belongs to the Maxwell-Boltzmann statistical operator. Next the symmetric Bose-Einstein and Fermi-Dirac statistical operators are constructed together with their associated point processes. Since these statistical operators are determined by their Campbell measures, and since the Campbell measures coincide for statistical operators and their point processes, we shall investigate the Campbell measure of these point processes.

As a result of the application of the method of Campbell measures we find that the point processes belonging to Bose-Einstein and Fermi-Dirac statistical operators respectively are given by Papangelou processes with explicitly given conditional intensity kernels. They are called here *Pólya sum* and *Pólya difference processes* respectively. The corresponding random fields are of first order and have independent increments. The distribution of the field variables, which represent the number of particles in a given region, are explicitly known. These results have been shown in [20]. Thus these processes have all characteristic properties of an ideal gas. In this way we obtain detailed informations about the point processes and thereby about the correponding statistical operators.

We stress here the point of view that for the developement of a full interacting theory of quantum gases one should start with the corresponding ideal gas and then modify this by means of a Boltzmann factor to include an interaction between the particles. (First steps in this direction can be found in [20].)

Historically the first attempts to unify quantum mechanics with point process theory can be found in the work of Fock [13], Segal [28], Cook [8] and Chaiken [7] and then, more systematically, in the work of Moyal [18]. For a more recent contribution to the construction of Bose and Fermi processes from the point of view of quantum mechanics we refer to Tamura and Ito [29].

Note added in February 2015. Unpublished versions of this work exist since 2008. We did not intend to publish it. But in the meantime several publications (see [20, 19, 26, 27] e.g.) referred to it so that it might be useful to make it available to the public.

2. Disintegration of statistical operators

We consider von Neumann's description of the measuring process of discrete quantum observables (cf. [22, 23]) and use it for a representation of statistical operators in terms of their conditional statistical operators and their laws.

Consider a countable set $Y \neq \emptyset$ together with an equivalence relation $\sim$ in Y. Represent $(Y, \sim)$ by means of (Γ, r) in such a way that Γ is a countable set and $r : Y \to \Gamma$ a surjective mapping satisfying

$$(2.1) \qquad \big(x \sim y \Longleftrightarrow r(x) = r(y)\big).$$

Given $\gamma \in \Gamma$ we set $Y_\gamma = \{r = \gamma\}$ for the associated equivalence class. In the sequel we assume always that

$$(2.2) \qquad 1 \leq \text{cd } Y_\gamma < +\infty \quad \text{for any } \gamma.$$

Let $\mathcal{H}$ be a complex separable Hilbert space of countable dimension $|Y|$. We identify the set Y with the complete othonormal system (*cons*) $\mathcal{Y} = \{e_y | y \in Y\}$ chosen in $\mathcal{H}$. Furthermore, we set $\mathcal{Y}_\gamma = \{e_y | y \in Y_\gamma\}$. The equivalence relation $\sim$ induces an equivalence relation in $\mathcal{Y}$ by means of $(e_x \sim e_y \Leftrightarrow x \sim y)$ with $\mathcal{Y}_\gamma$ as equivalence classes.

The set of *events of the system* described by the Hilbert space $\mathcal{H}$ can be identified with the collection of all orthogonal projections resp. all (closed) subspaces. The *state space* $\mathcal{S}(\mathcal{H})$ of the system is the collection of *(self-adjoint) bounded linear operators* $\mathcal{W}$ on $\mathcal{H}$ which are *positive* and have *trace one*, i.e. $\text{tr } \mathcal{W} = 1$. Such $\mathcal{W}$ are called *statistical operators*. They form a convex set whose extremal points, the socalled *pure states*, are defined by

$$h \circ h = \langle h, . \rangle \cdot h, \qquad h \in \mathcal{H}, \|h\| = 1.$$

By the spectral theorem every state $\mathcal{W}$ admits a representation

$$\mathcal{W} = \sum_{n=1}^{\infty} p_n \cdot h_n \circ h_n,$$

where $(p_n)_n$ is a probability on $\mathbb{N}$ and $(h_n)_n$ some *cons* in $\mathcal{H}$. (For more details we refer to [9].)

Our problem is how to associate to a given statistical operator $\mathcal{W} \in \mathcal{S}(\mathcal{H})$, admitting a spectral resolution with respect to a given *cons* $\mathcal{Y}$, a law, and, in particular situations, a point process κ, and vice versa.

In the above situation we are given a complex separable Hilbert space $\mathcal{H}$ with fixed basis $\mathcal{Y}$, indexed by Y. We consider

$$\mathcal{H}_\gamma = sp\{e_y | y \in Y_\gamma\},$$

the smallest subspace of $\mathcal{H}$ containing $\{e_y | y \in Y_\gamma\}$. The collection $(\mathcal{H}_\gamma)_{\gamma\in\Gamma}$ is an orthogonal decomposition of $\mathcal{H}$; and $\mathcal{H}$ is the direct sum of it. We have

$$1 \leq \dim \mathcal{H}_\gamma = |Y_\gamma| = \mathrm{cd}\, Y_\gamma < \infty.$$

Here cd denotes *cardinality*. Finally we write

$$P_\gamma = P^{\mathcal{H}_\gamma}$$

for the orthogonal projection onto $\mathcal{H}_\gamma$.

We start with a statistical operator $\mathcal{W} \in \mathcal{S}(\mathcal{H})$ which admits the spectral resolution

$$\mathcal{W} = \sum_{y\in Y} P_y\, \varrho(y) \tag{2.3}$$

for some law ϱ on Y with respect to the chosen *cons* $\mathcal{Y}$. Here $P_y = e_y \diamond e_y$ with $e_y \diamond e_y = (e_y, .) \cdot e_y$. Thus $\mathcal{W}$ is diagonalized by the given *cons* $\mathcal{Y}$. Set

$$\mathcal{W}_\gamma = \sum_{y\in Y_\gamma} P_y\, \varrho(y). \tag{2.4}$$

This defines self-adjoint linear operators on $\mathcal{H}$, leaving $\mathcal{H}_\gamma$ invariant s.th.

$$\mathcal{W}_\gamma = P_\gamma\, \mathcal{W}\, P_\gamma, \qquad \mathcal{W}_\gamma\, \mathcal{H}_\gamma^\perp = \{0\}.$$

Decomposition (2.4) is unique. If $\mathrm{tr}\, \mathcal{W}_\gamma = tr(P_\gamma \mathcal{W})$ is strictly positive, we can normalize $\mathcal{W}_\gamma$ to obtain the following statistical operator on $\mathcal{H}$:

$$\mathcal{W}(.|\gamma) = \frac{P_\gamma \mathcal{W} P_\gamma}{tr(P_\gamma \mathcal{W})}. \tag{2.5}$$

This is called the *conditional statistical operator of* $\mathcal{W}$ *given* P_γ. The notion of conditional statistical operators has been studied systematically by Cassinelli, Zanghi and Ozawa (cf. [6, 23] and the literature cited there).

Theorem 2.1. *Given an equivalence relation in* Y *which can be represented by means of* (Γ, r) *in such a way that conditions (2.1) and (2.2) are satisfied, any statistical* *operator* $\mathcal{W} \in \mathcal{S}(\mathcal{H})$, *admitting a spectral resolution (2.3) with respect to* $\mathcal{Y}$, *can be represented as*

$$(2.6) \qquad \mathcal{W} = \sum_{\gamma\in\Gamma} \mathcal{W}(.|\gamma) \cdot \kappa_{\mathcal{W}}(\gamma),$$

where $\mathcal{W}(.|\gamma) \in \mathcal{S}(\mathcal{H})$, *leaving* $\mathcal{H}_\gamma$ *invariant with* $\mathcal{W}(.|\gamma)\mathcal{H}_\gamma^\perp = \{0\}$, *and where* $\kappa_{\mathcal{W}}$ *is a probability on* Γ *having the following properties:*

$$(2.7) \qquad \kappa_{\mathcal{W}}(\gamma) = tr(P_\gamma \mathcal{W}), \qquad \gamma \in \Gamma.$$

This decomposition is unique.

In formula (2.6) and also later we use the convention that $\mathcal{W}(.|\gamma) \cdot \kappa_{\mathcal{W}}(\gamma) = 0$ if $\kappa_{\mathcal{W}}(\gamma) = 0$. We call $\kappa_{\mathcal{W}}$ the *law of the statistical operator* $\mathcal{W}$. It is some kind of partial trace of $\mathcal{W}$ with respect to γ, and we also write $\kappa_{\mathcal{W}}(\gamma) = tr_\gamma(\mathcal{W})$. This means that $tr_\gamma(\mathcal{W}) = \sum_{y\in Y_\gamma} \langle e_y, \mathcal{W}e_y\rangle$. We observe that for the calculation of the law $\kappa_{\mathcal{W}}$ we can use the *cons* which is most convenient, because a trace does not depend on the choice of a *cons*. Decomposition (2.6) is completely analogous to the decomposition of classical probabilities into conditional probabilities; and it is the starting point for the solution of our problem.

3. Disintegration of symmetric statistical operators

Consider next a finite group $\mathcal{G}$ acting on Y together with the equivalence relation $\sim$ induced by $\mathcal{G}$ in Y by means of $x \sim y \iff \exists g \in \mathcal{G} : y = g.x$. All orbits are finite, and $\mathcal{G}$ acts transitively on each of them. We assume also that $(Y, \sim)$ is represented by (Γ, r). As above $\mathcal{H}$ denotes a complex separable Hilbert space with a *cons* given by $\mathcal{Y}$. We consider then the unitary representation $\mathcal{U} = (\mathcal{U}_g)_{g\in\mathcal{G}}$ induced by $\mathcal{G}$ on $\mathcal{H}$ by means of

$$\mathcal{U}_g h = \sum_y \lambda_y \cdot e_{gy}, \quad h = \sum_y \lambda_y e_y.$$

It is obvious that $\mathcal{U}$ acts on $\mathcal{H}$ as well as on each $\mathcal{H}_\gamma$. Thus each $\mathcal{H}_\gamma$ as well as $\mathcal{H}_\gamma^\perp$ remains invariant under $\mathcal{U}$. The collection $\mathcal{U}_\gamma$ of restrictions of $\mathcal{U}_g, g \in \mathcal{G}$, to the subspaces $\mathcal{H}_\gamma$ is called an irreducible system, if *any closed subspace* S *of* $\mathcal{H}_\gamma$ *which remains invariant under* $\mathcal{U}_\gamma$ *is either* $\{0\}$ *or* $\mathcal{H}_\gamma$. This is equivalent to the condition that it does not commute with no non-trivial (self-adjoint) projection ([1], Exercise 1.3.D.) A statistical operator $\mathcal{W}$ is called *symmetric* (with respect to $\mathcal{G}$) if

$$(3.1) \qquad \mathcal{U}_g \mathcal{W} \mathcal{U}_{g^{-1}} = \mathcal{W} \text{ for any } g \in \mathcal{G}.$$

In the sequel we consider symmetric $\mathcal{W}$ admitting a spectral resolution for *cons* $\mathcal{Y}$.

Lemma 3.1. *W is symmetric if and only if each W_γ is symmetric.*

Proof. By (3.1) combined with decomposition (2.6) W is symmetric iff

$$\sum_\gamma W_\gamma = \sum_\gamma \mathcal{U}_g W_\gamma \mathcal{U}_g^* \text{ , for any } g \in \mathcal{G}.$$

The uniqueness of the decomposition combined with the fact that each $\mathcal{H}_\gamma$ resp. $\mathcal{H}_\gamma^\perp$ remains invariant under $\mathcal{U}$ immediately implies the result. □

We need also the following result which in our context is Schur's lemma ([4], Satz 7.1 b.):

Lemma 3.2. *Let W be symmetric. If the collection $\mathcal{U}_\gamma$ is irreducible then W_γ is of the form $W_\gamma = \kappa_W^*(\gamma) \cdot P_\gamma$. Here κ_W^* are non-negative functions on Γ, determined by the equation $\kappa_W^*(\gamma) = (e_y, We_y)$, $\quad y \in Y_\gamma$.*

The positivity of κ_W^* follows from the positivity of the statistical operator W. Thus we obtain the following disintegration of a symmetric statistical operator W.

Corollary 3.1. *If W is symmetric and if each $\mathcal{U}_\gamma$ is irreducible then*

$$W = \sum_{\gamma\in\Gamma} \kappa_W^*(\gamma) P_\gamma \quad \textit{and} \quad \sum_{\gamma\in\Gamma} \kappa_W^*(\gamma) \dim \mathcal{H}_\gamma = 1.$$

To summarize we have the following result.

Theorem 3.1. *Under the assumption that each $\mathcal{U}_\gamma, \gamma \in \Gamma$, is irreducible the equation*

$$W = \sum_{\gamma\in\Gamma} \frac{1}{\dim \mathcal{H}_\gamma} P_\gamma \cdot \kappa(\gamma) \tag{3.2}$$

induces a one-to-one correspondence between symmetric statistical operators W on $\mathcal{H}$, admitting a spectral resolution with respect to $\mathcal{Y}$, and probabilities κ on Γ.

This correspondence will be the main device in the sequel.

Corollary 3.2. *If W is a symmetric statistical operator on $\mathcal{H}$, admitting a spectral resolution with respect to $\mathcal{Y}$, and if $\mathcal{U}_\gamma$ is irreducible then the conditional statistical operator $W(.|\gamma)$, if well defined, coincides with the normalized projection onto $\mathcal{H}_\gamma$:*

$$W(.|\gamma) = \frac{1}{\dim \mathcal{H}_\gamma} \cdot P_\gamma. \tag{3.3}$$

Moreover, $\kappa_W(\gamma) = \dim \mathcal{H}_\gamma \cdot \kappa_W^*(\gamma)$, $\gamma \in \Gamma$, the law of W, determines the operator W completely.

From now on the underlying group $\mathcal{G}$ is given by a finite symmetric group $\mathcal{S}(E)$ of all permutations σ of some finite set E. In this case we consider the following operators:

$$\Pi_{\pm} = \frac{1}{|E|!} \cdot \sum_{\sigma \in \mathcal{S}(E)} sgn_{\pm}(\sigma) \cdot \mathfrak{U}_{\sigma}.$$

Here $sgn(\sigma) \in \{-1, +1\}$ denotes the sign of σ where sgn_+ is the identity and $sgn_- = sgn$. Both operators are orthogonal projections onto subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of $\mathcal{H}$ and satisfy

$$\text{(3.4)} \qquad \mathfrak{U}_{\sigma} \Pi_+ = \Pi_+, \ \mathfrak{U}_{\sigma} \Pi_- = sgn(\sigma) \cdot \Pi_- \quad \text{for any} \quad \sigma \in \mathcal{S}(E).$$

In particular the operators Π_+ and Π_- are *symmetric*. The elements of $\mathcal{H}_+$ are also called *symmetric*; the elements of $\mathcal{H}_-$ *antisymmetric*.

4. Examples

We consider the following standard finite setting (cf. [2, 24]). X is a finite, non-empty set of cardinality d; and $Y = X^n$. According to the convention of quantum mechanics the 1-particle space of a particle in X is given by $\mathbb{C}^X$, whereas the n-particle system is described by the complex Hilbert space $\mathcal{H} = \bigotimes^n \mathbb{C}^X$, i.e. the n-th tensor power of the 1-particle space. Note that $\mathcal{H}$ coincides with $\mathbb{C}^Y$, and if $n = 0$ then $\mathcal{H}$ is the one-dimensional complex plane. In $\mathbb{C}^X$ we choose some *cons* $(e_x)_{x \in X}$ conveniently. $\mathcal{Y} = \{e_y = \otimes_{j=1}^n e_{x_j} | y = (x_1, \ldots, x_n) \in Y\}$ then is a *cons* in $\mathcal{H}$ indexed by $\mathcal{Y}$. If $n = 0$ then $\mathcal{Y}$ is a singleton consisting of some unit vector $\mathbf{1}$ in $\mathbb{C}$ fixed once and for all. The underlying symmetric group is given by the collection $\mathcal{S}_n$ of bijections σ on $E = [n] = \{1, \ldots, n\}$. $\mathcal{S}_n$ acts on Y by means of

$$\sigma \longmapsto \big((x_1, \ldots, x_n) \longmapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})\big).$$

It operates on $\mathcal{H}$ by means of the collection of unitary representations consisting of

$$\mathfrak{U}_{\sigma} : e_{x_1} \otimes \cdots \otimes e_{x_n} \longmapsto e_{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{x_{\sigma^{-1}(n)}},$$

and is then extended by linearity. We shall be interested in statistical operators which are symmetric, i.e. commute with the above representation of $\mathcal{S}_n$, and which admit a spectral resolution with respect to $\mathcal{Y}$. Every observation W of a system of identical particles has this property. The Hilbert spaces $\mathcal{H}_+, \mathcal{H}_-$, appropriate for the description of particles obeying quantum statistics, are constructed by means of the projections Π_+, Π_- induced by the group $\mathcal{S}_n$.

A representation (Γ, r) of the equivalence relation induced by $\mathcal{S}_n$ on Y is given by

$$\Gamma = \mathcal{M}_n(X) := \{\delta_{x_1} + \cdots + \delta_{x_n} | (x_1, \ldots, x_n) \in Y\},$$
$$r : (x_1, \ldots, x_n) \longmapsto \delta_{x_1} + \cdots + \delta_{x_n}.$$

4.1. **The Maxwell-Boltzmann statistical operator.** In $\mathcal{H}$ we choose a *cons* indexed by Y in the following way: We are given a statistical operator w on the 1-particle space $\mathcal{H}_1 := \mathbb{C}^X$. Denote by ϱ the probability on X appearing in the spectral resolution of w, which at the same time gives a *cons* $(e_x)_{x \in X}$ in $\mathcal{H}_1$. This basis will be fixed also in the following examples and enables one to define the *cons* $\mathcal{Y}$ in $\mathcal{H}$ as above. Moreover, we always assume that ϱ is not a Dirac measure. This implies that $d = \text{cd}\ X \geq 2$. The *Maxwell-Boltzmann statistical operator for* w is defined by the tensor product of w: $\mathbf{M}_w^n = w^n$. Here w^n denotes the n-fold tensor product of w. Using proposition 16.3. in [24] this statistical operator can be expressed explicitly by

$$\mathbf{M}_w^n = \sum_{y \in Y} P_y \cdot \varrho^n(y), \tag{4.1}$$

where $P_y = e_y \circ e_y$, and ϱ^n is the product law $\varrho \otimes \cdots \otimes \varrho$ on Y. (4.1) is nothing else than the spectral resolution of $\mathbf{M}_w^n$ with respect to $\mathcal{Y}$. $\mathbf{M}_w^n$ is symmetric with respect to $\mathcal{S}_n$. By Theorem 2.1 there is associated the following law on $\mathcal{M}_n(X)$, which thus is a point process in X, namely

$$\kappa(\gamma) = \binom{n}{\gamma} \cdot \prod_{x \in X} \varrho(x)^{\gamma(x)}, \qquad \gamma \in \mathcal{M}_n(X). \tag{4.2}$$

Here

$$\binom{n}{\gamma} = \frac{n!}{\prod_{x \in X} \gamma(x)!}, \qquad \gamma \in \mathcal{M}_n(X).$$

(4.2) follows from the fact that $\dim \mathcal{H}_\gamma^n = \binom{n}{\gamma}$ and that, for $y = (x_1, \ldots, x_n) \in Y_\gamma$ and thereby $\gamma = \delta_{x_1} + \cdots + \delta_{x_n}$, by formula (4.1),

$$\kappa^*(\gamma) = \langle e_{x_1} \otimes \cdots \otimes e_{x_n}, \mathbf{M}_w^n\, e_{x_1} \otimes \cdots \otimes e_{x_n} \rangle = \prod_{j=1}^n \varrho(x_j).$$

The point process κ is called *Maxwell-Boltzmann process* for the parameters (ϱ, n), and will be denoted by P_ϱ^n.

4.2. **The Bose-Einstein statistical operator.** We start with the following observations: We are given a particle number $n \geq 0$. One can construct by means of $\mathcal{Y}$, as chosen above, a *cons* $\mathcal{Y}_+$ in $\mathcal{H}_+$ and $\mathcal{Y}_-$ in $\mathcal{H}_-$ respectively as follows:

$$\mathcal{Y}_+ = \left\{ e_+(\gamma) = \sqrt{\binom{n}{\gamma}} \cdot \Pi_+ \otimes_{a \in supp\,\gamma} e_a^{\otimes \gamma(a)} | \gamma \in \mathcal{M}_n(X) \right\},$$

$$\mathcal{Y}_- = \left\{ e_-(\gamma) = \sqrt{n!} \cdot \Pi_- \otimes_{a \in supp\,\gamma} e_a | \gamma \in \mathcal{M}_n(X) \right\}.$$

Here the tensor product is taken along a fixed numeration of X, and

$$\mathcal{M}_n^{\cdot}(X) = \left\{ \delta_{x_1} + \cdots + \delta_{x_n} | (x_1, \ldots, x_n) \in Y^{\cdot} \right\}.$$

$Y^{\cdot}$ is the collection of all $y \in Y$ with pairwise distinct components.

We work separately in each of the spaces $\mathcal{H}_\pm$ with these *cons*. In terms of $\mathcal{Y}_\pm$ the projections $\Pi_\pm$ can be written as $\Pi_\pm = \sum_{e \in \mathcal{Y}_\pm} Q_e^\pm$, where the one-dimensional projections are given by $Q_e^\pm = e \circ e$, $e \in \mathcal{Y}_\pm$. Since there is a bijection between $\mathcal{M}_n(X)$ and $\mathcal{Y}_+$ resp. $\mathcal{M}_n^{\cdot}(X)$ and $\mathcal{Y}_-$ we see immediately that (recall that $d = trX$)

$$cd\,\mathcal{Y}_+ = \binom{d+n-1}{n}; \quad cd\,\mathcal{Y}_- = \binom{d}{n}, \quad \text{if} \quad n \leq d; \quad cd\,\mathcal{Y}_- = 0, \quad \text{if} \quad n > d.$$

The *Bose-Einstein statistical operator for w* is given by the conditional Maxwell-Boltzmann statistical operator given the projection Π_+. This is an operator on $\mathcal{H}_+$ defined by $\mathbb{E}_w^n = \frac{1}{tr(\Pi_+ \mathbb{M}_w^n)} \cdot \Pi_+ \mathbb{M}_w^n$. Note that

$$tr(\Pi_+ \mathbb{M}_w^n) = \sum_{\mu \in \mathcal{M}_n(X)} \prod_{a \in X} \varrho(a)^{\mu(a)} > 0.$$

because the ϱ is assumed not to be a Dirac measure.

We choose a *cons* in $\mathcal{H}_+$ for which $\mathbb{E}_w^n$ can be diagonalized, namely $\mathcal{Y}_+$, which is indexed by the finite set $\mathcal{M}_n(X)$. The symmetric group now acts on the basis $\mathcal{Y}_+$ and is trivial, i.e. a singleton consisting of the identity. Thus the associated equivalence relation $\sim$ is given by the identity of elements in $\mathcal{Y}_+$; and the representation of $(\mathcal{Y}_+, \sim)$ is given by $\Gamma = \mathcal{M}_n(X)$ with $r : e_+(\gamma) \longrightarrow \gamma$. Theorem 3.1 then implies that the point process belonging to $\mathbb{E}_w^n$ is given by the following point process in X: For any $\gamma \in \mathcal{M}_n^{\cdot}(X)$

$$\mathsf{E}_\varrho^n(\gamma) = \frac{1}{\sum_{\mu \in \mathcal{M}_n(X)} \prod_{a \in X} \varrho(a)^{\mu(a)}} \cdot \prod_{a \in X} \varrho(a)^{\gamma(a)}. \tag{4.3}$$

Moreover, the Bose-Einstein statistical operator admits the representation

$$\mathbb{E}_w^n = \sum_{\gamma \in \mathcal{M}_n(X)} \mathsf{E}_\varrho^n(\gamma) \cdot Q_{e_+(\gamma)}^+.$$

We call E_ϱ^n the *Bose-Einstein point process in X for the parameters (n, ϱ)*

If ϱ is the uniform distribution on X, and thereby $w = \frac{1}{d} trI$, where I denotes the identity operator on $\mathbb{C}^X$, then

$$\mathbb{E}_w^n := \mathbb{E}_d^n := \frac{1}{\binom{d+n-1}{n}} \cdot \Pi_+,$$

and the Bose-Einstein process is then given by the uniform distribution on $\mathcal{M}_n^{\cdot\cdot}(X)$:

$$\mathsf{E}_d^n(\gamma) = \frac{1}{\binom{d+n-1}{n}}, \quad \gamma \in \mathcal{M}_n^{\cdot\cdot}(X).$$

4.3. **The Fermi-Dirac statistical operator.** For $n \leq d = \mathrm{cd}\, X$ the *Fermi-Dirac statistical operator for w* is given by the conditional Maxwell-Boltzmann statistical operator given the projection Π_-. This is a symmetric statistical operator on $\mathcal{H}_-$ defined by

$$\mathbb{D}_w^n = \frac{1}{tr(\Pi_- \mathbf{M}_w^n)} \cdot \Pi_- \mathbf{M}_w^n. \tag{4.4}$$

This operator admits a spectral resolution with respect to the *cons* $\mathcal{Y}_-$ in $\mathcal{H}_-$, where again the basis $(e_x)_{x \in X}$ is coming from the spectral resolution of w and ϱ is the corresponding law not being a Dirac measure. By Theorem 3.1 we then obtain as before the particle picture of the Fermi-Dirac statistical operator: It is given by the following simple point process, called *Fermi-Dirac process for* (n, ϱ) in X:

$$\mathsf{D}_\varrho^n(\gamma) = \frac{1}{Z}\, tr \prod_{a \in X} \varrho(a)^{\gamma(a)}, \qquad \gamma \in \mathcal{M}_n^{\cdot}(X), \ \textit{and } 0 \textit{ otherwise}. \tag{4.5}$$

The partition function now is given by $Z = \sum_{\mu \in \mathcal{M}_n^{\cdot}(X)} \prod_{a \in X} \varrho(a)^{\mu(a)}$. Thus D_ϱ^n is the conditional law of E_ϱ^n given $\mathcal{M}_n(X)$, i.e. given that the realization γ of the particle process is simple. We again have a representation of the Fermi-Dirac statistical operator which is parallel to the one for the Bose-Einstein statistical operator, namely

$$\mathbb{D}_w^n = \sum_{\gamma \in \mathcal{M}_n^{\cdot}(X)} \mathsf{D}_\varrho^n(\gamma) \cdot Q_{\psi_-(\gamma)}^-.$$

Note that in the special case where $w = \frac{1}{d} \cdot I$, thus ϱ being the uniform distribution on X, the Fermi-Dirac statistical operator is given by

$$\mathbb{D}_w^n := \mathbb{D}_d^n := \frac{1}{\binom{d}{n}} \cdot \Pi_-,$$

and the simple point process by the *Fermi-Dirac process* in X for the parameters (n, d). (Recall that $d = |X|$.)

$$\mathrm{D}_d^n(\gamma) = \frac{1}{\binom{d}{n}}, \qquad \gamma \in \mathcal{M}_n(X).$$

5. The method of the Campbell measure

In the situation of the last section we introduce the occupation number operator and the Campbell operator respectively Campbell measure of a statistical state.

The situation is the same as in the examples: $\mathcal{H}_1 = \mathbb{C}^X$ for some finite X; $(e_x)_{x \in X}$ is a *cons* in $\mathcal{H}_1$. Recall that $\Gamma = \mathcal{M}_n^{\cdot\cdot}(X)$, and $r : (x_1, \ldots, x_n) \longmapsto$ $\delta_{x_1} + \cdots + \delta_{x_n}$. Note that $r = M \circ \iota$, where $\iota : (x_1, \ldots, x_n) \longmapsto e_{x_1} \otimes \cdots \otimes e_{x_n}$ and $M(e_{x_1} \otimes \cdots \otimes e_{x_n}) = \delta_{x_1} + \cdots + \delta_{x_n}$.

We define for $x \in X$ the *occupation number operator in x* on $\mathcal{H} = \mathcal{H}_1^{\otimes n}$ as follows: If I is the identity operator on $\mathcal{H}_1$, let

$$\mathcal{N}_x = \sum_{j=1}^{n} I \otimes \cdots \otimes \underbrace{e_x \circ e_x}_{j} \otimes \cdots \otimes I. \tag{5.1}$$

(In the case $n = 0$ we set $\mathcal{N}_x^{(0)} = 0 \cdot I$.) And, more generally, $\mathcal{N}_B = \sum_{x \in B} \mathcal{N}_x$ the occupation number operator in $B \subset X$. It is evident that $\mathcal{N}_B = \zeta_B(M) I^n$, where for $x_1, \ldots, x_n \in X$ we set

$$\zeta_B(\delta_{x_1} + \cdots + \delta_{x_n}) = (\delta_{x_1} + \cdots + \delta_{x_n})(B).$$

Extend $\mathcal{N}_{(\cdot)}$ linearly to an operator-valued measure on $X \times \mathcal{M}_n^{\cdot\cdot}(X)$ by $\mathcal{N}_h = \zeta_h(M) \cdot I^n$, $h \in F_+(X \times \mathcal{M}_n^{\cdot\cdot}(X))$. Here $\zeta_h(\mu) = \int h(x, \mu)\, \mu(\mathrm{d}x)$, and F_+ denotes the collection of non-negative, measurable functions on the underlying domain. Thus in particular $\mathcal{N}_{B \times C} = \zeta_B(M) \cdot 1_C(M) \cdot I^{\otimes n}$. This shows: Any element $e_y = e_{y_1} \otimes \cdots \otimes e_{y_n}$ of the basis is an eigenvector of $\mathcal{N}_{B \times C}$ with eigenvalue $\zeta_B(M(e_y)) \cdot 1_C(M(e_y))$.

We are now in the position to define the Campbell measure for statistical operators on $\mathcal{H}$. Given a statistical operator W we call $W\mathcal{N}_{(\cdot)}$ on $\mathcal{H}$ the *Campbell operator measure* of W. Its trace $\mathcal{C}_W(.) = tr(W\mathcal{N}_{(\cdot)})$ is called the *Campbell measure of W on $X \times \mathcal{M}_n^{\cdot\cdot}(X)$*. Recall that the Campbell measure of the law κ_W of W is defined by

$$\mathcal{C}_{\kappa_W}(a, \gamma) = \gamma(a)\kappa_W(\gamma), \qquad a \in X, \gamma \in \mathcal{M}_n^{\cdot\cdot}(X).$$

It is obvious that such a Campbell measure is supported by the set $\{(a, \gamma) : \gamma(a) \geq 1\}$. Moreover, we see that the law κ_W of W is determined by its Campbell measure.

Proposition 5.1. *For any statistical operator $\mathcal{W}$ on the space $\mathcal{H}$ its Campbell measure coincides with the Campbell measure of its law, i.e. $\mathcal{C}_{\mathcal{W}} = \mathcal{C}_{\kappa_{\mathcal{W}}}$. The law of $\mathcal{W}$ is completely determined by $\mathcal{C}_{\mathcal{W}}$. If $\mathcal{W}$ is also symmetric then, under the additional irreducibility assumptions of Theorem 2. even $\mathcal{W}$ is completely determined by its Campbell measure.*

Proof.

$$tr(\mathcal{W}\mathcal{N}_h) = \sum_y \langle e_y, \mathcal{W}\mathcal{N}_h(e_y)\rangle = \sum_y \zeta_h(r(y))\langle e_y, \mathcal{W}(e_y)\rangle = \sum_\gamma \zeta_h(\gamma) \sum_{y\in Y_\gamma} \langle e_y, \mathcal{W}(e_y)\rangle.$$

The assertion now follows from the definition (2.7) of $\kappa_{\mathcal{W}}$. The remaining statement follows immediately from Theorem 3.1. □

We remark for later use that Proposition 5.1 remains true for statistical operators $\mathcal{W}$ acting on subspaces of $\mathcal{H}$ because the occupation number operators $\mathcal{N}_H$ act on them by restriction

6. States on Fock spaces and their Campbell measures

The above picture is now extended to systems with a random particle number.

Let X be a finite set of cardinality $d > 1$ and $\mathcal{H}_m = \bigotimes^m \mathbb{C}^X$, $m \geq 0$, with $\mathcal{H}_0 = \mathbb{C}$. The *cons* in $\mathbb{C}$ consists of some unit vector, denoted by $\mathbf{1}$. The direct sum of these Hilbert spaces is the Fock space over $\mathbb{C}^X$, denoted by $\mathbb{H}$. For each m the symmetric group $\mathcal{S}_m$ acts on X^m, and the corresponding unitary representation on $\mathcal{H}_m$ is denoted by $\mathcal{U}_m$. This family of representations gives rise to a unitary operator $\mathcal{U}$ on $\mathbb{H}$, defined by the direct sum $\mathcal{U} = \sum_{m=0}^{\infty} \mathcal{U}_m$. Thus $\mathcal{U}(g)h = \mathcal{U}_m(g)h$, if $g \in \mathcal{S}_m, h \in \mathcal{H}_m$. Given statistical operators $\mathcal{W}_m$ on $\mathcal{H}_m$ and scalars $p_m \geq 0, m \geq 0$, summing up to 1, then the direct sum

$$\mathcal{W} = \sum_{m=0}^{\infty} p_m \mathcal{W}_m \tag{6.1}$$

is a statistical operator on the Fock space $\mathbb{H}$. $\mathcal{W}$ is symmetric if and only if each $\mathcal{W}_m$ has this property. It is obvious that the point process belonging to this statistical operator is given by

$$\kappa_{\mathcal{W}} = \sum_{m=0}^{\infty} p_m \cdot \kappa_{\mathcal{W}_m}. \tag{6.2}$$

The simplest examples are obtained if $\mathcal{W}_m = w^m$ for some given statistical operator w on $\mathcal{H}_1 = \mathbb{C}^X$. Only them will be considered in the sequel in detail. In this framework the occupation number operator is given by the direct sum operator $\mathcal{N}_x = \sum_{m=0}^{\infty} \mathcal{N}_x^{(m)}$ on the Fock space over $\mathbb{C}^X$. Here $\mathcal{N}_x^{(m)}$ is the occupation number operator on $\mathcal{H}_m$

as defined above. And again $\mathcal{N}_B = \zeta_B(M) \cdot I, \quad B \subset X$, where I now denotes the identity operator on $\mathbb{H}$. Extending $\mathcal{N}$ to an operator valued measure on $X \times \mathcal{M}^{\cdot}(X)$ as above by $\mathcal{N}_h = \zeta_h(M) \cdot I, \quad h \in F_+(X \times \mathcal{M}^{\cdot}(X))$, we are now in the position to define the Campbell measure for statistical operators on $\mathbb{H}$ as we did already in a special situation. Recall that $\zeta_h(\mu) = \int h(x, \mu)\, \mu(\mathrm{d}x)$.

Given a statistical operator $\mathcal{W}$ on $\mathbb{H}$ we call $\mathcal{W}\mathcal{N}_{(.)}$ the *Campbell operator measure* of $\mathcal{W}$. By Theorem 2.1 we know that $\mathcal{W}\mathcal{N}_h = \sum_{\gamma \in \Gamma} \zeta_h(\gamma)\kappa_{\mathcal{W}}(\gamma) \cdot \mathcal{W}(\cdot|\gamma)$, $h \in F_+$. Define $\mathcal{C}_{\mathcal{W}}(.) = tr(\mathcal{W}\mathcal{N}_{(.)})$. This object is called the *Campbell measure of* $\mathcal{W}$. Arguing as above we obtain

Theorem 6.1. *For any statistical operator $\mathcal{W}$ on the Fock space $\mathbb{H}$ one has $\mathcal{C}_{\mathcal{W}} = \mathcal{C}_{\kappa_{\mathcal{W}}}$. Thus the law of $\mathcal{W}$ is completely determined by $\mathcal{C}_{\mathcal{W}}$. If $\mathcal{W}$ is also symmetric then, under the additional irreducibility assumptions of Theorem 3.1, even $\mathcal{W}$ is completely determined by its Campbell measure.*

Consider now the direct sums $\Pi_\pm = \sum_{m=0}^{\infty} \Pi_\pm^{(m)}$, where $\Pi_\pm^{(m)}$ is the orthogonal projection onto the *BE*- resp. *FD symmetric subspace of* $\mathcal{H}_m$. $\Pi_\pm$ is then the orthogonal projection onto the *BE*- resp. *FD symmetric subspace* $\mathbb{H}_\pm$ *of* $\mathbb{H}$. It follows (see [2]) that $\Pi_\pm$ satisfy

$$\mathcal{U}_\sigma \Pi_\pm = sgn_\pm(\sigma)\Pi_\pm, \quad \sigma \in \mathcal{S}_\infty := \bigcup_{m \geq 0} \mathcal{S}_m. \tag{6.3}$$

We are mainly interested in statistical operators $\mathcal{W}$ living on the symmetric subspaces $\mathbb{H}_\pm$. By this we mean that $\mathcal{W}$ satisfies the conditions $\mathcal{W} = \Pi_\pm \mathcal{W} \Pi_\pm$. In case $+$ this is equivalent to say that $\mathcal{W}$ is Bose-Einstein symmetric, i.e. $\mathcal{U}_\sigma \mathcal{W} = \mathcal{W}$, $\sigma \in \mathcal{S}_\infty$; and in case $-$ that $\mathcal{W}$ is Fermi-Dirac symmetric, i.e. $\mathcal{U}_\sigma \mathcal{W} = sgn(\sigma)\mathcal{W}$, $\sigma \in \mathcal{S}_\infty$. Moreover, these conditions imply the symmetry of the statistical operator. (All this can be found in [2])

Theorem 6.1 remains true for statistical operators acting on the Fock spaces $\mathbb{H}_\pm$ because the $\mathcal{N}_B$ act on $\mathbb{H}_\pm$ by restriction. Note also that one obtains by means of a basis in $\mathcal{H}_1$ a basis in the Fockspaces $\mathbb{H}, \mathbb{H}_\pm$ by taking unions $\bigcup_{m \geq 1} \mathcal{Y}^{(m)}, \bigcup_{m \geq 1} \mathcal{Y}_\pm^{(m)}$, augmented in each case by the basis in $\mathcal{H}_0$, which consists of $\mathbf{1}$. Considered as an element of the Fock spaces $\mathbf{1}$ is called ground state and corresponds to the empty particle configuration.

7. States with random particle numbers

The method of second quantization is recalled which permits to lift an operator on a 1-particle space to a Fock space.

7.1. **The method of second quantization.** We recall the method of the so-called second quantization. The idea behind is to lift operators H on $\mathcal{H}$ to one of the Fock spaces. The method goes back to the work of Fock [13], Cook [8] and Berezin [3] (cf. also [5]). If H is a statistical operator on $\mathcal{H}$, one can define a operator H_m on the tensor product $\mathcal{H}_m$ by setting $H_0\mathbf{1} = 0$ and

$$H_m(e_{a_1} \otimes \cdots \otimes e_{a_m}) = \sum_{j=1}^{m} e_{a_1} \otimes \cdots \otimes H\, e_{a_j} \otimes \cdots \otimes e_{a_m}, \qquad a_1, \ldots, a_m \in X.$$

Denoting by δ_{jk} the Kronecker symbol,

$$H_m = \sum_{j=1}^{m} H^{\delta_{j1}} \otimes \cdots \otimes H^{\delta_{jm}}.$$

The direct sum of the H_m is denoted by

$$d\Gamma(H) = \sum_{m=0}^{\infty} H_m.$$

Note that we used this method already for the operator $e_x \circ e_x$ and obtained in chapter 6 for the operator $d\Gamma(e_x \circ e_x)$ the occupation number operator $\mathbb{N}_x$ on the Fock space over $\mathbb{C}^X$.

If w is a statistical operator on $\mathcal{H}$, the *second quantization of* w then is defined by

$$\Gamma(w) = \sum_{m=0}^{\infty} \frac{1}{m!} w^m.$$

This is an operator on the full Fock space $\mathbb{H}$ having finite trace e.

An important observation is given in terms of such *trace class operators* These are multiples of statistical operators, i.e. operators of the form $w = z\bar{w}^m$, where $z > 0$ and $\bar{w}$ is some statistical operator. In this case

$$\Gamma(w) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \bar{w}^m \ \textit{with}\ tr\,\Gamma(w) = e^z\,.$$

Lemma 7.1. *Let H be a bounded, self adjoint operator such that $w = \exp(-\beta H)$ is a trace class operator with $\beta \in \mathbb{R}_+$. Then*

$$\exp\,(-\beta\, H)^m = \exp\left(-\beta \sum_{j=1}^{m} H^{\delta_{j1}} \otimes \cdots \otimes H^{\delta_{jm}}\right)$$

Recall here that the left hand side of this equation is given by $e^{-\beta H} \otimes \cdots \otimes e^{-\beta H}$. For a proof of the lemma we refer to Cook [8].

Lemma 7.2. *Let H be a self adjoint operator such that $w = \exp(-\beta H)$ is a trace class operator with $\beta \in \mathbb{R}$. Defining the associated Gibbs state*

$$G = \frac{1}{tr\,\exp(-\beta H)}\,\exp(-\beta H) \tag{7.1}$$

and $z = cd\,\exp(-\beta H)$ we obtain

$$\Gamma\left(\exp(-\beta H)\right) = \sum_{m=0}^{\infty} \frac{z^m}{m!} G^m \tag{7.2}$$

$\Gamma(\exp(-\beta H))$ is trace class with trace e^z.

As a consequence we see that $\mathbb{M}_{zG} := e^{-z}\,\Gamma(\exp(-\beta H))$ is a statistical operator on the Fock space.

According to Lemmas 7.1 and 7.2 there are two representations of this operator:

$$\mathbb{M}_{zG} = e^{-z} \sum_{m=0}^{\infty} \frac{z^m}{m!} G^m = e^{-z} \sum_{m=0}^{\infty} \frac{1}{m!} \exp\left(-\beta \sum_{j=1}^{m} H^{\delta_{j1}} \otimes \cdots \otimes H^{\delta_{jm}}\right).$$

To summarize in a slightly modified way: Given some trace class operator $w = z\widetilde{w}$ with corresponding spectral measure $\varrho = z\widetilde{\varrho}$, then w^m has trace $\mathrm{tr}\, w^m = z^m$. In this case the associated second quantization of w is given by

$$\mathbb{M}_w = \frac{1}{\Xi_w} \sum_{m=0}^{\infty} \frac{tr\,w^m}{m!} \cdot \frac{w^m}{cd\,w^m} = \frac{1}{\Xi_w} \sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \widetilde{w^m}. \tag{7.3}$$

Ξ_w is the normalizing constant. In this way the trace class operator w is lifted to some symmetric statistical operator on the full Fock space $\mathbb{H}$.

The construction principle behind the *method of second quantization* is: Given m, the trace class operator w^m is normalized to some statistical operator $\widetilde{w^m}$, then weighted by the factor $\frac{cd\,w^m}{m!}$ and summed up; finally it is normalized so that the resulting operator becomes statistical.

One also uses this quantization method in a sligthly generalized form to lift the underlying w on the subspaces $\mathbb{H}_\pm$ and obtain the statistical operators

$$\mathbb{E}_w = \frac{1}{\Xi_w^+} \sum_{m=0}^{\infty} tr(\Pi_+^{(m)} w^m) \cdot \frac{\Pi_+^{(m)} w^m}{\mathrm{tr}(\Pi_+^{(m)} w^m)},$$

$$\mathbb{D}_w = \frac{1}{\Xi_w^-} \sum_{m=0}^{\infty} tr(\Pi_-^{(m)} w^m) \cdot \frac{\Pi_-^{(m)} w^m}{\mathrm{tr}(\Pi_-^{(m)} w^m)}.$$

Note here that the normalizing constants $\Xi_w^\pm = \sum_{m=0}^{\infty} tr(\Pi_\pm^{(m)} \mathbb{M}_w^m)$ are termwise strictly positive and convergent on account of the assumption that ϱ is not a Dirac measure. $\mathbb{E}_w$ is called the *Bose-Einstein operator for w*, $\mathbb{D}_w$ the *Fermi-Dirac operator* *for* w and $p_{\varrho}^{\pm} : m \longmapsto \frac{1}{z^{\pm}} \cdot tr(\Pi_{\pm}^{(m)} \mathbf{M}_w^m)$ the *particle number distribution of* $\mathbb{E}_w$ *or* $\mathbb{E}_w$ *respectively*. Thus the operators $\mathbf{M}_w$, $\mathbb{E}_w$ and $\mathbb{D}_w$ are the second quantizations of w for the different Fock spaces $\mathbb{H}$, $\mathbb{H}_{\pm}$. One question then is to calculate the correponding laws and to characterize them.

7.2. **Maxwell-Boltzmann statistical operators with Poissonian random particle number.** The Maxwell-Boltzmann statistical operator is described as a solution of an integration-by-parts formula.

We are in the framework of section 4: $\bar{w}$ is a statistical operator on $\mathbb{C}^X$, X being a finite set of cardinality d. As above we choose a cons $e_x, x \in X$, the one coming from the spectral decomposition of $\bar{w}$ with law $\bar{\varrho}$. We are interested in the symmetric statistical operator given by the second quantization of the trace class operator $w = z\bar{w}$

$$\mathbf{M}_w = e^{-z} \sum_{m=0}^{\infty} \frac{z^m}{m!} \cdot \mathbf{M}_{\bar{w}}^m. \tag{7.4}$$

This is the *Maxwell-Boltzmann statistical operator for* $z, \bar{w}$. We remark that, instead of the Poisson law, any law $(p_m)_m$ can be taken to get some statistical operator. By formula (6.2) the corresponding point process is the Poisson process P_ϱ with intensity $\varrho = z\bar{\varrho}$. Thus $\kappa_{\mathbf{M}_w} = P_\varrho$, where

$$P_\varrho(\varphi) = \kappa_{\mathbf{M}_w}(\varphi) = e^{-z} \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{(x_1,\dots,x_m)\in X^m} \varphi(\delta_{x_1} + \dots + \delta_{x_m})\ \bar{\varrho}(x_1)\dots\bar{\varrho}(x_m).$$

P_ϱ is supported by $\mathcal{M}^{\cdot\cdot}(X) = \bigcup_{n=0}^{\infty} \mathcal{M}_n^{\cdot\cdot}(X)$. Note that this formula is completely parallel to (7.4), namely

$$\kappa_{\mathbf{M}_w} = e^{-z} \sum_{m=0}^{\infty} \frac{z^m}{m!} (L_{\bar{\varrho}})^{*m}, \quad \text{where} \quad L_{\bar{\varrho}} = \sum_{x\in X} \delta_x\, \bar{\varrho}(x),$$

and $*$ denotes convolution of laws.

It is well-known by Mecke's characterization of the Poisson process (see [17]) that P_ϱ is characterized as the unique solution Q of the equation

$$\mathcal{C}_{\mathsf{Q}}(h) = \sum_{x\in X} \sum_{\gamma\in\mathcal{M}} h(x, \gamma + \delta_x)\, \varrho(x) \mathsf{Q}(d\gamma), \quad h \in F_+. \tag{7.5}$$

To say it in another way. Q is the unique solution of the equation $\mathcal{C}_{\mathsf{Q}}(x, \gamma) = \varrho(x)\mathsf{Q}(\gamma - \delta_x)$, $x \in X$, $\gamma \in \mathcal{M}^{\cdot\cdot}(X)$, $\gamma(x) \geq 1$. Another very useful view to equation (7.6) is

$$\mathcal{C}_{\mathsf{Q}} = \mathcal{C}_{L_\varrho} \star \mathsf{Q}. \tag{7.6}$$

(Note that the operation $\star$ differs from the convolution operation $*$.) To summarize: The first part of Theorem 6.1 implies

Corollary 7.1. *Let $\bar{w}$ be a statistical operator on $\mathbb{C}^X$ with spectral law $\bar{\varrho}$ and $z > 0$ a parameter. Then $\mathbb{M}_{z,\bar{w}}$ is a solution $\mathbb{W}$ of the equation $\mathcal{C}_{\mathbb{W}} = \mathcal{C}_{L_{z\bar{\varrho}}} \star \kappa_{\mathbb{W}}$.*

This result is a version of Lemma 4.12 of Liebscher [16].

7.3. **Bose-Einstein statistical operators with random particle number.** We consider the *Bose-Einstein statistical operator on the Fock space $\mathbb{H}_+$ with one-particle statistical operator w*. It is clear that $\mathbb{E}_w$ is symmetric and thereby also BE-symmetric. By the results obtained in Bл4. $\mathbb{E}_w$ is given by the following direct sum

$$\mathbb{E}_w = \frac{1}{\Xi_w^+} \sum_{m=0}^{\infty} tr(\Pi_+^{(m)} \mathbb{M}_w^m) \cdot \sum_{\gamma \in \mathcal{M}_{+}(X)} \mathsf{E}_\varrho^m(\gamma) \cdot Q^{+,m}_{e_+(\gamma)}. \tag{7.7}$$

Here we denote now the dependence on the particle number m in $Q^{+,m}_{e_+(\gamma)}$.

Example 7.1. *Consider a statistical operator w with ϱ being the uniform distribution on X, i.e. $\varrho \equiv \frac{1}{d}$. Recall that $d \geq 2$. In this case*

$$tr(\Pi_+^{(m)} \mathbb{M}_w^m) = \binom{d+m-1}{m} \cdot \frac{1}{d^m},$$

and $\Xi_w^+ = \Xi_w^+(d) = \frac{1}{(1-\frac{1}{d})^d}$. Thus the particle number distribution is given by the following negative binomial distribution

$$p_d^+(m) = \binom{d+m-1}{m} \cdot \left(1 - \frac{1}{d}\right)^d \cdot \frac{1}{d^m}. \tag{7.8}$$

We want to calculate the Campbell measure $\mathcal{C}_{\mathbb{E}_w}$. Thus we first calculate its law: formulas (6.1) and (6.2) immediately imply that

$$\kappa_{\mathbb{E}_w} = \mathsf{E}_\varrho := \frac{1}{\Xi_w^+(d)} \sum_{m=0}^{\infty} tr(\Pi_+^{(m)} \mathbb{M}_w^m) \cdot \mathsf{E}_\varrho^m. \tag{7.9}$$

This point process is called here the *Bose-Einstein process* and denoted by E_ϱ. This enables us to represent $\mathbb{E}_w$ as

$$\mathbb{E}_w = \sum_{\gamma \in \mathcal{M}_{+}(X)} \mathsf{E}_\varrho(\gamma) \cdot Q^+_{e_+(\gamma)}.$$

The Campbell measure of the Bose-Einstein statistical operator $\mathbb{E}_w$ is given by the usual Campbell measure of the Bose-Einstein process. Moreover, $\mathbb{E}_w$ is completely determined by the Campbell measure of its law E_ϱ. So we have to study the Campbell measure $\mathcal{C}_{\mathsf{E}_\varrho}$ which will be done in the Bл8.

7.4. Fermi-Dirac statistical operators with random particle number. Consider now the *Fermi-Dirac statistical operator on $\mathbb{H}_-$ with one-particle statistical operator w.* Analogously to the case of the Bose-Einstein operator it is FD-symmetric and can be represented as

$$\mathbb{D}_w = \frac{1}{\Xi_w^-} \sum_{m=0}^{\infty} \operatorname{tr}(\Pi_-^{(m)} \mathbb{M}_w^m) \cdot \sum_{\gamma \in \mathcal{M}_m(X)} \mathsf{D}_\varrho^m(\gamma) \cdot Q_{e_-(\gamma)}^{-,m}. \tag{7.10}$$

Example 7.2. *Consider a statistical operator w with ϱ being the uniform distribution on X, i.e. $\varrho \equiv \frac{1}{d}$ with $d \geq 2$. Then*

$$\operatorname{tr}(\Pi_-^{(m)} \mathbb{M}_w^m) = \binom{d}{m} \cdot \frac{1}{d^m};$$

and $\Xi_w^- = \Xi_w^-(d) = (1 + \frac{1}{d})^d$. Thus the particle number distribution is given by the following binomial distribution

$$p_d^-(m) = \binom{d}{m} \cdot \left(\frac{1}{d+1}\right)^m \cdot \left(1 - \frac{1}{d+1}\right)^{d-m}. \tag{7.11}$$

Observe here the symmetry between Bose-Einstein and Fermi-Dirac statistical operators:

$$\Xi_w^-(d) = \Xi_w^+(-d).$$

We want to calculate its Campbell measure $\mathcal{C}_{\mathbb{D}_w}$. Again we calculate first its law: This is given by

$$\kappa_{\mathbb{D}_w} = \mathsf{D}_\varrho := \frac{1}{\Xi_w^-} \sum_{m=0}^{\infty} \operatorname{tr}(\Pi_-^{(m)} \mathbb{M}_w^m) \cdot \mathsf{D}_\varrho^m. \tag{7.12}$$

This point process is called the *Fermi-Dirac process* and is denoted by D_ϱ. Again we have a representation of the form

$$\mathbb{D}_w = \sum_{\gamma \in \mathcal{M}(X)} \mathsf{D}_\varrho(\gamma) \cdot Q_{e_-(\gamma)}^-.$$

Now we have the problem to study $\mathcal{C}_{\mathsf{D}_\varrho}$ and to analyze D_ϱ. This problem will be solved in Bzb8 by using again the method of the Campbell measure.

8. Characterizations of Bose-Einstein and Fermi-Dirac processes

The question is, what are the properties of the Boson resp. Fermion point processes. The answer is given by means of the method of the Campbell measure. For this aim we derive integration-by-parts formulas for E_ϱ resp. D_ϱ in terms of its Campbell measures. The arguments are only sketched. For the details we refer to [15, 20, 21, 25].

8.1. **Bosons**. Recall that the law ϱ on X is not a Dirac measure. Recall that for a given $\mu \in \mathcal{M}\ (X)$

$$\mathsf{E}_\varrho(\mu) = \frac{1}{\Xi^+_w} \prod_{a \in X} \varrho(a)^{\mu(a)}.$$

If $\mu(X) = m$, this can be written as

$$\mathsf{E}_\varrho(\mu) = \frac{1}{\Xi^+_w} \frac{1}{\binom{m}{\mu}} \mathsf{P}^m_\varrho(\mu).$$

In terms of the Poisson process in X with intensity measure ϱ. which is defined by

$$\mathsf{P}_\varrho(\mu) = e^{-\varrho(X)} \frac{\varrho(X)^m}{m!} \mathsf{P}^m_\varrho(\mu),$$

we obtain a representation of E_ϱ in terms of P_ϱ:

$$\mathsf{E}_\varrho(\mu) = \frac{1}{\Xi^+_w} \frac{1}{\binom{m}{\mu}} \frac{m!}{\varrho(X)^m} e^{\varrho(X)} \mathsf{P}_\varrho(\mu). \tag{8.1}$$

Now we start to calculate the Campbell measure of E_ϱ, i.e.

$$\mathcal{C}_{\mathsf{E}_\varrho}(a, \mu) = \mu(a)\mathsf{E}_\varrho(\mu - \delta_a), \quad \mu(a) \geq 1.$$

Using representation (8.1) in combination with Mecke's characterization (7.5) of the latter yields a recurrence which immediately leads to

Lemma 8.1. *For* $(a, \mu) \in C = \{(a, \mu) : \mu(a) \geq 1\}$

$$\mathcal{C}_{\mathsf{E}_\varrho}(a, \mu) = \sum_{j=1}^{\mu(a)} \varrho(a)^j \cdot \mathsf{E}_\varrho(\mu - j\delta_a). \tag{8.2}$$

Observe that (8.2) is an equation for E_ϱ. To solve this equation we look at it in the following way:

Proposition 8.1. *For any* $h \in F_+$

$$\mathcal{C}_{\mathsf{E}_\varrho}(h) = \sum_{a \in X} \sum_{\gamma \in \mathcal{M}\ (X)} \sum_{j \geq 1} h(a, \gamma + j\delta_a)\varrho(a)^j \lambda(a)\mathsf{E}_\varrho(\gamma). \tag{8.3}$$

Here λ *denotes the counting measure on* X.

Equation (8.3) has the same structure as equation (7.6):

$$\mathcal{C}_{\mathsf{E}_\varrho} = \mathcal{C}_{L^+_\varrho} \star \mathsf{E}_\varrho, \tag{$\Sigma_{L^+_\varrho}$}$$

where the operation $\star$ is a version of a convolution operation defined by the right hand side of (8.3); and L^+_ϱ is given by the following positive measure on $\mathcal{M}_f^{\cdot}(X)$.

$$L^+_\varrho(\varphi) = \sum_{j \geq 1} \sum_{a \in X} \frac{1}{j} \varphi(j\delta_a)\varrho(a)^j, \quad \varphi \in F_+.$$

This implies that E_ϱ is the so called *random KMM measure in X for L_ϱ^+* in the sense of [21].

As Mathias Rafler [25] has shown in full generality E_ϱ then coincides with the *Pólya sum process $\mathsf{S}_{\varrho,\lambda}$ for (ϱ,λ)*. This process is by definition a *Papangelou process with the kernel π^+* defined by

$$\pi^+(\mu,a) = \varrho(a)\cdot\big(\lambda(a)+\mu(a)\big), \qquad a\in X, \mu\in \mathcal{M}\;(X). \tag{8.4}$$

And this means that $\mathsf{S}_{\varrho,\lambda}$ is the unique solution S of the following integration by parts formula:

$$\mathcal{C}_{\mathsf{S}}(h) = \sum_\mu\sum_a h(a,\mu+\delta_a)\pi^+(\mu,a)\mathsf{S}(\mu), \qquad h\in F_+. \tag{8.5}$$

This process has been called in [20] the *Pólya sum process for the parameters (ϱ,λ)*. Thus we see that the characteristic properties of the Bose-Einstein process are twofold: It is a KMM process as well as a Pólya sum process.

The argument for the equality of E_ϱ and $\mathsf{S}=\mathsf{S}_{\varrho,\lambda}$ is as follows: If one iterates the last equation (8.5) one obtains for any $N\in\mathbb{N}$

$$\begin{aligned}
\mathcal{C}_{\mathsf{S}}(h) &= \sum_\mu\sum_a h(a,\mu+\delta_a)\varrho(a)(1+\mu(a))\mathsf{S}(\mu)\\
&= \sum_{j=1}^{N}\sum_\mu\sum_a \varrho(a)^j h(a,\mu+j\delta_a)\mathsf{S}(\mu) +\\
&+\sum_\mu\sum_a \varrho(a)^N h(a,\mu+N\delta_a)\mu(a)\mathsf{S}(\mu)\\
&\longrightarrow_{N\to+\infty} \sum_{j\geq 1}\sum_\mu\sum_a \varrho(a)^j h(a,\mu+j\delta_a)\mathsf{S}(\mu).
\end{aligned}$$

Here we used again that ϱ is not a Dirac measure and also that S is of first order. This shows that S solves equation (8.3) or equivalently $(\Sigma_{L_\varrho^+})$. One can show that this equation has only one solution. (Cf. [21]) To summarize we obtained the

Proposition 8.2. *Given a probability ϱ on X which is not a Dirac measure then the Bose-Einstein process E_ϱ coincides with the random KMM measure in X for L_ϱ^+ as well as the Polya sum process $\mathsf{S}_{\varrho,\lambda}$ for the parameters (ϱ,λ). Moreover, this process is infinitely divisible and uniquely determined as a solution of the integration-by-parts formula (8.3).*

We know also from [20] that the property of E_ϱ being a Papangelou process for π^+ allows to calculate explicitly its particle number distribution. In the case where ϱ is the uniform distribution on X this coincides with p_ϱ^+ which we calculated above by completely different quantum mechanical methods. This implies that the point process in this case is *of first order*, i.e. the mean particle number is finite. (All this can be found in [20].) This shows that E_ϱ has all properties of an ideal gas.

Moreover, equation $\left(\Sigma_{L_+^*}\right)$ implies that E_ϱ is a socalled permanental process. This means that its reduced density matrix has a permanental structure. A proof based on $\left(\Sigma_{L_+^*}\right)$ can be found in [21, 15] and the references therein.

Finally, using the above developed method of the Campbell measure, in particular Theorem 4, we obtain immediately characterizations of the Bose-Einstein statistical operator for w: The fact that $\kappa_{\mathbb{E}_w} = \mathsf{E}_\varrho$ solves equation $\left(\Sigma_{L_+^*}\right)$ immediately implies

Theorem 8.1. *Let w be a statistical operator on $\mathbb{C}^X$ with spectral law ϱ which is not a Dirac measure. A symmetric statistical operator $\mathbb{W}$ on the Fock space $\mathbb{H}_+$, admitting a spectral resolution with respect to $\mathcal{Y}_+$, coincides with $\mathbb{E}_w$ iff it is a solution of equation $\mathcal{C}_{\mathbb{W}} = \mathcal{C}_{L_+^*} \star \kappa_{\mathbb{W}}$.*

Moreover, $\kappa_{\mathbb{E}_w} = \mathsf{E}_\varrho$ being also a solution to equation (8.5), implies

Theorem 8.2. *Under the assumptions of Theorem 8.1, $\mathbb{W}$ coincides with $\mathbb{E}_w$ iff it is the solution of the equation*

$$\mathcal{C}_{\mathbb{W}}\, h = \sum_{(x,\gamma)} h(x, \gamma + \delta_x)\, \pi^+(\gamma, x) \kappa_{\mathbb{W}}(\gamma), \quad h \in F_+. \tag{8.6}$$

Statistical operators $\mathbb{W}$ which solve equation (8.6) can be called Pólya sum statistical operators specified by π_+.

8.2. **Fermions.** The Campbell measure of D_ϱ is concentrated on C and given there by

$$\mathcal{C}_{\mathsf{D}_\varrho}(a, \mu) = \varrho(a) \cdot \mathsf{D}_\varrho(\mu - \delta_a), \quad \mu(a) = 1.$$

This implies that D_ϱ is a Papangelou process for the kernel

$$\pi^-(a, \mu) = \varrho(a) \cdot \Big(\lambda(a) - \mu(a)\Big), \qquad \mu(a) \leq 1;$$

(and $\pi^- \equiv 0$ else.) Recall here that λ denotes the counting measure. In the terminology of [20], D_ϱ is a *Pólya difference process for* (λ, ϱ). As for Bosons the distribution of the particle number is explicitly known, and the process is of first order. Again D_ϱ is completely determined by its kernel π_-. D_ϱ is a simple process, i.e. concentrated on $\mathcal{M}^{\cdot}(X)$, and thus respects Pauli's exclusion principle. Furthermore, D_ϱ has independent increments. Thus it has all properties of an ideal gas. (For more details we refer to [20].) We observe here that the same reasoning we did above for the Papangelou process E_ϱ yields that

Proposition 8.3. *The Papangelou process D_ϱ is the unique solution of the following equation for simple point processes* Q.

$$(8.7)\qquad \mathcal{C}_{Q}(h) = \sum_{j=1}^{+\infty}(-1)^{j-1}\sum_{a,\mu}\varrho(a)^j h(a,\mu + j\delta_a)\ Q(\mu), \quad h \in F_+.$$

(The proof is exactly the same as above.) Again equation (8.10), which has D_ϱ as a unique solution, is of the form

$$\left(\Sigma_{L_\varrho^-}\right)\qquad C_Q = \mathcal{C}_{L_\varrho^-} \star Q,$$

but now for the signed measure

$$L_\varrho^-(\varphi) = \sum_{j\geq 1}\sum_{a\in X}\frac{(-1)^{j-1}}{j}\varphi(j\delta_a)\varrho(a)^j, \quad \varphi \in F_+.$$

In this case one can show (see [21, 15]) that $\left(\Sigma_{L_\varrho}\right)$ implies that D_ϱ is a so called determinantal process.

As above for Bosons we obtain a characterization of symmetric statistical operators for Fermions: A symmetric statistical operator W, admitting a spectral resolution with respect to $\mathcal{Y}_-$, coincides with $\mathbb{D}_w$ iff it is the unique solution of the equation $\mathcal{C}_W = \mathcal{C}_{L_\varrho^-} \star \kappa_W$; or equivalently, iff it is the solution of the equation

$$\mathcal{C}_W h = \sum_{(x,\gamma)} h(\gamma + \delta_x)\,\pi_-(\gamma, x)\kappa_W(\gamma), \quad h \in F_+.$$

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