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FINE PROPERTIES OF FUNCTIONS FROM HAJŁASZ–SOBOLEV CLASSES M_α^p, $p > 0$, I. LEBESGUE POINTS

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Abstract. Let X be a metric measure space satisfying the doubling condition of order $\gamma > 0$. For a function $f \in L^p_{loc}(X)$, $p > 0$ and a ball $B \subset X$ by $I_B^{(p)} f$ we denote the best approximation by constants in the space $L^p(B)$. In this paper, for functions f from Hajłasz–Sobolev classes $M_\alpha^p(X)$, $p > 0$, $\alpha > 0$, we investigate the size of the set E of points for which the following limit exists: $\lim_{r\to+0} I^{(p)}_{B(x,r)} f = f^*(x)$. We prove that the complement of the set E has zero outer measure for some general class of outer measures (in particular, it has zero capacity). A sharp estimate of the Hausdorff dimension of this complement is given. Besides, it is shown that for $x \in E$ $\lim\limits_{r\to+0} \fint_{B(x,r)} |f - f^*(x)|^q \, d\mu = 0$, $1/q = 1/p - \alpha/\gamma$. Similar results are also proved for the sets where the "means" $I^{(p)}_{B(x,r)} f$ converge with a specified rate.

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1. Introduction

The classical Lebesgue theorem (see [1, Chapter 1, §1]) states that for *any function $f \in L^p_{loc}(\mathbb{R}^n)$, $p \geq 1$, almost all points are Lebesgue points, that is, for μ-almost everywhere $x \in \mathbb{R}^n$ the limit*

$$\lim_{r\to+0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu = f^*(x) \tag{1.1}$$

exists and the function f^* is equivalent to f, where μ is the Lebesgue measure on $\mathbb{R}^n$.

For more regular functions, the sizes of the Lebesgue exceptional set (that is, the set of points for which the limit (1.1) does not exist), can be estimated more precisely. For instance, for functions from Sobolev space $W_k^p(\mathbb{R}^n)$ with $1 < p < n/k$, this exceptional set has vanishing W_k^p-capacity, and its Hausdorff dimension is at most $n - kp$. Such type questions first were considered in [2], and then were generalized in [3]–[5]. A detailed history of similar results on $\mathbb{R}^n$ can be found in [6, Chapter 6.2].

Recently similar problems in more general situation, for Hajłash Sobolev classes $M^p_\alpha(X)$, $p \geq 1$, $\alpha > 0$, on metric spaces with measure (see definition below) have been studied by a number of authors (see [7]–[14], and references therein).

In the present paper, we also study estimates of the size of the Lebesgue exceptional set for functions from Hajlasz Sobolev classes $M^p_\alpha(X)$, but for $p > 0$, in which case the functions from $M^p_\alpha(X)$ can be non-summable. To this end, we first introduce a generalization of the notion of Lebesgue points, which does not use the integral averages over balls.

2. The basic notions and notation

Let (X, d, μ) be a metric space with a regular Borel measure μ and a metric d. We assume that the measure μ satisfies the doubling condition, that is, there exists a number $a_\mu > 0$, such that

$$\mu(B(x,2r)) \leq a_\mu \mu(B(x,r)), \quad x \in X, \quad r > 0. \tag{2.1}$$

Note that the condition (2.1) can be given the following quantitative form: under (2.1) for some $\gamma > 0$ (can be taken $\gamma = \log_2 a_\mu$) the following inequality holds:

$$\mu(B(x,R)) \leq a_\mu \left(\frac{R}{r}\right)^\gamma \mu(B(x,r)), \quad x \in X, \ 0 < r \leq R. \tag{2.2}$$

The constant γ plays the role of the dimension of the space X.

For a ball $B \subset X$, by r_B and x_B we denote the radius and center of B, respectively, and by λB we denote the concentric with B ball of radius λr_B.

Throughout the paper we will use the notation

$$f_B = \fint_B f \, d\mu = \frac{1}{\mu(B)} \int_B f \, d\mu$$

for the average of a function $f \in L^1_{\mathrm{loc}}(X)$ over the ball $B \subset X$.

By letter c will be denoted various positive constants, possibly depending on some parameters, but these dependence will not be essential for us. Besides, the notation $A \lesssim B$ will mean $A \leq cB$.

It is easy to check (see, e.g., [15, Lemma 3]) that for any number $p > 0$, a ball $B \subset X$ and a function $f \in L^p(B)$ there exists a number $I^{(p)}_B f$ to satisfy

$$\left(\fint_B |f(y) - I^{(p)}_B f|^p \, d\mu(y)\right)^{1/p} = \inf_I \left(\fint_B |f(y) - I|^p \, d\mu(y)\right)^{1/p}.$$

Note that the number $I_B^{(y)} f$ generally is not uniquely defined. In this case we take any of the possible values of $I_B^{(y)} f$. The numbers $I_B^{(y)} f$ we play the role of integral averages f_B for nonsummable functions.

For a function $f \in L^p(X)$ by $D_\alpha[f]$ we denote the class of all nonnegative μ-measurable functions g, for each of which there exists a set $E \subset X$ with $\mu(E) = 0$, such that

$$|f(x) - f(y)| \le \big[d(x,y)\big]^\alpha [g(x) + g(y)], \quad x, y \in X \setminus E.$$

The elements of $D_\alpha[f]$ are called generalized α-gradients of function f. We list some simple properties of the generalized α-gradients $D_\alpha[\cdot]$ that will be used below without additional comments and references:

$$D_\alpha[f] \subset D_\alpha\big[|f|\big], \quad D_\alpha[f + \mathrm{const}] = D_\alpha[f],$$

$$g_f \in D_\alpha[f] \text{ and } g_v \in D_\alpha[v] \quad \Longrightarrow \quad g_f + g_v \in D_\alpha[f + v],$$

$$g_f \in D_\alpha[f] \quad \Longrightarrow \quad c g_f \in D_\alpha[cf], \quad c > 0,$$

if $g_i \in D_\alpha[f_i]$ and $\sup_i f_i < +\infty$ μ-almost everywhere, then

$$\sup_{i \in \mathbb{N}} g_i \in D_\alpha\big[\sup_{i \in \mathbb{N}} f_i\big]. \tag{2.3}$$

We introduce the scale of Hajłasz Sobolev classes $M_\alpha^p(X) = \{f \in L^p(X) : D_\alpha[f] \cap L^p(X) \neq \varnothing\}$, $0 < p < \infty$, $\alpha > 0$. These spaces are normed as follows:

$$\|f\|_{M_\alpha^p(X)} = \|f\|_{L^p(X)} + \inf\left\{\|g\|_{L^p(X)} : g \in D_\alpha[f] \cap L^p(X)\right\} \tag{2.4}$$

(notice that for $0 < p < 1$ the expression (2.4) is only a pre-norm). For $\alpha = 1$ these spaces were introduced in the paper by P. Hajłasz [7], where it was shown that $M_1^p(\mathbb{R}^n)$ coincides with the classical Sobolev space $W_1^p(\mathbb{R}^n)$. For $\alpha > 0$ these spaces first were appeared in [16, 17].

For $\alpha > 0$ define the Hölder classes:

$$H_\alpha(X) = \left\{\phi \in C(X) : \|\phi\|_{H_\alpha(X)} = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{[d(x,y)]^\alpha} < +\infty\right\}.$$

Observe that in contrast to the classical cases $X = [0,1]^n$ and $X = \mathbb{R}^n$, the class $H_\alpha(X)$ can be nonempty for some $\alpha > 1$. Notice also that the class $H_\alpha(X)$ is everywhere dense in $M_\alpha^p(X)$ for $p > 0$ and $0 < \alpha \le 1$ (see [7, Theorem 5] for $p > 1$ and $\alpha = 1$), in the case $p > 0$ and $0 < \alpha \le 1$ the proof is similar.

The classes M_α^p generate the capacities:

$$\mathrm{Cap}_{\alpha,p}(E) = \inf\left\{\|f\|^p_{M_\alpha^p(X)} : f \in M_\alpha^p(X), f \ge 1 \text{ in the neighborhood of } E \subset X\right\}.$$

The capacities, together with the measure, the content and the Hausdorff dimension will play the role of "measuring instruments" of the sizes of exceptional sets.

Now we consider another way of estimating the complement of an exceptional set, which is based on the use of the so-called modified Hausdorff contents (see [19] and [22]).

Let $h : (0, 1] \to (0, 1]$ be a given increasing function with $h(+0) = 0$, which will be called a gauge function. The modified Hausdorff R-content of codimension h for a set $E \subset X$ is defined to be

$$\mathcal{H}^h_R(E) = \inf\left\{\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{h(r_i)} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \quad r_i < R\right\},$$

where the infimum is taken over all possible coverings of the set E by countable collections of balls, while the quantity $\mathcal{H}^h(E) = \lim_{R\to+0} \mathcal{H}^h_R(E)$ is called the modified Hausdorff measure of codimension h for E.

Recall that for a gauge function h and $0 < R \le 1$ the classical Hausdorff (h, R)-content of a set $E \subset X$ is defined as follows:

$$H^h_R(E) = \inf\left\{\sum_{i=1}^{\infty} h(r_i) : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \quad r_i < R\right\},$$

where the infimum is taken over all possible coverings of the set E by countable collections of balls, while the quantity $H^h(E) = \lim_{R\to+0} H^h_R(E)$ is called the Hausdorff h-measure for E. For $h(t) = t^\alpha$, $\alpha > 0$, we write H^α instead of H^{t^α}. The Hausdorff dimension is defined as follows:

$$\dim_H E = \inf\left\{s : H^s_1(E) = 0\right\}.$$

3. The main results

In the definition of Lebesgue points of a function $f \in L^p(X)$, $p > 0$, instead of the integral averages f_B, we use the best approximations $I^{(p)}_B f$. The next theorem shows that the results, proved for averages f_B, remain valid for such defined Lebesgue points.

Theorem 3.1. *Let $\alpha \in (0, 1]$, $0 < p < \gamma/\alpha$ and $f \in M^p_\alpha(X)$. Then there exists a set $E \subset X$ such that $\mathrm{Cap}_{\alpha,p}(E) = 0$ and for any $x \in X \setminus E$ the following limit exists:*

(3.1) $$\lim_{r\to+0} I^{(p)}_{B(x,r)} f = f^*(x).$$

Besides, we have

$$\lim_{r\to+0} \fint_{B(x,r)} |f - f^*(x)|^q \, d\mu = 0, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}. \tag{3.2}$$

The result of Theorem 3.1 for classical Lebesgue points is well-known. In the case $X = \mathbb{R}^n$, $p > 1$, $\alpha = 1$ it was proved in [2], for $p > 1$, $\alpha = 1$ Theorem 3.1, without relation (3.2), was established in [8], for the case $p > 1$, $\alpha = 1$ and $1/q > 1/p - \alpha/\gamma$ it was proved in [9], for $p > 1$, $\alpha > 0$ in [10]–[12], and for $p = 1$, $\alpha = 1$ in [13].

Remark 3.1. In addition to the result of Theorem 3.1, for any $x \in X \setminus E$ the following assertions hold:

1) if $0 < \theta \leq q$, then

$$\lim_{r\to+0} \fint_{B(x,r)} |f - I^{(\theta)}_{B(x,r)} f|^\theta \, d\mu = 0,$$

2) if $0 < \theta \leq q$, then

$$\lim_{r\to+0} I^{(\theta)}_{B(x,r)} f = f^*(x),$$

3) if $p \geq \frac{\gamma}{\gamma+\alpha}$ (in this case $q \geq 1$), then

$$\lim_{r\to+0} \fint_{B(x,r)} |f - f_{B(x,r)}|^q \, d\mu = 0,$$

in particular, we have $\lim_{r\to+0} f_{B(x,r)} = f^*(x)$.

When preparing the present paper, the preprint [18] was published, where a different approach to the definition of Lebesgue points was proposed. This approach is based on the use of the quantities

$$m^\delta_f(E) = \inf\{a \in \mathbb{R} : \mu(\{x \in E : f(x) > a\}) < \delta\mu(E)\}, \quad 0 < \delta \leq 1/2, \tag{3.3}$$

which are called δ-medians of a measurable function f over the set $E \subset X$ of finite measure.

The main result of [18] concerning the spaces $M^p_\alpha(X)$ (in [18] also was considered Besov and Triebel-Lizorkin type spaces) is that the result of Theorem 3.1 holds when the best approximations $I^{(\theta)}_{B(x,r)} f$ are replaced by medians $m^\delta_f(B(x,r))$ for any $0 < \delta \leq 1/2$, that is, in Theorem 3.1 the relation (3.1) can be replaced by

$$\lim_{r\to+0} m^\delta_f(B(x,r)) = f^*(x). \tag{3.4}$$

We show that in a sense these two approaches are equivalent, and hence the above cited result from [18] can easily be deduced from our Theorem 3.1.

Corollary 3.1. *Let $\alpha \in (0, 1]$, $0 < p < \gamma/\alpha$ and $f \in M_\alpha^p(X)$. Then there exists a set $E \subset X$, such that $\operatorname{Cap}_{\alpha,p}(E) = 0$ and for any $x \in X \setminus E$ and any $0 < \delta \le 1/2$ the relation* (3.4) *holds.*

A nonnegative function ν, defined on the σ-algebra of Borel sets from X, is called an outer measure if it is monotone (that is, $E_1 \subset E_2$ implies $\nu(E_2) \le \nu(E_2)$) and is subadditive with some constant a_ν, meaning that for any sequence of Borel sets E_k the following inequality holds: $\nu\left(\bigcup_k E_k\right) \le a_\nu \sum_k \nu(E_k)$.

Let h be a given gauge function. We will use the following condition that connect the underlying measure μ and the outer measure ν: there exists a constant c_ν, such that

$$\nu(B) \le c_\nu \frac{\mu(B)}{h(r_B)} \quad \text{for all balls} \quad B \subset X,\ r_B \le 1. \tag{3.5}$$

In the theorem that follows, in terms of the above condition, we give an estimate of the complement of the set of points at which the relation (3.1) is satisfied for functions from the classes $M_\alpha^p(X)$ for all $p > 0$ and $\alpha > 0$. We will consider gauge functions h of the following form:

$$h(t) = \left(\frac{t^\alpha}{\varphi(t)}\right)^p, \tag{3.6}$$

where $\varphi : (0, 1] \to (0, 1]$ is a positive increasing function for which $\varphi(t)t^{-\alpha}$ decreases.

Theorem 3.2. *Let $\alpha > 0$, $0 < p < \gamma/\alpha$ and $f \in M_\alpha^p(X)$. Let ν be an outer measure satisfying condition* (3.5) *with a gauge function h of the form* (3.6)*, such that*

$$\sum_{i=0}^{\infty} \varphi(2^{-i}) < \infty. \tag{3.7}$$

Then there exists a set $E \subset X$ such that $\nu(E) = 0$ and for any $x \in X \setminus E$ the limit in (3.1) *exists and the relation* (3.2) *is satisfied.*

Observe that $\nu = \operatorname{Cap}_{\alpha,p}$ satisfies the condition (3.5) for $\varphi = 1$, for which (3.7) is not satisfied. Hence Theorem 3.1 cannot be deduced from Theorem 3.2. As an example of outer measure ν satisfying conditions of the theorem can be considered the modified Hausdorff measure $\mathcal{H}^h$, from which we obtain the following result.

Corollary 3.2. *Let $\alpha > 0$, $0 < p < \gamma/\alpha$ and $f \in M_\alpha^p(X)$, and let a gauge function given by* (3.6) *satisfy the condition* (3.7)*. Then there exists a set $E \subset X$ such that $\mathcal{H}^h(E) = 0$ and for any $x \in X \setminus E$ the limit in* (3.1) *exists and the relation* (3.2) *is satisfied.*

A similar result can be stated in terms of medians $m_f^\delta(B(x,r))$, instead of the best approximations $I^{(p)}_{B(x,r)}f$.

Corollary 3.3. *Let $\alpha > 0$, $0 < p < \gamma/\alpha$ and $f \in M^p_\alpha(X)$, and let a gauge function given by* (3.6) *satisfy the condition* (3.7). *Then there exists a set $E \subset X$ such that $\mathcal{H}^h(E) = 0$ and for any $x \in X \setminus E$ the limit in* (3.4) *exists and the relation* (3.2) *is satisfied.*

Note that this result, without relation (3.2), for $\alpha \in (0,1]$ and $p \in (0,1)$ was proved in [22].

The classical content H_1^β satisfies the condition (3.5) with function $h(t) = t^{\gamma-\beta}$, at least locally. Indeed, we fix the ball $B_0 = B(x_0, R_0)$ and let B be a ball that is contained in B_0, then by doubling condition we have $\mu(B) \geq \frac{\mu(B_0)}{R_0^\gamma} r_B^\gamma$. Therefore

$$H_1^\beta(B) \leq r_B^\beta = r_B^{\beta-\gamma} r_B^\gamma \lesssim \frac{R_0^\gamma}{\mu(B_0)} r_B^{-(\gamma-\beta)} \mu(B).$$

Hence taking into account the subadditivity of the content H_1^β we can state the following result.

Corollary 3.4. *Let $\alpha > 0$, $0 < p < \gamma/\alpha$ and $f \in M^p_\alpha(X)$. Then there exists a set $E \subset X$, such that $\dim_{\mathrm{H}}(E) \leq \gamma - \alpha p$ and for any $x \in X \setminus E$ the limit in* (3.1) *exists and the relation* (3.2) *is satisfied.*

Our next result estimates the sizes of the complement of a set on which the "averages" $I^{(p)}_{B(x,r)}f$ converge with a specified rate.

Theorem 3.3. *Let $0 < \alpha$, $0 < p < \gamma/\alpha$ be given, and let ν be an outer measure satisfying the condition* (3.5) *with h of the form* (3.6) *and a function φ satisfying* (3.7). *Then for any function $f \in M^p_\alpha(X)$ there exists a set $E \subset X$, such that $\nu(E) = 0$ and for all $x \in X \setminus E$ the following relations hold:*

$$\lim_{r\to+0} [\varphi(r)]^{-1} [f^*(x) - I^{(p)}_{B(x,r)}f] = 0, \tag{3.8}$$

$$\lim_{r\to+0} [\varphi(r)]^{-1} [m_f^\delta(B(x,r)) - f^*(x)] = 0, \quad 0 < \delta \leq 1/2, \tag{3.9}$$

$$\lim_{r\to+0} [\varphi(r)]^{-1} \left(\fint_{B(x,r)} |f - f^*(x)|^q \, d\mu \right)^{1/q} = 0, \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}. \tag{3.10}$$

Notice that the set E in Theorem 3.3 satisfies the condition

$$\mathcal{H}^{(\alpha-\beta)p}(E) = H^{\gamma-(\alpha-\beta)p}(E) = 0$$

for $0 < \beta < \alpha$ (see [14, Theorem 2] in the case $p > 1$ and $\varphi(t) = t^{\beta}$).

Remark 3.2. All above stated results remain valid for the homogeneous versions $\dot{M}^p_\alpha(X)$ of Hajłasz–Sobolev spaces, which are defined to be the sets of measurable functions f satisfying $D_\alpha[f] \cap L^p(X) \neq \varnothing$.

4. Auxiliary notions and results

We first examine the behavior of the best approximations $I^{(p)}_{B(x,r)}f$.

Lemma 4.1. *If $f \in C(X)$, $x \in X$ and $p > 0$, then $\lim\limits_{r \to +0} I^{(p)}_{B(x,r)} f = f(x)$. If $f \in L^p(X)$, then the limit $\lim_{r\to+0} I^{(p)}_{B(x,r)} f$ exists μ-almost everywhere.*

The first assertion is obvious, while the second was proved in [15, Lemma 7] (see also [24, Theorem 2] and [25, Lemma 2]). Lemma 4.1 shows that for nonsummable functions, in the definition of Lebesgue points the integral averages $f_{B(x,r)}$ can be replaced by the best approximations $I^{(p)}_{B(x,r)}f$.

The basic tools in the proofs of the main results play the L^p- oscillations:

$$A_p(f, B) = \left(\fint_B |f - I^{(p)}_B f|^p \, d\mu \right)^{1/p},$$

where $B \subset X$ is a ball, and the corresponding maximal operators:

$$A^{(p)}_\alpha f(x) = \sup_{B \ni x} r_B^{-\alpha} A_p(f, B),$$

where the supremum is taken over all balls B containing the point x, and $\alpha \geq 0$, $f \in L^p_{\mathrm{loc}}(X)$, $p > 0$.

Lemma 4.2. *Let α, p, $\theta > 0$. The following assertions hold:*

1) if $B_1, B_2 \subset X$ are two balls such that $B_1 \subset B_2$ and $0 < r_{B_1} \leq r_{B_2}$, then

$$|I^{(p)}_{B_1} f - I^{(p)}_{B_2} f| \lesssim A_p(f, B_1) + \left(\frac{r_{B_2}}{r_{B_1}}\right)^{\gamma/p} A_p(f, B_2); \tag{4.1}$$

2) for any ball $B \subset X$ we have

$$A_\theta(f, B) \lesssim r_B^\alpha \left(\fint_B |A^{(\theta)}_\alpha f|^p \, d\mu \right)^{1/p}. \tag{4.2}$$

if, in addition, $p < \gamma/\alpha$, *then*

$$\left(\fint_B |f - I_B^{(\theta)} f|^q \, d\mu\right)^{1/q} \lesssim r_B^\alpha \left(\fint_{2B} |A_\alpha^{(\theta)} f|^p \, d\mu\right)^{1/p}, \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}. \tag{4.3}$$

Proof. The inequality (4.1) was proved in [15, Lemma 5]. The inequality (4.2) can be obtained by averaging over $y \in B$ of the obvious inequality

$$A_\theta(f, B) \le r_B^\alpha A_\alpha^{(\theta)} f(y) \quad \text{for} \quad y \in B,$$

while (4.3) is proved in [15, Theorem 2].

The next lemma follows from the so-called self-inprovement property of the Poincaré inequality (see [15, Theorem 6]).

Lemma 4.3. *Let* $1/\theta < 1/p + \alpha/\gamma$. *Then for* $f \in M_\alpha^p(X)$ *and* $g \in D_\alpha[f]$

$$A_p(f, B) \lesssim r_B^\alpha \left(\fint_{2B} g^\theta \, d\mu\right)^{1/\theta}$$

Next, we present results containing descriptions of classes in terms of maximal functions $A_\alpha^{(\theta)}$ (see [23, Corollary 3.1], and also [15, Theorem 4]).

Lemma 4.4. *If* $\alpha, p > 0$ *and* $f \in L^p(X)$, *then*

1) from $A_\alpha^{(\theta)} f \in L^p(X)$ *for some* $\theta > 0$ *follows* $f \in M_\alpha^p(X)$,

2) from $f \in M_\alpha^p(X)$ *follows* $A_\alpha^{(\theta)} f \in L^p(X)$ *for* $1/\theta > \max\{1/p - \alpha/\gamma, 0\}$.

Lemma 4.5. *Let* $\alpha, p > 0$. *If* $f \in M_\alpha^p(X)$, $\varphi \in H_\alpha(X)$ *and is bounded, then* $f\varphi \in M_\alpha^p(X)$. *Besides, if* $\phi(x) = 0$ *for* $x \in X \setminus E$, *then for any function* $g \in D_\alpha[f] \cap L^p(X)$ *we have*

$$\left(g \cdot \|\phi\|_\infty + |f| \cdot \|\phi\|_{H_\alpha(X)}\right) \chi_E \in D_\alpha[f\phi] \cap L^p(X).$$

The result of Lemma 4.5 is known for $p > 1$ (see [8, Lemma 5.20] and [9, Lemma 2.5]). The proof is similar in the case $p > 0$. The next classical lemma can be found in [26] (see also [27, Lemma 1.6]).

Lemma 4.6. *From each covering of the set* $E \subset X$ *by balls of bounded diameters can be selected at most countable set of mutually disjoint balls* $\{B_i\}$ *satisfying* $E \subset \bigcup_i 5B_i$.

For a measurable function h we introduce the h-"fractional" maximal function:

$$\mathcal{M}_{h,p} g(x) = \sup_{B \ni x} \left(h(r_B) \fint_B |g|^p \, d\mu\right)^{1/p}$$

In the special case $h(t) \equiv 1$ we obtain the usual modified Hardy-Littlewood maximal function:

$$\mathcal{M}_p g(x) = \sup_{B \ni x} \left(\fint_B |g|^p \, d\mu \right)^{1/p},$$

for which for $0 < \theta < p$ the following standard estimates hold (see, e.g., [1, §1.3]):

$$\|\mathcal{M}_\theta f\|_{L^p(X)} \lesssim \|f\|_{L^p(X)} \tag{4.4}$$

In the next lemma we collect the necessary properties of h-"fractional" maximal function.

Lemma 4.7. *Let $p > 0$ and ν be an outer measure satisfying the condition* (3.5). *Then the following weak type inequality holds:*

$$\nu \{x \in X : \mathcal{M}_{h,p} g(x) > \lambda\} \lesssim \left(\frac{1}{\lambda} \|g\|_{L^p(X)} \right)^p, \tag{4.5}$$

and $\nu(E_{h,p}[g]) = 0$ for any function $g \in L^p(X)$, where

$$E_{h,p}[g] := \left\{ x \in X : \varlimsup_{r \to +0} h(r) \fint_{B(x,r)} |g|^p \, d\mu > 0 \right\}.$$

Proof. We first prove the second assertion of the lemma. By subadditivity of measure ν, it is enough to show that $\nu(E_\lambda) = 0$ for $\lambda > 0$, where

$$E_\lambda = \left\{ x \in X : \varlimsup_{r \to +0} h(r) \fint_{B(x,r)} g^p \, d\mu > \lambda \right\}.$$

Let $0 < \delta \le 1$. If $x \in E_\lambda$, then there exists a ball $B_x = B(x, r_x)$, where $r_x \in (0, \delta)$, such that

$$h(r_x) \fint_{B_x} g^p \, d\mu > \lambda. \tag{4.6}$$

By Lemma 4.6, from the covering $\{B_x : x \in E_\lambda\}$ it can be selected at most countable subset of mutually disjoint balls $\{B_i = B_{x_i}\}$ satisfying $E_\lambda \subset \bigcup_i 5B_i$.

Hence taking into account the subadditivity of ν, and conditions (2.2), (3.5) and (4.6), we can write

$$\nu(E_\lambda) \le \nu\left(\bigcup_i 5B_i \right) \lesssim \sum_i \nu(5B_i) \lesssim \sum_i \mu(B_i)/h(r_i) \lesssim$$
$$\lesssim \frac{1}{\lambda} \sum_i \int_{B_i} g^p \, d\mu \lesssim \frac{1}{\lambda} \int_{\bigcup_i B_i} g^p \, d\mu \to 0 \quad \text{as} \quad \delta \to 0.$$

The last relation follows from absolute continuity of the integral and continuity of the gauge function h at zero, because by (4.6) we have

$$\mu\left(\bigcup_i B_i\right) \le \sum_i \frac{h(r_i)}{\lambda} \int_{B_i} g^p\, d\mu \le \frac{h(\delta)}{\lambda} \int_{\mathcal{E}} g^p\, d\mu \to 0 \quad \text{as} \quad \delta \to 0.$$

Repeating the above arguments for the set $\{\mathcal{M}_{h,p} g > \lambda\}$, we obtain (4.5).

Lemma 4.8. *Let $p > 0$, $0 \le \beta < \alpha$, and let the outer measure ν satisfy the condition (3.5), where $h(t) = t^{(\alpha-\beta)p}$. Then for $f \in L^p_{\mathrm{loc}}(X)$ the following inequality holds:*

$$\int_0^\infty \lambda^{p-1} \nu\{A_\beta^{(p)} f > \lambda\}\, d\lambda \lesssim \|A_\alpha^{(p)} f\|^p_{L^p(X)}.$$

This result can be found in [20] (see Theorem 2 and the remark at the end of §2 in [20]).

Lemma 4.9 ([10, 11]). *Let $E \subset X$, $0 < \alpha \le 1$, $\gamma > \alpha p$ and $p > 0$. Then the following assertions hold:*

1) the capacity $\mathrm{Cap}_{\alpha,p}$ is an outer measure and

$$\mathrm{Cap}_{\alpha,p}(E) = \inf\left\{\mathrm{Cap}_{\alpha,p}(O) : E \subset O, O \text{ is open}\right\}.$$

2) for $x \in X$, $0 < r \le 1$ $\mathrm{Cap}_{\alpha,p}(B(x,r)) \lesssim r^{-\alpha p} \mu(B(x,r))$.

3) for $0 < \beta \le \alpha$ $\mathrm{Cap}_{\alpha,p}(E) = 0 \Rightarrow \mathrm{Cap}_{\beta,p}(E) = 0$.

5. A WEAK INEQUALITY FOR CAPACITIES

To prove the main result of the paper, we need a number of results, which can also represent an independent interest. Such results are weak estimates of capacities, which will be obtained using a discrete maximal function. To define this notion, we need some preliminary work. We start with lemmas on coverings and on existence of partition of unity (see [9])

Lemma 5.1. *There exists a number $N \in \mathbb{N}$, such that for any $r > 0$ a family of finite or countable balls $\left\{B(x_i, r)\right\}_{i=1}^\infty$ can be found to satisfy*

$$X \subset \bigcup_{i=1}^\infty B(x_i, r), \quad \sum_{i=1}^\infty \chi_{B(x_i, 6r)} \le N.$$

Lemma 5.2. *Let $0 < \alpha \le 1$ and $r > 0$. For the balls $\{B(x_i, r)\}$ from Lemma 5.1 there exists a collection of functions $\{\phi_i\} \subset H_\alpha(X)$ possessing the following properties:*

1) $0 \le \phi_i \le 1$.

2) $\phi_i(x) = 0$, $x \in X \setminus B(x_i, 6r)$.

3) $\phi_i(x) \geq c$, $x \in B(x_i, 3r)$, *where constant* $c > 0$ *depends only on* a_μ *from* (2.1).

4) $\|\phi_i\|_{H_\alpha(X)} \lesssim r^{-\alpha}$,

5) $\sum_{i=1}^{\infty} \phi_i(x) \equiv 1$.

Now we are ready to define the discrete maximal function. Let $r > 0$ and $\{B_i\}$ be a covering of X by balls $B(x_i, r)$ from Lemma 5.1, and let ϕ_i be the functions from Lemma 5.2. Define the discrete convolution of function f by

$$f_r(x) = \sum_{i=1}^{\infty} \phi_i(x) |I^{(p)}_{B(x_i, 3r)} f|, \tag{5.1}$$

and the discrete maximal operator by

$$M^* f(x) = \sup_j f_{r_j}(x),$$

where $\{r_j\}$ are in some way enumerated sequence of rational numbers from $(0, 1)$.

Note that in the standard definition of the discrete maximal function (see [9], [13]) is used the integral averages. In our case, these averages are replaced by the elements of the best approximation $I^p_B f$. Now we proceed to prove a number of properties of the discrete convolution and discrete maximal function. We will follow the scheme from [13].

Lemma 5.3. *The operator of discrete convolution is bounded in the spaces* $L^p(X)$ *and* $M^p_\alpha(X)$, *that is,*

$$\|f_r\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}, \tag{5.2}$$

$$\|f_r\|_{M^p_\alpha(X)} \lesssim \|f\|_{M^p_\alpha(X)}. \tag{5.3}$$

Proof. We first prove the inequality (5.2). We have

$$\int_X |f_r|^p \, d\mu \leq \sum_{i=1}^{\infty} \int_X \phi_i^p |I^{(p)}_{3B_i} f|^p \, d\mu \leq \sum_{i=1}^{\infty} \mu(6B_i) |I^{(p)}_{3B_i} f|^p.$$

Using the following easy verified inequality $\left|I^{(p)}_{3B_i} f\right|^p \lesssim f_{3B_i} |f|^p \, d\mu$ and taking into account the bounded multiplicity of the intersection of balls $6B_i$ (see Lemma 5.1), we obtain

$$\|f_r\|^p_{L^p(X)} \lesssim \sum_{i=1}^{\infty} \int_{3B_i} |f|^p \, d\mu \lesssim \int_X |f|^p \, d\mu,$$

and the inequality (5.2) follows.

To prove the inequality (5.3), we follow [13], and produce the α-gradient for f_r and prove that this gradient and the function f_r belong to $L^p(X)$. We use the notation $B_i = B(x_i, r)$, and represent f_r in the form

$$f_r(x) = |f(x)| + \sum_{i=1}^{\infty} \phi_i(x) \left[|I^{(p)}_{3B_i} f| - |f(x)| \right].$$

It is easy to see that $g + \sum_{i=1}^{\infty} g_i \in D[f_r]$, where $g \in D[f]$ and g_i is the generalized αgradient for $\phi_i \left[|I^{(p)}_{3B_i} f| - |f| \right]$. Using Lemma 4.5, with some constant $c > 0$ we have

$$\left(\frac{c}{r^\alpha} |f - I^{(p)}_{3B_i} f| + g \right) \chi_{6B_i} \in D \left[\phi_i \left(|I^{(p)}_{3B_i} f| - |f| \right) \right].$$

Let $x \in 6B_i$, then $3B_i \subset B(x, 9r)$. We write the obvious inequality

$$|f(x) - I^{(p)}_{3B_i} f| \le |f(x) - I^{(p)}_{B(x,9r)} f| + |I^{(p)}_{B(x,9r)} f - I^{(p)}_{3B_i} f|,$$

and estimate the terms on the right-hand side separately.

If at a point x the relation (3.1) is satisfied, then using Lemma 4.3, we can write the following chain of inequalities

$$|f(x) - I^{(p)}_{B(x,9r)} f| \le \sum_{j=0}^{\infty} |I^{(p)}_{B(x,3^{2-j}r)} f - I^{(p)}_{B(x,3^{1-j}r)} f| \lesssim$$

$$\le \sum_{j=0}^{\infty} A_p(f, B(x, 3^{2-j} r)) \lesssim \sum_{j=0}^{\infty} \frac{r^\alpha}{3^{\alpha j}} \mathcal{M}_\theta g.$$

The parameter θ we choose to satisfy the condition of Lemma 4.3, that is,

$$\frac{1}{p} < \frac{1}{\theta} < \frac{1}{p} + \frac{\alpha}{\gamma}. \tag{5.4}$$

To estimate the second term, we again use Lemma 4.3 to obtain

$$|I^{(p)}_{B(x,9r)} f - I^{(p)}_{3B_i} f| \lesssim A_p(f, B(x, 9r)) \lesssim r^\alpha \mathcal{M}_\theta g.$$

Thus, for almost all $x \in X$ (see Lemma 4.1) the inequality holds:

$$|f(x) - I^{(p)}_{3B_i} f| \lesssim r^\alpha \mathcal{M}_\theta g, \quad \frac{1}{p} < \frac{1}{\theta} < \frac{1}{p} + \frac{\alpha}{\gamma},$$

implying that $c(g + \mathcal{M}_\theta g) \in D[f_r]$. Taking into account (4.4), we obtain inequality (5.3). □

Lemma 5.4. *The discrete maximal function acts boundedly in the space $M^p_\alpha(X)$, that is,*

$$\|M^* f\|_{M^p_\alpha(X)} \lesssim \|f\|_{M^p_\alpha(X)}. \tag{5.5}$$

Proof. In Lemma 5.3 it is shown that the function $c(g + \mathcal{M}_\theta g)$ (where g is any α-gradient of function f, and θ satisfies (5.4)) is an α-gradient for the discrete convolution f_r for any r. Hence, by (2.3) this function is also an α gradient for $M^* f$. Write the inequality

$$f_r(x) \le \sum_{i'} |I^{(p)}_{3B_i} f|,$$

where i' indicates that the sum is over those indices i, for which $x \in 6B_i$ (by Lemma 5.1, the number of such indices is bounded by a constant, depending only on γ). We choose a number θ to satisfy (5.4), and estimate each term on the right-hand side of the inequality

$$\left|I^{(p)}_{3B_i} f\right| \le A_p(f, 3B_i) + \left(\fint_{3B_i} |f|^\theta \, d\mu \right)^{1/\theta} \lesssim A_p(f, 6B_i) + \mathcal{M}_\theta f(x).$$

Using the self-improvement property of the Poincaré inequality and Lemma 4.3, we obtain

$$f_r(x) \lesssim \mathcal{M}_\theta g(x) + \mathcal{M}_\theta f(x),$$

implying the inequality (5.5). The proof of Lemma 5.4 is complete. □

Now we introduce one more maximal operator:

$$M_I f(x) = \sup_{B \in x, r_B < 1} \left|I^{(p)}_B f\right|.$$

The next lemma asserts that for the operators $M_I f$, $\mathcal{M}_p f$ and $\mathcal{A}_0 f$ a weak type inequality by capacity holds.

Lemma 5.5. *Let $f \in M^p_\alpha(X)$, then*

$$\operatorname{Cap}_{\alpha,p}\{x : (Tf)(x) > \lambda\} \lesssim \frac{\|f\|^p_{M^p_\alpha(X)}}{\lambda^p}, \quad \lambda > 0, \tag{5.6}$$

where as T can be taken any of the operators $M_I f$, $\mathcal{M}_p f$ or $\mathcal{A}_0 f$.

Proof. Observe first that the inequality (5.6) for $T = \mathcal{A}^{(p)}_0$ follows from Lemma 4.8. Indeed, for given $\lambda > 0$, by Lemmas 4.3 and 4.8 we get

$$\lambda \left(\operatorname{Cap}_{\alpha,p} \left\{ \mathcal{A}^{(p)}_0 f > \lambda \right\} \right)^{1/p} \lesssim \left(\int_{\lambda/2}^{\lambda} t^{p-1} \operatorname{Cap}_{\alpha,p} \left\{ \mathcal{A}^{(p)}_0 f > t \right\} dt \right)^{1/p} \lesssim$$

$$\lesssim \|\mathcal{A}^{(p)}_\alpha f\|_{L^p(X)} \lesssim \|f\|_{M^p_\alpha(X)}.$$

To prove the inequality (5.6) for $T = M_I$, observe that

$$M_I f(x) \le c\left(\mathcal{A}^{(p)}_0 f(x) + M^* f(x)\right).$$

From the previous inequality, using the fact that the assertion is already proved for $A_0^{(p)}$, we obtain

$$\mathrm{Cap}_{\alpha,p}(M_I f > \lambda) \le \mathrm{Cap}_{\alpha,p}\left(A_0^{(p)} f > \frac{\lambda}{2c}\right) + \mathrm{Cap}_{\alpha,p}\left(M^* f > \frac{\lambda}{2c}\right) \lesssim$$
$$\lesssim \frac{\|f\|^p_{M^p_\alpha}}{\lambda^p} + \mathrm{Cap}_{\alpha,p}\left(M^* f > \frac{\lambda}{2c}\right).$$

Thus, it remains to show that

$$\mathrm{Cap}_{\alpha,p}\left(M^* f > \frac{\lambda}{2c}\right) \lesssim \frac{\|f\|^p_{M^p_\alpha}}{\lambda^p}.$$

To prove the last inequality, we use the definition of capacity to obtain

$$\mathrm{Cap}_{\alpha,p}\left(M^* f > \frac{\lambda}{2c}\right) \lesssim \left\|\frac{M^* f}{\lambda}\right\|^p_{M^p_\alpha} \lesssim \frac{\|f\|^p_{M^p_\alpha}}{\lambda^p}.$$

Therefore

$$\mathrm{Cap}_{\alpha,p}(M_I f > \lambda) \lesssim \frac{\|f\|^p_{M^p_\alpha}}{\lambda^p}.$$

Finally, the inequality (5.6) for operator $\mathcal{M}_p f$ follows from the above proved inequality and $\mathcal{M}_p f(x) \lesssim A_0^{(p)} f(x) + M_I f(x)$. Lemma 5.5 is proved.

6. Proof of Theorem 3.1 and its corollaries

We define

$$\Omega f(x) = \overline{\lim_{0<r<R\to 0}} |I^{(p)}_{B(x,r)} f - I^{(p)}_{B(x,R)} f|,$$

and show that

$$\mathrm{Cap}_{\alpha,p}\{x \in X : \Omega f(x) > 0\} = 0.$$

Taking into account that the Hölder class $H_\alpha(X)$ is everywhere dense in $M^p_\alpha(X)$, for $\varepsilon > 0$ the function f can be represented in the form $f = f_1 + f_2$, where $f_1 \in H_\alpha(X)$ and $\|f_2\|_{M^p_\alpha} < \varepsilon$.

For any $y, z \in X$ we write the obvious inequality

$$|I^{(p)}_{B(x,r)} f - I^{(p)}_{B(x,R)} f| \le |I^{(p)}_{B(x,r)} f - f(y)| + |I^{(p)}_{B(x,r)} f_1 - f_1(y)| +$$
$$+ |I^{(p)}_{B(x,r)} f_2 - f_2(y)| + |I^{(p)}_{B(x,R)} f - f(z)| + |I^{(p)}_{B(x,R)} f_1 - f_1(z)| +$$
$$+ |I^{(p)}_{B(x,R)} f_2 - f_2(z)| + |I^{(p)}_{B(x,r)} f_1 - I^{(p)}_{B(x,R)} f_1| + |I^{(p)}_{B(x,r)} f_2 - I^{(p)}_{B(x,R)} f_2|,$$

and average it by $y \in B(x,r)$ and $z \in B(x,R)$ to obtain

$$|I^{(p)}_{B(x,r)} f - I^{(p)}_{B(x,R)} f| \lesssim A_p(f, B(x,r)) + A_p(f_1, B(x,r)) +$$
$$+ A_p(f_2, B(x,r)) + A_p(f, B(x,R)) + A_p(f_1, B(x,R)) +$$
$$+ A_p(f_2, B(x,R)) + |I^{(p)}_{B(x,r)} f_1 - I^{(p)}_{B(x,R)} f_1| + |I^{(p)}_{B(x,r)} f_2 - I^{(p)}_{B(x,R)} f_2|.$$

The first six terms can be estimated similarly as follows. Using the inequality (4.2) and Lemmas 4.2, 4.7 and 4.9, we conclude that the following relation is fulfilled $\mathrm{Cap}_{\alpha,p}$-almost everywhere:

$$A_p(f, B(x,r)) \le r^\alpha \left(\fint_{B(x,r)} |A_\alpha^{(p)} f|^p \, d\mu \right)^{1/p} \to 0 \quad \text{as} \quad r \to +0.$$

Besides, in view of continuity of f_1 and Lemma 4.1, for any $x \in X$ we have

$$\varlimsup_{0<r<R\to 0} |I^{(p)}_{B(x,r)} f_1 - I^{(p)}_{B(x,R)} f_1| = 0.$$

Therefore, $\mathrm{Cap}_{\alpha,p}$-almost everywhere

$$\Omega f(x) \lesssim \Omega f_2(x) \lesssim M_I f_2(x).$$

Also, by Lemma 5.5 for any $\lambda > 0$ we have

$$\mathrm{Cap}_{\alpha,p}\{\Omega f > \lambda\} \lesssim \mathrm{Cap}_{\alpha,p}\{M_I f_2 > \lambda\} \lesssim \left(\frac{\|f_2\|_{M_\alpha^p}}{\lambda} \right)^p \lesssim \left(\frac{\varepsilon}{\lambda} \right)^p.$$

Hence for any $\lambda > 0$ we have $\mathrm{Cap}_{\alpha,p}\{\Omega f > \lambda\} = 0$, and the result follows. Theorem 3.1 is proved.

Now we prove the assertions in Remark 3.1. Observe first that the assertion 1) immediately follows from Holder inequality. To prove the assertion 2), the inequality

$$|I^{(\theta)}_{B(x,r)} f - I^{(q)}_{B(x,r)} f|^\theta \lesssim |f(y) - I^{(q)}_{B(x,r)} f|^\theta + |f(y) - I^{(\theta)}_{B(x,r)} f|^\theta$$

we average by $y \in B(x,r)$ to obtain

$$|I^{(\theta)}_{B(x,r)} f - I^{(q)}_{B(x,r)} f| \lesssim A_q(f, B(x,r)).$$

The right-hand side of the last inequality tends to 0 as $r \to +0$, which implies 2), since in view of (3.1) and (3.2)) we have $I^{(q)}_{B(x,r)} f \to f^*(x)$.

To prove the assertion 3), for $q \ge 1$ we write the obvious inequality

$$\left(\fint_{B(x,r)} |f - f_{B(x,r)}|^q \, d\mu \right)^{1/q} \le \left(\fint_{B(x,r)} |f - I^{(p)}_{B(x,r)} f|^q \, d\mu \right)^{1/q} +$$

$$+ |I^{(p)}_{B(x,r)} f - f_{B(x,r)}| \lesssim \left(\fint_{B(x,r)} |f - I^{(p)}_{B(x,r)} f|^q \, d\mu \right)^{1/q}$$

and observe that by Theorem 3.1, the right-hand side of the last inequality tends to 0 as $r \to +0$, implying the assertion 3).

To prove Corollary 3.1, we need an inequality for the difference of the best approximations $I_B^{(p)} f$ and medians $m_f^\delta(B)$.

Lemma 6.1. *For any ball $B \subset X$ and any function $f \in L^p_{\mathrm{loc}}(X)$ the following inequality holds:*

$$|m_f^\delta(B) - I_B^{(p)} f| \le \left(\frac{1}{\delta} \fint_B |f - I_B^{(p)} f|^p \, d\mu\right)^{1/p}. \tag{6.1}$$

Proof. The result easily follows from the following elementary properties of medians m_f^δ (see [21], and also [18, Lemma 2.7]):

$$m_f^\delta(B) + a = m_{f+a}^\delta(B) \quad \text{for} \quad a \in \mathbb{R}, \quad |m_f^\delta(B)| \le m_{|f|}^\delta(B),$$

$$m_{|f|}^\delta(B) \le \left(\frac{1}{\delta} \fint_B |f|^p \, d\mu\right)^{1/p}, \quad p > 0.$$

Indeed, using the above properties, for any ball $B \subset X$ we have

$$|m_f^\delta(B) - I_B^{(p)} f| \le m^\delta_{|f - I_B^{(p)} f|}(B) \le \left(\frac{1}{\delta} \fint_B |f - I_B^{(p)} f|^p \, d\mu\right)^{1/p},$$

implying (6.1).

Now, the result of Corollary 3.1 follows from Theorem 3.1, since by the inequality (4.2) of Lemma 4.2, and Lemmas 4.7 and 6.1 (see also property 2) of capacities from Lemma 4.9), for all $0 < \delta \le 1/2$ we have

$$\mathrm{Cap}_{\alpha,p}\{x \in X : \varlimsup_{r \to +0} |m_f^\delta(B(x,r)) - I_{B(x,r)}^{(p)} f| > 0\} = 0.$$

7. Proof of Theorem 3.2

Let the numbers $0 < r < 1$ and $n \in \mathbb{N}$ be such that $2^{-n} < r \le 2^{-n+1}$. Then by Lemma 4.2 (see the inequalities (4.1) and (4.2)) we have

$$|I_{B(x,r)}^{(p)} f - I_{B(x,2^{-n})}^{(p)} f| \lesssim r^\alpha \left(\fint_{B(x,r)} |A_\alpha^{(p)} f|^p \, d\mu \right)^{1/p}.$$

Hence, we can use Lemma 4.7 to obtain

$$\nu\{x \in X : \varlimsup_{r \to +0} |I_{B(x,r)}^{(p)} - I_{B(x,2^{-n})}^{(p)}| > 0\} = 0,$$

showing that it is enough to check that the sequence $\{I_{B(x,2^{-n})}^{(p)}\}$ converges ν-almost everywhere.

Let $m > n$ and $B_n = B(x, 2^{-n})$. We again use the inequalities (4.1) and (4.2), to estimate the difference $|I^{(p)}_{B_n} f - I^{(p)}_{B_m} f|$, to obtain

$$|I^{(p)}_{B_n} f - I^{(p)}_{B_m} f| \le \sum_{i=n}^{m-1} |I^{(p)}_{B_i} f - I^{(p)}_{B_{i+1}} f| \lesssim \sum_{i=n}^{m-1} 2^{-\alpha i} \left(\fint_{B_i} |A^{(p)}_\alpha f|^p \, d\mu \right)^{1/p} \lesssim$$

$$\lesssim \sum_{i=n}^{m-1} 2^{-\alpha i} \left(\fint_{B_i} |A^{(p)}_\alpha f|^p \, d\mu \right)^{1/p} \lesssim \sum_{i=n}^{m-1} \varphi(2^{-i}) \cdot \left(h(2^{-i}) \fint_{B_i} |A^{(p)}_\alpha f|^p \, d\mu \right)^{1/p} \lesssim$$

$$\lesssim \mathcal{M}_{h,p} f(x) \sum_{i=n}^{\infty} \varphi(2^{-i}) \to 0$$

as $n \to \infty$ at each point $x \in X$ for which $\mathcal{M}_{h,p} f(x) < \infty$, and hence, by Lemma 4.7, the convergence is ν-almost everywhere. Theorem 3.2 is proved.

To prove Corollary 3.2 it is enough to show that the limit (3.1) exists $\mathcal{H}^h_1$-almost everywhere, since the conditions $\mathcal{H}^h_1(E) = 0$ and $\mathcal{H}^h(E) = 0$ are equivalent. So, it remains to apply Theorem 3.2 with $\nu = \mathcal{H}^h_1$. Corollary 3.3 immediately follows from Lemma 6.1.

8. Proof of Theorem 3.3

We prove the convergence for the elements of the best approximation, that is, the equality (3.8). Then the result for medians (3.9) will follow from (3.8) and Lemma 6.1.

Let E_1 be the complement of the set of points $x \in X$ satisfying the relation:

$$\lim_{r \to +0} \left[f(x) - I^{(p)}_{B(x,r)} f \right] = 0.$$

It follows from Theorem 3.2 that $\nu(E_1) = 0$.

Let $x \in X \setminus E_1$, $0 < r < 1$ and $B_j = B(x, 2^{-j} r)$. Applying the inequalities (4.1) and (4.2), from Lemma 4.2 we obtain

$$[\varphi(r)]^{-1} |f(x) - I^{(p)}_{B(x,r)} f| \le [\varphi(r)]^{-1} \sum_{j=0}^{\infty} |I^{(p)}_{B_{j+1}} f - I^{(p)}_{B_j} f| \lesssim$$

$$\lesssim [\varphi(r)]^{-1} \sum_{j=0}^{\infty} A_p(f, B_j) \lesssim \frac{r^\alpha}{\varphi(r)} \sum_{j=0}^{\infty} 2^{-j\alpha} \left(\fint_{B_j} |A^{(p)}_\alpha f|^p \, d\mu \right)^{1/p} \lesssim$$

$$\lesssim \sup_{t < r} \left(h(t) \fint_{B(x,t)} |A^{(p)}_\alpha f|^p \, d\mu \right)^{1/p},$$

and the last term tends to 0 as $r \to +0$, provided that

$$\lim_{r \to +0} h(r) \fint_{B(x,r)} |\mathcal{A}_\alpha^{(p)} f|^p \, d\mu = 0.$$

Applying Lemma 4.7 with $g = \mathcal{A}_\alpha^{(p)} f$, we obtain $\nu(E_2) = 0$, where

$$E_2 = \left\{ x \in X : \overline{\lim_{r \to +0}}\, h(r) \fint_{B(x,r)} |\mathcal{A}_\alpha^{(p)} f|^p \, d\mu > 0 \right\}.$$

Thus, on the complement of the set $E = E_1 \cup E_2$ the relation (3.8) is fulfilled.

Now we prove the relation (3.10). It follows from inequality (4.3) that

$$[\varphi(r)]^{-1} \left(\fint_{B(x,r)} |f - I_{B(x,r)}^{(p)} f|^q \, d\mu \right)^{1/q} \lesssim \left(h(r) \fint_{B(x,2r)} \left[\mathcal{A}_\alpha^{(p)} f\right]^p d\mu \right)^{1/p},$$

and for $x \in X \setminus E_2$ the right-hand side of the last inequality tends to 0 as $r \to +0$.

Therefore for any $x \in X \setminus E$ we can write

$$\overline{\lim_{r \to +0}}\, [\varphi(r)]^{-1} \left(\fint_{B(x,r)} |f - f(x)|^q \, d\mu \right)^{1/q} \lesssim \lim_{r \to +0} [\varphi(r)]^{-1} |f(x) - I_{B(x,r)}^{(p)} f| +$$

$$+ \lim_{r \to +0} [\varphi(r)]^{-1} \left(\fint_{B(x,r)} |f - I_{B(x,r)}^{(p)} f|^q \, d\mu \right)^{1/q} = 0,$$

and (3.10) follows. Theorem 3.3 is proved.

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