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ANALYTIC SUMMABILITY OF REAL AND COMPLEX FUNCTIONS

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Abstract. Gamma-type functions satisfying the functional equation $f(x+1) = g(x)f(x)$ and limit summability of real and complex functions were introduced by Webster (1997) and Hooshmand (2001). However, some important special functions are not limit summable, and so other types of such summability are needed. In this paper, by using Bernoulli numbers and polynomials $B_n(z)$, we define the notions of analytic summability and analytic summand function of complex or real functions, and prove several criteria for analytic summability of holomorphic functions on an open domain D. As consequences of our results, we give some criteria for absolute convergence of the functional series $\sum_{n=0}^{\infty} c_n\sigma(z^n)$, where $\sigma(z^n) = S_n(z) = \frac{B_{n+1}(z+1)-B_{n+1}(1)}{n+1}$. Finally, we state some open problems for future study of analytic and limit summability of functions.

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1. Introduction and preliminaries

The notion of *limit summability of real functions* was introduced and studied in [3,4] as a generalization of the Gamma-type functions satisfying the functional equation $f(x+1) = g(x)f(x)$ from [6]. Below we summarize some definitions and results from [3,4]. Let f be a real or complex function with domain $D_f \supseteq \mathbb{N}^* := \{1, 2, 3, \cdots\}$. Put

$$\Sigma_f = \{x | x + \mathbb{N}^* \subseteq D_f\},$$

and then for any $x \in \Sigma_f$ and $n \in \mathbb{N}^*$ set

$$R_n(f, x) = R_n(x) := f(n) - f(x+n),$$

$$f_{\sigma_n}(x) = f_{\sigma_{f,n}}(x) := xf(n) + \sum_{k=1}^{n} R_k(x).$$

The function f is called limit summable at $x_0 \in \Sigma_f$ if the functional sequence $\{f_{\sigma_n}(x)\}$ is convergent at $x = x_0$. The function f is called limit summable on a set $S \subseteq \Sigma_f$ if it is limit summable at all points of S.

Now, put

$$f_\sigma(x) = f_{\sigma_\ell}(x) = \lim_{n\to\infty} f_{\sigma_n}(x)\ ,\ R(x) = R(f,x) = \lim_{n\to\infty} R_n(f,x),$$

and observe that $D_{f_\sigma} = \{x \in \Sigma_f | f \text{ is limit summable at } x\}$, and $f_{\sigma_\ell} = f_\sigma$ is the same limit function f_{σ_n} with domain D_{f_σ}.
The function f is called limit summable if it is summable on Σ_f, $R(1) = 0$ and $D_f \subseteq D_f - 1$. In this case the function f_σ is referred to as the limit summand function of f. Notice that if f is limit summable, then $D_{f_\sigma} = D_f - 1$ and

$$f_\sigma(x) = f(x) + f_\sigma(x-1)\ ;\ \forall x \in D_f.$$

Therefore, if f is limit summable, then its limit summand function f_σ satisfies the well-known difference functional equation $\varphi(x) - \varphi(x-1) = f(x)$ (see [2 - 4]). Hence, we have

$$f_\sigma(m) = f(1) + \cdots + f(m) = \sum_{j=1}^{m} f(j)\ ;\ \forall m \in \mathbb{N}^*.$$

If f is limit summable, then one may use the notation $\sigma_\ell(f(x))$ instead of $f_{\sigma_\ell}(x)$.
In [3,4] were obtained some criteria for existence of unique solutions of the above functional equation. For instance, if $|a| < 1$, then the complex (resp. real) exponential function a^z is limit summable and $\sigma_\ell(a^z) = \frac{a}{a-1}(a^z - 1)$.
Often if a real function f is limit summable on an interval of length 1 and $R(1) = 0$, then f is limit summable (see [3,4]).

Example 1.1. If $0 < b \neq 1$ and $0 < a < 1$, then the real function $f(x) = ca^x + \log_b x$ is limit summable and

$$f_\sigma(x) = \frac{ca}{a-1}(a^x - 1) + \log_b \Gamma(x+1).$$

However, some important special functions, such as nonconstant polynomials and trigonometric functions are not limit summable according to the above definition. So, we need to introduce other types of summability. To this end, we first recall the Bernoulli polynomials and numbers.

The Bernoulli polynomial $B_n(x)$ is generated by the identity

$$\frac{te^{zt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(z)}{n!} t^n\ ;\ |t| < 2\pi,\ z \in \mathbb{C}.$$

Denote by $B_n := B_n(0)$ and $b_n := B_n(1)$ the first and second Bernoulli numbers, respectively. Recall that $b_n = B_n$ for all $n \geq 2$, and $b_n = (-1)^n B_n$, $|b_n| = |B_n|$ for all

$n \geq 0$ ($b_{2k+1} = B_{2k+1} = 0$ for all $k \geq 1$ and $b_1 = -B_1 = \frac{1}{2}$, $b_0 = B_0 = 1$).

Also, we have

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}, \sum_{n=0}^{\infty} \frac{b_n}{n!} t^n = \frac{te^t}{e^t - 1} \ ; \ |t| < 2\pi.$$

We refer the readers to [1, 5] for more properties of Bernoulli polynomials and numbers. Now, put

$$\sigma_A(z^n) = \sigma(z^n) := S_n(z) = \frac{B_{n+1}(z+1) - b_{n+1}}{n+1} \ ; \ z \in \mathbb{C}, n \geq 0. \tag{1.1}$$

Note that the notation $S_n(x)$ was used in many references (see, e.g., [1, 5], and references therein).

Since $B_n(z+1) - B_n(z) = nz^{n-1}$ ($z \in \mathbb{C}$, $n \geq 1$), then $S_n(m) = \sum_{k=1}^{m} k^n$ for all $m \in \mathbb{N}^*$, and

$$\sigma(z^n) = z^n + \sigma((z-1)^n) \ ; \ z \in \mathbb{C}, n \geq 0. \tag{1.2}$$

On the other hand, we can write

$$\sigma(z^n) = \sum_{k=1}^{n+1} \beta_{nk} z^k \ ; \ z \in \mathbb{C}, n \geq 0, \tag{1.3}$$

where

(1.4)
$$\beta_{nk} = \beta_{n,k} = \binom{n+1}{k} \frac{b_{n+1-k}}{n+1} = \frac{n!}{k!(n+1-k)!} b_{n+1-k} \ ; \ n \geq 0, 1 \leq k \leq n+1.$$

Note that we can define $\beta_{nk} = 0$ for all $k \geq n+2$, but β_{n0} is not defined. Simple calculations show that $\beta_{n,n+1} = \frac{1}{n+1}$, $\beta_{n,n} = b_0 = \frac{1}{2}$, $\beta_{n,1} = b_n$, $\beta_{n,k} = \frac{n}{k}\beta_{n-1,k-1}$ and $\sum_{k=1}^{n+1} \beta_{nk} = 1$. Also, if $n-k$ is an even number ≥ 2, then $\beta_{nk} = 0$. Hence we have

(1.5)
$$\sigma(z^n) = \sum_{k=1}^{n+1} \beta_{nk} z^k = \sum_{k=1}^{n+1} \frac{n!}{k!(n+1-k)!} b_{n+1-k} z^k = \frac{1}{n+1} \sum_{k=0}^{n}{}' \binom{n+1}{k} b_k z^{n+1-k}.$$

2. Analytic summability and analytic summand functions

Now, we are ready to introduce the notion of analytic summability of complex and real functions. For simplicity, we define the analytic summability for analytic functions around $c = 0$, the case $c \neq 0$ is similar.

Definition 2.1. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a complex or real analytic function defined on an open domain D. We call f "analytic summable at z_0"(resp. "absolutely analytic summable at z_0"), if the functional series

$$f_{\sigma_A}(z_0) = f_\sigma(z_0) = \sum_{n=0}^{\infty} c_n \sigma(z_0^n)$$

is convergent (resp. is absolutely convergent). We call f "analytic summable on a set $E \subseteq D$ if it is analytic summable at every point of E. The function $f_{\sigma_A} = f_\sigma$ (with the largest possible domain) is called "analytic summand (function) of f". If f is analytic summable on the whole $\mathbb{C}$, then we call f "entire analytic summable".

Remark 2.1. In the cases where we use both concepts (analytic and limit summable functions), we will use the symbols f_{σ_l} and f_{σ_A} to denote the limit summand and the analytic summand functions of f, respectively.

We will use the following identity for iterated series of double complex sequences, which represents the sum of all arrays of the lower triangle of the $(N+1) \times (N+1)$ matrix $[C_{nk}]$ by two different ways:

$$\sum_{n=0}^{N} \sum_{k=1}^{n+1} C_{nk} = \sum_{n=1}^{N+1} \sum_{k=n-1}^{N} C_{kn}. \tag{2.1}$$

It is known that the natural exponential function e^z is not limit summable. Indeed, the function a^z is limit summable if and only if $|a| \leq 1$ (see [3,4]). The following example shows that e^z is analytic summable.

Example 2.1. The exponential function $\exp(z) = e^z$ is entire analytic summable and

$$\exp_\sigma(z) = \frac{e}{e-1}(e^z - 1) \quad : \quad z \in \mathbb{C}.$$

Indeed, using (2.1) we can write

$$\begin{aligned}
\exp_\sigma(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\sigma(z^n) &= \lim_{N\to\infty} \sum_{n=0}^{N} \sum_{k=1}^{n+1} \frac{1}{n!} \frac{n!}{k!(n+1-k)!} b_{n+1-k} z^k \\
&= \lim_{N\to\infty} \sum_{n=1}^{N+1} \sum_{k=n-1}^{N} \frac{1}{n!} \frac{b_{k+1-n}}{(k+1-n)!} z^n = \sum_{n=1}^{\infty} \frac{1}{n!} \Big(\sum_{j=0}^{\infty} \frac{b_j}{j!}\Big) z^n \\
&= \frac{e}{e-1} \sum_{n=1}^{\infty} \frac{z^n}{n!} = \frac{e}{e-1}(e^z - 1).
\end{aligned}$$

For the last equality, we used the identity $\sum_{j=0}^{\infty} \frac{b_j}{j!} = \sum_{j=0}^{\infty} (-1)^j \frac{B_j}{j!} = \frac{e}{e-1}$.

Now we are in position to state some basic properties of analytic summability of real and complex functions. One can see that these properties are similar to that of limit summability of functions.

Theorem 2.1. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and $g(z) = \sum_{n=0}^{\infty} d_n z^n$ be analytic functions defined on an open domain D. The following assertions hold.*
(a) If $z, z-1 \in D$, then f is analytic summable at z if and only it is analytic summable at $z-1$. So, if f is analytic summable on D, then

$$f_\sigma(z) = f(z) + f_\sigma(z-1) \ ; \ \forall z \in D \cap (D+1). \tag{2.2}$$

(b) If f is analytic summable on D and $D \subseteq D+1$, then

$$f_\sigma(z) = f(z) + f_\sigma(z-1) \ ; \ \forall z \in D. \tag{2.3}$$

(c) If f and g are analytic summable at z (resp. on D), then every linear combination of f and g is also analytic summable, and we have $(af+bg)_\sigma(z) = af_\sigma(z) + bg_\sigma(z)$ (resp. for all $z \in D$).

Proof. Put $f_{\sigma_N}(z) := \sum_{n=0}^{N} c_n \sigma(z^n)$. If $z, z-1 \in D$, then by using (1.2) we have

$$f_{\sigma_N}(z) = \sum_{n=0}^{N} c_n z^n + f_{\sigma_N}(z-1).$$

Also, a simple calculation shows that

$$(af+bg)_{\sigma_N}(z) = af_{\sigma_N}(z) + bg_{\sigma_N}(z).$$

Now, one easily can get the results. □

3. Some upper bounds for $\sigma_A(z^n)$

Since the analytic summand function is generated by the sequence $\{\sigma_A(z^n)\}_{n=1}^{\infty}$, upper bounds for $\sigma_A(z^n)$ should be useful in establishing criteria about analytic summability. We first consider the following bounds for Bernoulli numbers:

$$\frac{1}{1-2^{-2r}} \cdot \frac{2(2r)!}{(2\pi)^{2r}} < |B_{2r}| = |b_{2r}| < \frac{2(2r)!}{(2\pi)^{2r}} \cdot \frac{1}{1-2^{1-2r}} \ ; \ r = 1,2,3,\ldots. \tag{3.1}$$

The inequality (3.1) together with $B_{2r+1} = b_{2r+1} = 0$ (for all $r \geq 1$) imply

$$|B_n| = |b_n| < \frac{2n!}{(2\pi)^n} \cdot \frac{1}{1-2^{1-n}} \ ; \ n = 2,3,4,5,\ldots. \tag{3.2}$$

Applying the identity (1.4), for every positive integer n and $1 \leq k \leq n-1$, we obtain

$$|\beta_{nk}| < \frac{n!}{k!(n-k+1)!} \cdot \frac{2(n-k+1)!}{(2\pi)^{n-k+1}} \cdot \frac{1}{1-2^{k-n}} = \frac{n!}{k!\pi^{n-k+1}} \cdot \frac{1}{2^{n-k}-1}.$$

Since $n-k \geq 1$, then $\frac{1}{2^{n-k}-1} \leq 1$, and hence we have

$$|\beta_{nk}| \leq \frac{n!}{k!\pi^{n-k+1}} \cdot \frac{1}{2^{n-k}-1} \leq \frac{n!}{k!\pi^{n-k+1}} \ ; \ 1 \leq k \leq n-1 \text{ or } k = n+1 \tag{3.3}$$

Observe that the inequality (3.3) does not hold for $k = n$, but the next inequality holds for all $1 \le k \le n+1$

$$|\beta_{nk}| \le \frac{2n!}{k!\pi^{n-k+1}} \; ; \; 1 \le k \le n+1, \tag{3.4}$$

and for $k = n$ we have $\beta_{nn} = \frac{1}{2} \le \frac{2n!}{n!\pi^{n-n+1}} = \frac{2}{\pi}$.

Now, using (1.5) and (3.3), we obtain

$$|\sigma(z^n)| \le \frac{|z|^{n+1}}{n+1} + \frac{|z|^n}{2} + \sum_{k=1}^{n-1} \frac{n!}{k!\pi^{n-k+1}}|z|^k = \frac{|z|^{n+1}}{n+1} + \frac{|z|^n}{2} + \frac{n!}{\pi^{n+1}}\sum_{k=1}^{n-1}\frac{(\pi|z|)^k}{k!}$$

$$= \frac{\pi-2}{2\pi}|z|^n + \frac{n!}{\pi^{n+1}}\sum_{k=1}^{n+1}\frac{(\pi|z|)^k}{k!}.$$

Therefore

$$|\sigma(z^n)| \le \frac{\pi-2}{2\pi}|z|^n + \frac{n!}{\pi^{n+1}}\sum_{k=1}^{n+1}\frac{(\pi|z|)^k}{k!} \le \frac{\pi-2}{2\pi}|z|^n + \frac{n!}{\pi^{n+1}}(e^{\pi|z|}-1). \tag{3.5}$$

In similar way, by using (3.4), we can derive the following inequality

$$|\sigma(z^n)| \le \frac{2n!}{\pi^{n+1}}\sum_{k=1}^{n+1}\frac{(\pi|z|)^k}{k!} \le \frac{2n!}{\pi^{n+1}}(e^{\pi|z|}-1). \tag{3.6}$$

4. Some criteria for analytic summability of complex and real functions

The inequalities for $\sigma_A(z^n)$, stated in Section 3, together with some previous results allow to prove a number of criteria for analytic summability.

Theorem 4.1. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an analytic function defined on an open domain D. If $\sum_{n=0}^{\infty} \frac{n!}{\pi^n} c_n$ is absolutely convergent (for example if $\limsup_{n\to\infty} \sqrt[n]{n!|c_n|} < \pi$), then f is absolutely analytic summable on D. Moreover, by putting $\sigma_{n,N} := \sum_{k=n-1}^{N} \beta_{kn}c_k$, $Abs(f(z)) := \sum_{n=0}^{\infty} |c_n||z|^n$ and $Abs_{1/\pi}(f) := \sum_{n=0}^{\infty} \frac{n!}{\pi^n}|c_n|$, we have the following assertions.*

(a) The analytic summand function f_σ is analytic on D. Indeed, the limit $\sigma_n := \lim_{N\to\infty} \sigma_{n,N}$ exists (for all n).

$$|\sigma_n| \le \frac{2\pi^{n-1}}{n!} Abs_{1/\pi}(f) \; ; \; \forall n. \tag{4.1}$$

and f_σ admits the representation:

$$f_\sigma(z) = \sum_{n=1}^{\infty} \sigma_n z^n = \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\sum_{j=0}^{\infty} \frac{(j+n-1)!}{j!} b_j c_{j+n-1}\Big)z^n \; ; z \in D. \tag{4.2}$$

(which provides an important explicit formula for computing the analytic summand function $f_{\sigma_A}(z)$.)

(b) The following upper bounds for f_{σ_A} hold:

$$|f_\sigma(z)| \leq \frac{2}{\pi}(e^{\pi|z|} - 1)Abs_{1/\pi}(f) \ ; \ z \in D \tag{4.3}$$

and

$$|f_\sigma(z)| \leq \frac{1}{\pi}\{(\frac{\pi}{2} - 1)Abs(f(z)) + (e^{\pi|z|} - 1)Abs_{1/\pi}(f)\} \ ; \ z \in D. \tag{4.4}$$

Proof. By using (3.6), for every $z \in D$ and a positive integer N, we can write

$$\sum_{n=0}^{N} |c_n\sigma(z^n)| \leq \sum_{n=0}^{N} |c_n| \frac{2n!}{\pi^{n+1}}(e^{\pi|z|} - 1) = \frac{2}{\pi}(e^{\pi|z|} - 1)\sum_{n=0}^{N} \frac{n!}{\pi^n}|c_n|$$

$$\Rightarrow |f_{\sigma_N}(z)| \leq \sum_{n=0}^{N} |c_n\sigma(z^n)| \leq \frac{2}{\pi}(e^{\pi|z|} - 1)\sum_{n=0}^{\infty} \frac{n!}{\pi^n}|c_n|.$$

Therefore, f is absolutely analytic summable on D and (4.3) holds. Similarly, using (3.5), we can obtain (4.4).

Next, by applying (2.1), we can write

$$f_{\sigma_N}(z) = \sum_{n=0}^{N} c_n\sigma(z^n) = \sum_{n=0}^{N} c_n \sum_{k=1}^{n+1} \beta_{nk}z^k = \sum_{n=0}^{N}\sum_{k=1}^{n+1} \beta_{nk}c_n z^k = \sum_{n=1}^{N+1}\sum_{k=n-1}^{N} \beta_{kn}c_k z^n.$$

Therefore

$$f_{\sigma_N}(z) = \sum_{n=1}^{N+1} \sigma_{n,N}z^n = \sum_{n=1}^{N+1}\sum_{k=n-1}^{N} \beta_{kn}c_k z^n = \sum_{n=0}^{N} c_n\sigma(z^n) = \sum_{n=0}^{N}\sum_{k=1}^{n+1} \beta_{nk}c_n z^k. \tag{4.5}$$

Taking into account that

$$|\sigma_{n,N}| \leq \sum_{k=n-1}^{N} |\beta_{kn}||c_k| \leq \frac{2\pi^{n-1}}{n!}\sum_{k=n-1}^{N} \frac{k!}{\pi^k}|c_k| \leq \frac{2\pi^{n-1}}{n!}\sum_{k=0}^{N} \frac{k!}{\pi^k}|c_k|.$$

we obtain (4.1), and conclude that $\lim_{N\to\infty}\sigma_{n,N}$ exists (for all n). Since the series $\sum_{n=0}^{\infty} \frac{n!}{\pi^n}c_n$ is absolutely convergent, in view of (4.5), we get

$$f_\sigma(z) = \lim_{N\to\infty} f_{\sigma_N}(z) = \lim_{N\to\infty} \sum_{n=1}^{N+1} \sigma_{n,N}z^n = \sum_{n=1}^{\infty} \sigma_n z^n.$$

Finally, noting that

$$\sigma_n = \lim_{N\to\infty} \sigma_{n,N} = \sum_{k=n-1}^{\infty} \beta_{kn}c_k = \frac{1}{n!}\sum_{j=0}^{\infty} \frac{(j+n-1)!}{j!}b_j c_{j+n-1},$$

we complete the proof. Theorem 4.1 is proved. □

Corollary 4.1. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an analytic function defined on an open domain D, and let $z_0 \in D$. If the iterated series $\sum_{n=1}^{\infty}\sum_{k=n-1}^{\infty} \beta_{kn}c_k z^n$ is absolutely* *convergent on D (resp. at z_0), then f is absolutely analytic summable on D (resp. at z_0).*

Proof. Since the iterated series $\sum_{n=1}^{\infty} \sum_{k=n-1}^{\infty} \beta_{kn} c_k z^n$ is absolutely convergent, then $\sum_{k=n-1}^{\infty} \beta_{kn} c_k$ is convergent (for all n), and

$$\lim_{N\to\infty} \sum_{n=1}^{N+1} \sum_{k=n-1}^{N} \beta_{kn} c_k z^n = \sum_{n=1}^{\infty} \sum_{k=n-1}^{\infty} \beta_{kn} c_k z^n$$

(note that the absolute convergence is enough for the above equality). Hence, we can apply the identity (4.5) to conclude that f is analytic summable at z, and

$$f_\sigma(z) = \sum_{n=1}^{\infty} \sum_{k=n-1}^{\infty} \beta_{kn} c_k z^n.$$

Corollary 4.2. *Analytic summand function of every polynomial of degree n exists, and it is a polynomial of degree $n+1$ without a constant term.*

Corollary 4.3. *If the series $\sum_{n=0}^{\infty} \frac{n!}{\pi^n} c_n$ is absolutely convergent, then*

$$\lim_{n\to\infty} \sum_{j=0}^{\infty} (-1)^j \frac{(j+n-1)!}{j!n!} B_j c_{j+n-1} = 0.$$

Theorem 4.2. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an analytic function defined on an open domain D. If $\sqrt[n]{n!|c_n|} \le \delta < \pi$ for all n, then f is absolutely analytic summable on D, and the following inequalities hold:*

$$|f_\sigma(z)| \le \frac{1}{\pi-\delta}(e^{\pi|z|}-1) + \frac{\pi-2}{2\pi} e^{\delta|z|} \quad ; \quad z \in D \tag{4.6}$$

and

$$|f_\sigma(z)| \le \frac{2}{\pi-\delta}(e^{\pi|z|}-1) \quad ; \quad z \in D. \tag{4.7}$$

Proof. By applying Theorem 4.1 and (3.5), for all $z \in D$ we can write

$$\begin{aligned}|f_\sigma(z)| &\le \sum_{n=0}^{\infty} |c_n||\sigma(z^n)| \le \sum_{n=0}^{\infty} |c_n|\left(\frac{\pi-2}{2\pi}|z|^n + \frac{n!}{\pi^{n+1}}(e^{\pi|z|}-1)\right)\\ &\le \frac{\pi-2}{2\pi} \sum_{n=0}^{\infty} \frac{(\delta|z|)^n}{n!} + \frac{e^{\pi|z|}-1}{\pi} \sum_{n=0}^{\infty} \left(\frac{\delta}{\pi}\right)^n\\ &= \frac{\pi-2}{2\pi} e^{\delta|z|} + \frac{1}{\pi-\delta}(e^{\pi|z|}-1),\end{aligned}$$

and thus (4.6) is proved. The proof of (4.7) is similar. □

Example 4.1. If $f(z) = e^z$, then $\delta = 1$, and we have

$$|exp_\sigma(z)| = \left|\frac{e}{e-1}(e^z-1)\right| \le \frac{1}{\pi-1}(e^{\pi|z|}-1) + \frac{\pi-2}{2\pi} e^{|z|} \quad ; \quad z \in \mathbb{C}.$$

Hence

$$\left|\frac{e}{e-1}(e^z-1)\right| \le \frac{2}{\pi-1}(e^{\pi|z|}-1) \ ; \ z \in \mathbb{C}.$$

5. Analytic summand of exponential and trigonometric functions

As it was mentioned before, polynomials of degree at least one, the trigonometric functions (sin and cos) and the exponential functions a^z with $|a| > 1$ are not limit summable. However, they are analytic summable. Indeed, observe first that in view of Corollary 4.3 and Example 2.2, we have

$$\sigma_{\mathcal{A}}\Big(\sum_{n=0}^{N} c_n z^n\Big) = \sum_{n=1}^{N+1} \sigma_{n,N} z^n \quad \text{and} \quad \sigma_{\mathcal{A}}(e^z) = \frac{e}{e-1}(e^z-1).$$

Next, we consider analytic summability of functions a^z, $\sin(z)$, $\cos(z)$, etc.
Note that $a^z = \exp(z\ln a)$ is an entire function for a fixed value of $\ln a$, and $f(z) = a^z = \sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} z^n$. If $|\ln a| < \pi$, then in view of Theorem 4.1, a^z is (absolutely) entire analytic summable and

$$\sigma_n = \frac{1}{n!}\sum_{j=0}^{\infty} \frac{(j+n-1)!}{j!} b_j \frac{(\ln a)^{j+n-1}}{(j+n-1)!} = \frac{1}{n!}(\ln a)^{n-1}\sum_{j=0}^{\infty}\frac{b_j}{j!}(\ln a)^j$$

$$= \frac{1}{n!}(\ln a)^{n-1}\cdot\frac{(\ln a)a}{a-1} = \frac{a}{a-1}\cdot\frac{(\ln a)^n}{n!}.$$

Therefore, $\sigma_{\mathcal{A}}(a^z) = \sum_{n=1}^{\infty} \frac{a}{a-1}\cdot\frac{(\ln a)^n}{n!} z^n$, and hence

$$\sigma_{\mathcal{A}}(a^z) = \frac{a}{a-1}(a^z-1) \ ; \ |\ln a| < \pi, \ z \in \mathbb{C}.$$

To determine the analytic summand function of $\sin(z)$, let $\{\epsilon_n\}$ be a sequence such that $\epsilon_n = 0$ if n is even and $\epsilon_n = 1$ if n is odd. Then, we have

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^{[\frac{n}{2}]}\epsilon_n}{n!} z^n = \sum_{n=0}^{\infty} c_n z^n,$$

and hence

$$\sigma_n = \frac{1}{n!}\sum_{j=0}^{\infty}(-1)^{[\frac{j+n-1}{2}]}\epsilon_{j+n-1}\frac{b_j}{j!} = \begin{cases} \frac{(-1)^{\frac{n-1}{2}}}{2.n!}, & n \in 2\mathbb{Z}+1 \\ \frac{(-1)^{\frac{n-2}{2}}}{n!}\sum_{k=0}^{\infty}(-1)^k\frac{b_{2k}}{(2k)!}, & n \in 2\mathbb{Z}. \end{cases}$$

Taking into account that

$$\sum_{k=0}^{\infty}(-1)^k\frac{b_{2k}}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k\frac{B_{2k}}{(2k)!} = \frac{\frac{1}{2}\sin(1)}{1-\cos(1)},$$

we can write

$$\sin_\sigma(z) = \sum_{n=1}^{\infty}\sigma_n z^n = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!} - \frac{\frac{1}{2}\sin(1)}{1-\cos(1)}\sum_{k=1}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!}$$

$$= \frac{1}{2}\sin(z) + \frac{\frac{1}{2}\sin(1)}{1-\cos(1)}(1-\cos(z)) = \frac{\sin(z)+\sin(1)-\sin(z+1)}{2-2\cos(1)}.$$

The function $\cos_\sigma(z)$ can be calculated analogously, or by using the identity

$$\frac{e^i}{e^i - 1}(e^{iz} - 1) = \cos_\sigma(z) + i\sin_\sigma(z).$$

Finally, we have

$$\sin_\sigma(z) = \frac{\sin(z)+\sin(1)-\sin(z+1)}{2-2\cos(1)}\ , \ \cos_\sigma(z) = \frac{\cos(z)+\cos(1)-\cos(z+1)-1}{2-2\cos(1)}$$

Using the properties of analytic summability, some trigonometric identities and the above results, we obtain

$$\sigma_A(\sin(az+b)) = \frac{\sin(az+b)+\sin(a+b)-\sin(az+a+b)-\sin(b)}{2-2\cos(a)}$$

$$\sigma_A(\cos(az+b)) = \frac{\cos(az+b)+\cos(a+b)-\cos(az+a+b)-\cos(b)}{2-2\cos(a)},$$

where a, b are real or complex constants and $a \neq 0$.

Now, we pose a number of questions that are very important for future study of analytic and limit summability of functions.

Open problem I. Let f be an analytic function defined on an open domain $D = D_f$ with the property $\mathbb{N}^* \subsetneq D \subsetneq \Sigma_f$. If f is both limit and analytic summable, then is it true that $f_{\sigma_l} = f_{\sigma_A}$ on D?

Open problem II. If f is analytic summable on $D = D_f$, then under what conditions is it a unique solution of the functional equation $f_\sigma(z) = f(z) + f_\sigma(z-1)$ on D with the initial condition $f_\sigma(0) = 0$? Compare with the uniqueness Theorem 3.1, Corollary 3.4 of [3] and Theorem A, Corollary 3.4 of [3]).

Open problem III. Is $f(z) = \sum_{n=0}^{\infty} c_n z^n$ absolutely analytic summable (on D) whenever $\pi \le \limsup_{n\to\infty} \sqrt[n]{n!|c_n|} < 2\pi$? A special interest represents the case when it is equal to π.

Open problem IV. Is the inequality (4.3) (or (4.4)) sharp? If no, find a sharp upper bound for the analytic summand of f.

Finally, as another direction of research, one may study intersection of the spaces of limit and analytic summable functions.

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