Известия НАН Арменци, Математика, том 51, н. 5. 2016, стр. 49-62. ON THE ROBUSTNESS TO SMALL TRENDS OF PARAMETER ESTIMATION FOR CONTINUOUS-TIME STATIONARY MODELS WITH MEMORY

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Abstract. The paper deals with a question of robustness of inferences, carried out on a continuous-time stationary process contaminated by a small trend, to this departure from stationarity. We show that a smoothed periodogram approach to parameter estimation is highly robust to the presence of a small trend in the model. The obtained result is a continuous version of that of Hede and Dai (Journal of Time Series Analysis, 17, 141-150, 1996) for discrete time processes.

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1. INTRODUCTION

Much of statistical inferences about unknown spectral parameters is concerned with the discrete-time stationary models, in which case it is assumed that the model is centered, or has a constant mean (see Beran et al. [7]. Dzhaparidze [12]. Giraitis et al. [24], Taniguchi and Kakizawa [28], and references therein). In this paper we are concerned with the robustness of inferences, carried out on a continuous-time stationary process contaminated by a small trend, to this departure from stationarity.

Specifically, let $\{Y(t), t \in \mathbb{R}\}$ be a a zero mean stationary process with spectral density $f(\lambda, \theta)$, where $\theta := (\theta_1, \ldots, \theta_p) \in \Theta \subset \mathbb{R}^p$ is an unknown vector parameter. We want to make inferences about θ in the case where the actual observed data are in the contaminated form:

(1.1)
$$X(t) = Y(t) + M(t), \quad 0 \le t \le T.$$

where M(t) is a deterministic trend.

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We assume that the trend M(t) is small, that is, we consider the situation in which the major trend is removed from the model and a certain component that remains in the model has only minor effect. In these cases standard inferences can be carried on the basis of the stationary model Y(t), and we are interested in question whether the conclusions are robust against this kind of departure from the stationary model.

A sufficiently developed inferential theory is now available for a continuous-time stationary model Y(t). For instance, in Anh et al. [3, 4]. Avram et al. [5], Casas and Gao [9], Gao [14], Gao et al. [15, 16], Leonenko and Sakhno [26] were obtained sufficient conditions ensuring consistency and asymptotic normality of various statistical estimators of θ , including quasi maximum likelihood (Whittle) and minimum contrast estimators of θ constructed on the basis of a finite realization $\mathbf{Y}_T := \{Y(t), 0 \le t \le T\}$ of the process Y(t).

In this paper we show that under some conditions on the process Y(t) and the deterministic trend M(t) the above asymptotic properties of Whittle and minimum contrast estimators remains valid for the model X(t), that is, the estimating procedure is relatively robust against replacing the stationary model Y(t) by the non-stationary model X(t) of the form (1.1). We will be concerned with this question for models which may exhibit long memory, short memory or intermediate memory.

Throughout the paper the letters C and c are used to denote positive constants, the values of which can vary from line to line.

The paper is structured as follows. In Section 2 we describe the statistical model. Section 3 contains the approach and the main result of the paper - Theorem 3.1. Section 4 is devoted to the proof of Theorem 3.1.

2. The model: long memory, short memory and intermediate memory processes

Let $\{Y(t), t \in \mathbb{R}\}$ be a centered, real-valued, continuous-time second-order stationary process with covariance function r(t), possessing a spectral density $f(\lambda), \lambda \in \mathbb{R}$, that is, $E[|Y(t)|^2] < \infty$. E[Y(t)] = 0, r(t) = E[Y(t+u)Y(u)] $(u, t \in \mathbb{R})$, and r(t) and $f(\lambda)$ are connected by the Fourier integral:

(2.1)
$$r(t) = \int_{\mathbb{R}} e^{i\lambda t} f(\lambda) d\lambda, \quad t \in \mathbb{R}.$$

There are several possible definitions of the notion of "memory" of a stationary process, and they are not necessarily identical (see Beran et al. [7], Gao [14]. Giraitis et al. [24], Heyde and Dai [25]. Taniguchi and Kakizawa [28]). In this paper, we define the memory concept basing on the integrability property of covariance function r(t), and depending on the memory structure we will distinguish the following types of stationary models: (a) short memory or short-range dependent, (b) long memory or long-range dependent, (c) intermediate memory or anti-persistent.

We will say that the process Y(t) displays short memory (SM) or short-range dependence (SRD) if the covariance function r(t) is integrable: $r \in L^1(\mathbb{R})$ and $\int_{-\infty}^{+\infty} r(t)dt \neq 0$. In this case the spectral density $f(\lambda)$ is bounded away from zero and infinity at frequency $\lambda = 0$, that is, $0 < f(0) < \infty$.

A typical continuous-time short memory model example is the stationary continuoustime autoregressive moving average (CARMA) process whose spectral density is a rational function (sec, e.g., Brockwell [8]).

Much of statistical inference is concerned with the short memory stationary models. However, data in many fields of science (economics, hydrology, telecommunications, etc.) is well modeled by a stationary process with unbounded or vanishing at the origin spectral density (see Beran et al. [7], Casas and Gao [9]. Gao [14], Tsai and Chan [29] and references therein).

The process Y(t) is said to be anti-persistent or exhibits intermediate memory (IM) if the covariance function r(t) is integrable: $r \in L^1(\mathbb{R})$ and $\int_{-\infty}^{+\infty} r(t)dt = 0$. In this case the spectral density $f(\lambda)$ vanishes at frequency zero: f(0) = 0.

We say that the process Y(t) displays long memory (LM) or long-range dependence (LRD) if the covariance function r(t) is not integrable: $r \notin L^1(\mathbb{R})$. In this case the spectral density $f(\lambda)$ has a pole at frequency zero, that is, it is unbounded at the origin.

The memory property of a stationary process can also be characterized by the behavior of spectral density $f(\lambda)$ in the neighborhood of zero, or by the behavior of covariance function $r(t) \ge 1$ infinity (see Beran et al. [7], Section 1.3.4).

An example of a continuous-time model that displays the above defined memory structures is the continuous-time autoregressive fractionally integrated moving-average (CARFIMA) process (see Chambers [10], Tsai and Chan [29]).

In the continuous context, a basic process which has commonly been used to model LRD is fractional Brownian motion (fBm) $B_H(t)$ with Hurst index H. This is a Gaussian process with stationary increments and spectral density of the form

(2.2)
$$f(\lambda) \sim c |\lambda|^{1-2H}, \quad c > 0, \quad 1/2 < H < 1$$

as $\lambda \to 0$, and covariance function:

(2.3)
$$r(t) \sim c t^{2H-2}, \ 1/2 < H < 1,$$

as $t \to \infty$, where the symbol "~" indicates that the ratio of left- and right-hand sides tends to 1. Notice that the form (2.2) can be understood in a limiting sense, since the fBm B_H is a nonstationary process (see, e.g., Solo [27]. Gao et al. [15]).

A proper stationary model in lieu of fBm is the fractional Riesz-Bessel motion (fRBm), introduced in Anh et al. [1], and then extensively discussed in a number of papers (see Anh et al. [2], Gao et al. [15], Leonenko and Sakhno [26], and references therein). The fRBm is defined to be a continuous-time Gaussian stationary process with spectral density of the form

(2.4)
$$f(\lambda) = \frac{c}{|\lambda|^{2v} (1+\lambda^2)^v}, \quad \lambda \in \mathbb{R}, \ 0 < c < \infty, \ 0 < u < 1/2, \ v > 0.$$

Observe that the spectral density (2.4) behaves as $O(|\lambda|^{-2u})$ as $|\lambda| \to 0$ and as $O(|\lambda|^{-2(u+v)})$ as $|\lambda| \to \infty$. Thus, under the conditions 0 < u < 1/2, v > 0 and u + v > 1/2 the function $f(\lambda)$ in (2.4) is well-defined for both $|\lambda| \to 0$ and $|\lambda| \to \infty$ due to the presence of the component $(1 + \lambda^2)^{-v}$, which is the Fourier transform of the Bessel potential. Note that in the spacial case 0 < u < 1/2, v > 1/2 the condition u + v > 1/2 holds automatically. The exponent u determines the LRD, while the exponent v indicates the second-order intermittency of the fRBm (see Anh et al. [2] and Gao et al. [15]).

Comparing (2.2) and (2.4), we observe that the spectral density of IBm is the limiting case as $v \to 0$ that of fRBm with Hurst index H = u + 1/2. Thus, the form (2.4) means that fRBm may exhibit both LRD and second-order intermittency.

The next result, which was proved in Ginovyan and Sahakyan [22], gives an asymptotic formula for covariance function of an fRBm: Let $f(\lambda)$ be as in (2.4) with 0 < u < 1/2 and v > 1/2, and let $\tau(t) := f(t)$ be the Fourier transform of $f(\lambda)$, then

(2.5)
$$r(t) = Ct^{2u-1}\sin(\pi\alpha)\Gamma(1-2u)\cdot(1+o(1)) \quad \text{as} \quad t \to \infty.$$

3. THE APPROACH AND RESULTS

The basic approach in estimating unknown spectral parameters, originated by Whittle [30], is based on the smoothed periodogram analysis on a frequency domain, involving approximation of the likelihood function and asymptotic distributions of empirical spectral functionals.

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The Whittle estimation procedure, originally devised for discrete-time short memory stationary processes, has played a major role in the parametric estimation in the frequency domain, and was the focus of interest of many statisticians. Their aim was to weaken the conditions needed to guarantee the validity of the Whittle approximation for short memory models, to find analogues for long and intermediate memory models, and to show that the Whittle estimator is asymptotically equivalent to exact maximum likelihood estimator (see Dahlhaus [11], Dzhaparidze [12], Fox and Taqqu [13], Giraitis and Surgailis [23], Giraitis et al. [24] and references therein). In particular, it was shown that for Gaussian and linear stationary models the Whittle approach leads to consistent and asymptotically normal estimators with the standard rate of convergence under short, intermediate and long memory assumptions.

Continuous versions of Whittle estimation procedure have been considered, for example, in Anh et al. [3, 4], Avram et al. [5], Casas and Gao [9], Gao [14], Gao et al. [15, 16], Leonenko and Sakhno [26].

The procedure of estimation of a parameter θ involved in the spectral density $f(\lambda, \theta)$ of the model, based on a finite realization $\mathbf{Y}_T := \{Y(t), 0 \leq t \leq T\}$ of the centered stationary process Y(t), is to choose the estimator θ_W to minimize the weighted Whittle functional:

(3.1)
$$U_{w,T}(\theta) := \frac{1}{4\pi} \int_{-\infty}^{+\infty} \left[\log f(\lambda,\theta) + \frac{I_{T,Y}(\lambda)}{f(\lambda,\theta)} \right] \cdot w(\lambda) \, d\lambda,$$

where

$$(3.2) I_{T,Y}(\lambda) = \frac{1}{2\pi T} \left| \int_0^T e^{i\lambda t} Y(t) dt \right|$$

is the "continuous" periodogram of Y(t), and $w(\lambda)$ is an even weight function (that is, $w(-\lambda) = w(\lambda)$, $w(\lambda) \ge 0$, and $w(\lambda) \in L^1(\mathbb{R})$) for which the integral in (3.1) is well defined. The choice of an appropriate weight function depends on the specific form of the spectral density (see Anh et al. [4]). An example of common used weight function is $w(\lambda) = 1/(1 + \lambda^2)$.

Thus, the Whittle estimator $\hat{\theta}_W$ with weight function $w(\lambda)$ is defined to be a solution of the following estimating equation

(3.3)
$$\int_{-\infty}^{+\infty} \left[I_{T,Y}(\lambda) - f(\lambda,\theta) \right] \frac{\partial}{\partial \theta} f^{-1}(\lambda,\theta) \cdot w(\lambda) \, d\lambda = 0.$$

obtained by differentiating under the integral sign in (3.1).

The asymptotic properties of the Whittle estimator $\hat{\theta}_{W}$ then can be obtained using the standard Taylor expansion methods based on the following smoothed periodogram convergence results:

(3.4)
$$\int_{-\infty}^{+\infty} g(\lambda,\theta) I_{T,Y}(\lambda) d\lambda \xrightarrow{P} \int_{-\infty}^{+\infty} g(\lambda,\theta) f(\lambda,\theta) d\lambda \quad \text{as} \quad T \to \infty,$$

and

(3.5)
$$T^{1/2} \int_{-\infty}^{+\infty} g(\lambda,\theta) \left[I_{T,Y}(\lambda) - f(\lambda,\theta) \right] d\lambda \xrightarrow{d} \xi \sim N(0,\sigma^2) \text{ as } T \to \infty,$$

where $g(\lambda, \theta) = \frac{1}{\partial \theta} f^{-1}(\lambda, \theta) w(\lambda)$, $I_{T,Y}(\lambda)$ is the periodogram of Y(t) given by (3.2), $N(0, \sigma^2)$ denotes the normal law with mean zero and variance σ^2 , and \rightarrow and \rightarrow stand for convergence in distribution and in probability, respectively.

Using this approach, statistical properties of Whittle minimum contrast estimators for continuous-time stationary processes were studied in Anh et al. [3], Avram et al. [5], Casas and Gao [9], Gao [14], Gao et al. [15, 16], Leonenko and Sakhno [26]. In particular, consistency and asymptotic normality of Whittle minimum contrast estimator θ_W was established for some classes of stationary models, including the fractional Riesz-Bessel motion model, specified by spectral density $f(\lambda) = f(\lambda; \theta)$ given by (2.4) with $\theta = (u, v, c)$.

In our analysis we will use a general even integrable smoothing function $g(\lambda; \theta)$ rather than the specific form $g(\lambda, \theta) = \frac{1}{100} f^{-1}(\lambda, \theta)w(\lambda)$ which is suggested by the Whittle procedure in (3.3). The general estimator θ_G of θ is then obtained as a solution of the estimating equation

(3.6)
$$\int_{-\infty}^{+\infty} \left[I_{T,Y}(\lambda) - f(\lambda,\theta) \right] g(\lambda,\theta) d\lambda = 0.$$

Then the asymptotic properties of the estimator $\hat{\theta}_G$ can be obtained from smoothed periodogram convergence results of type (3.4) and (3.5) with general smoothing function $g(\lambda; \theta)$.

Notice that in the continuous context the basic tool for derivation of limit theorems for empirical spectral functionals of the form

(3.7)
$$J(g) := \int_{-\infty}^{+\infty} g(\lambda, \theta) I_{T,Y}(\lambda) d\lambda$$

is a central limit theorem for Toeplitz type quadratic functionals of stationary processes (see Ginovyan [18, 19]. Ginovyan and Sahakyan [21] for Gaussian processes, and Bai et al. [5, 6] for linear processes). It can be shown that the standard Taylor expansion methods based on the smoothed periodogram convergence results of type (3.4) and(3.5) with a general smoothing function $g(\lambda; \theta)$ and with the contaminated periodogram $I_{T,X}(\lambda)$ instead of $I_{T,Y}(\lambda)$, lead consistent and asymptotically normally distributed estimators of θ . We will not oursue this matter here (the details will be reported elsewhere), however, notice that in the special case of Whittle procedure, where $g(\lambda; \theta) = \frac{\partial}{\partial} \frac{1}{f(\lambda, \theta)} \cdot w(\lambda)$ the results of Anh et al. [3], Avram et al. [5], Casas and Gao [9], Gao [14], Gao et al. [15, 16]. Leonenko and Sakhno [26] concerning consistency and asymptotic normality of the Whittle minimum contrast estimators constructed on the basis of the periodogram $I_{T,Y}(\lambda)$, continue to hold without change for estimators calculated on the basis of the contaminated periodogram $I_{T,X}(\lambda)$ under appropriate assumptions imposed on the model Y(t) on the smoothing function $g(\lambda, \theta)$ and on the trend M(t).

In Theorem 3.1 that follows we show that a small trend of the form $|M(t)| \leq C|t|^{-\beta}$ does not effect the asymptotic properties (3.4) and (3.5) of the smoothed periodogram, and hence, the asymptotic properties of the estimator $\hat{\theta}_{G_1}$ even if $I_{T,Y}(\lambda)$ is replaced by the contaminated periodogram $I_{T,X}(\lambda)$.

Theorem 3.1. Suppose that the stationary mean zero process $\{Y(t), t \in \mathbb{R}\}$ in (1.1) is such that the asymptotic relations (3.4) and (3.5) are satisfied with general even integrable smoothing function $g(\lambda)$ and σ^2 as in (??). If the trend M(t) and the Fourier transform a(t) := g(t) of smoothing function $g(\lambda)$ are such that M(t) is locally integrable on \mathbb{R} and

$$(3.8) |M(t)| \le C|t|^{-\beta}, |a(t)| \le C|t|^{-\gamma}, t \in \mathbb{R}, \quad 2\beta + \gamma > \frac{3}{2}.$$

with some constants C > 0, $\gamma > 0$ and $\beta > 1/4$, then

(3.9)
$$T^{1/2} \int_{-\infty}^{+\infty} g(\lambda, \theta) \left[I_{T,X}(\lambda) - I_{T,Y}(\lambda) \right] d\lambda \xrightarrow{P} 0 \text{ as } T \to \infty,$$

and hence the asymptotic relations (3.4) and (3.5) are satisfied with $J_{T,Y}(\lambda)$ replaced by the contaminated periodogram $I_{T,X}(\lambda)$, provided that one of the following conditions holds:

- (i) the process Y(t) has SM or IM, that is, the covariance function r(t) of Y(t) satisfies $r \in L^1(\mathbb{R})$, and $\beta + \gamma > 1$.
- (ii) the process Y(t) has LM with covariance function r(t) satisfying

(3.10)
$$|r(t)| \le C|t|^{-\alpha}, \quad t \in \mathbb{R}, \quad \alpha + \gamma \ge \frac{3}{2}$$

with some constants C > 0, $0 < \alpha \le 1$. and $\alpha + 2\beta > 1$ if $\beta < 1 < \gamma$.

Remark 3.1. It is easy to check that the statement of Theorem 3.1 holds, in particular, if the parameters α , β and γ satisfy the following conditions:

in the case (i): $\beta > 1/2$, $\gamma \ge 1/2$,

in the case (ii): $\alpha \ge 3/4$. $\beta > 3/8$, $\gamma \ge 3/4$.

Remark 3.2. The discrete version of Theorem 3.1 (with additional conditions $\gamma = 1$ in the case (i), and $\gamma > 1$, $\alpha < 1/2$ in the case (ii)), was proved by Heyde and Dai [25] (see also Taniguchi and Kakizawa [28]. Theorems 6.4.1 and 6.4.2). Using the same arguments applied in the proof of Theorem 3.1 one can prove that the complete discrete analog of Theorem 3.1 is also true.

Remark 3.3. Convergence results of type (3.4) and (3.5) holds under broad circumstances of SM. IM and LM. For detailed conditions see, for example, Avram et al. [5], Ginovyan [17] [20], Ginovyan and Sahakyan [21], and Leonenko and Sakhno [26].

Remark 3.4. The conditions imposed on the Fourier transform of generating function g(t) in (3.8) and on the covariance function r(t) in (3.10) ensure central limit theorem for empirical functionals of Gaussian and linear long memory processes. This can be seen from the considerations of Theorem 5 of Ginovyan and Sahakyan [21] (for Gaussian processes), and Theorem 2.1 and Corollary 2.1 of Bai et al. [6] (for linear processes).

4. PROOF OF THE MAIN RESULT

Proof of Theorem 3.1. In view of (1.1) and (3.2) we can write

$$\begin{split} I_{T,X}(\lambda) - I_{T,Y}(\lambda) &= \frac{1}{2\pi T} \left(\left| \int_0^T e^{i\lambda t} X(t) \, dt \right|^2 - \left| \int_0^T e^{i\lambda t} Y(t) \, dt \right|^2 \right) \\ &= \frac{1}{2\pi T} \left(\left| \int_0^T e^{i\lambda t} [Y(t) + M(t)] \, dt \right|^2 - \left| \int_0^T e^{i\lambda t} Y(t) \, dt \right|^2 \right) \\ &= \frac{1}{2\pi T} \int_0^T \int_0^T e^{i\lambda (t-s)} [Y(t)M(s) + Y(s)M(t) + M(t)M(s)] \, dt ds \end{split}$$

and

$$\int_{-\infty}^{+\infty} g(\lambda,\theta) \left[I_{T,X}(\lambda) - I_{T,Y}(\lambda) \right] d\lambda$$
$$= \frac{1}{T} \int_{0}^{T} \int_{0}^{T} \left[Y(t)M(s) + Y(s)M(t) + M(t)M(s) \right] a(t-s) dt ds.$$

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Thus, to complete the proof it is enough to prove that under the conditions of the theorem we have

(4.1)
$$T^{-1/2} \int_0^T \int_0^T M(t) M(s) a(t-s) dt ds \to 0 \quad \text{as} \quad T \to \infty$$

and

(4.2)
$$T^{-1/2} \int_0^T \int_0^T Y(t) M(s) a(t-s) dt ds \xrightarrow{P} 0 \text{ as } T \to \infty.$$

Proof of (4.1). For T > 2 we set

(4.3)
$$I(T) = \int_0^T \int_0^T |M(t)M(s)a(t-s)| dt ds$$
$$= \int_0^1 \int_0^2 + \int_0^1 \int_2^T + \int_1^T \int_0^{1/2} + \int_1^T \int_{1/2}^T =: I_1(\underline{T}) + I_2(T) + I_3(T) + I_4(T).$$

and estimate the integrals $I_i(T)$, i = 1, 2, 3, 4, separately.

Observe first that the Fourier transform $a(t) = \overline{g}(t)$ is a bounded function on \mathbb{R} , since g is integrable on \mathbb{R} . Hence, taking into account that by assumption the trend M(t) is locally integrable on \mathbb{R} , for $I_1(T)$ we obtain the estimate

(4.4)
$$I_1(T) \le C ||a||_{\infty} \int_0^1 |M(s)| \, ds \int_0^2 |M(t)| \, dt \le C < \infty, \quad T > 2.$$

Next, in view of (3.8), for 0 < s < 1 and t > 2 we have $|a(t-s)| \le C(t-s)^{-\gamma} \le Ct^{-\gamma}$, and hence, taking into account that $\beta + \gamma > 1$, $I_2(T)$ can be estimated as follows

(4.5)
$$I_2(T) \leq C \int_0^1 |M(s)| \, ds \int_2^T \frac{1}{t^{n+1}} \, dt \leq C < \infty, \quad T > 2.$$

Similarly, for $I_3(T)$ we have

$$(4.6) I_3(T) \le C < \infty, T > 2.$$

To estimate $I_4(T)$ observe first that, in view of (3.8), for 1 < s < T we can write

$$\begin{split} h(s) &= \int_{1/2}^{T} |M(t)a(t-s)| \ dt \leq C \left[\int_{(s-1)}^{s+1} \frac{|a(t-s)|}{t^{s}} dt + \int_{s+1}^{s} \frac{1}{t^{s}(t-s)^{\gamma}} dt \\ &+ \int_{2s}^{T} \frac{1}{t^{\beta}(t-s)^{\gamma}} dt + \int_{1/2}^{s/2} \frac{1}{t^{\beta}(s-t)^{\gamma}} dt + \int_{s/2}^{s-1} \frac{1}{t^{\beta}(s-t)^{\gamma}} dt \right] \\ &\leq C \left[\|a\|_{\infty} \cdot s^{-\beta} + s^{-\beta} \int_{1}^{s} \frac{1}{\tau^{\gamma}} d\tau + \int_{2s}^{T} \frac{1}{t^{\beta+\gamma}} dt + s^{-\gamma} \int_{1/2}^{s/2} \frac{1}{t^{\beta}} dt + s^{-\beta} \int_{1}^{s/2} \frac{1}{\tau^{\gamma}} d\tau \right] \\ &\leq C \left[s^{-\beta} + L(\gamma, T)^{s-1-\beta} + L(\beta+\gamma, T) \left(T^{1-\beta-\gamma} + s^{1-\beta-\gamma} \right) \\ &+ L(\beta, T) \left(s^{1-\beta-\gamma} + s^{-\beta} + s^{-\gamma} \right) \right] \end{split}$$
4.7)

where the function L(u, T) is defined by

$$L(u,T) = \begin{cases} \log T & \text{if } u = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Taking into account that $\beta + \gamma > 1$, from (4.7) we get

(4.8)
$$h(s) \le C \log T \cdot \left(s^{1-\beta-\gamma} + s^{-\beta} + s^{-\gamma} \right), \qquad 1 < s < T,$$

and hence for T > 2, $I_4(T)$ can be estimated as follows

$$\begin{split} I_{4}(T) &= \int_{1}^{T} |M(s)h(s)| \, ds \\ &\leq C \log T \left[\int_{1}^{T} \frac{1}{s^{2\beta+\gamma-1}} \, ds + \int_{1}^{T} \frac{1}{s^{2\beta}} \, ds + \int_{1}^{T} \frac{1}{s^{\beta+\gamma}} \, ds \right] \\ &\leq C \log T \left[1 + L(2\beta+\gamma-1,T) \, T^{2-2\beta-\gamma} + L(2\beta,T) \, T^{1-2\beta} + T^{1-\beta-\gamma} \right] \\ 4.9) &\leq C \log^{2} T \left(1 + T^{2-2\beta-\gamma} + T^{1-2\beta} \right) . \end{split}$$

Finally, taking into account that by assumption $2\beta + \gamma > 3/2$ and $\beta > 1/4$, from (4.3)-(4.6) and (4.9) we obtain

$$T^{-1/2} \cdot I(T) \le C \log^2 T \left(T^{-1/2} + T^{3/2 - 2\beta - \gamma} + T^{1/2 - 2\beta} \right) \to 0 \text{ as } T \to \infty,$$

which implies (4.1).

Proof of (4.2). Observe first that the inequality

 $(4.10) \qquad \qquad \beta + \gamma > 1$

holds also in the case (ii), since by (3.8) and (3.10), we have $2\beta + 2\gamma \ge 2\beta + \gamma + 3/2 - \alpha > 3/2 + 1/2 = 2$.

Denote

$$\nu(s) = \nu(T, s) := \int_0^T M(t)a(t-s) dt, \quad 0 < s < T,$$

and observe that

 $\int_{0}^{1/2} |M(t)a(t-s)| \, dt \le C \int_{0}^{1/2} |M(t)| \frac{1}{(s-1/2)^{\gamma}} \, dt \le C \cdot s^{-\gamma}, \qquad 1 < s < T,$ and by (4.8),

$$(4.11) \qquad |\nu(s)| \leq C \log T \cdot \left(s^{1-\beta-\gamma}+s^{-\beta}+s^{-\gamma}\right), \qquad 1 < s < T.$$

On the other hand, by (3.8), for T > 2 and 0 < s < 1 we have

$$(4.12) \quad |\nu(s)| \le C \left[\|a\|_{\infty} \int_{0}^{2} |M(t)| \, dt + \int_{2}^{T} \frac{1}{t^{\beta}(t-1)^{\gamma}} dt \right] \le C \log T.$$

Observe that from (3.8) and (4.11) it follows that (4.12) holds for 0 < s < T.

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Now, we denote

$$Q(T) := T^{-1/2} \int_0^T \int_0^T Y(s) M(t) a(t-s) dt ds = T^{-1/2} \int_0^T Y(s) \nu(s) ds,$$

and observe that

$$E\{Q^{2}(T)\} = T^{-1} \int_{0}^{T} \int_{0}^{T} E\{Y(s)Y(\tau)\}\nu(s)\nu(\tau)dsd\tau$$

= $T^{-1} \int_{0}^{T} \int_{0}^{T} \nu(s)\nu(\tau)r(s-\tau)dsd\tau.$

Hence, to prove (4.2) it is enough show that

(4.13)
$$J(T) := \int_0^T \int_0^T |\nu(s)\nu(\tau)r(s-\tau)| \, ds d\tau = o(T) \quad \text{as} \quad T \to \infty.$$

In the case (i), when the process Y(t) has SM or IM, and hence $r \in L^1(\mathbb{R})$, from (4.12) for T > 2 we get

(4.14)
$$|J(T)| \le C \log T \int_0^T |\nu(s)| \int_0^T |r(s-\tau)| d\tau ds \le C \log T \int_0^T |\nu(s)| ds.$$

In view of (4.11), the last integral in (4.14) can be estimated as follows:

$$\int_{0}^{T} |\nu(s)| \, ds \leq C \log^{2} T \left[\int_{0}^{1} ds + \int_{1}^{T} \left(s^{1-\beta-\gamma} + s^{-\beta} + s^{-\gamma} \right) \, ds \right]$$

$$\leq C \log^{2} T \left[1 + L(\beta+\gamma-1,T) \, T^{2-\beta-\gamma} + L(\beta,T) \, T^{1-\beta} + L(\gamma,T) \, T^{1-\gamma} \right]$$

$$(4.15) \leq C \log^{3} T \left(1 + T^{1-\beta} + T^{1-\gamma} + T^{2-\beta-\gamma} \right).$$

Hence, taking into account that $\beta + \gamma > 1$, from (4.14) and (4.15) we obtain

J(T) = o(T) as $T \to \infty$.

In the case (ii), when the process Y(t) has LM, using (3.10), (4.10) - (4.12), for $1 < \tau < T$ we obtain

$$q(\tau) := \int_{0}^{T} |\nu(s)r(s-\tau)| ds \le C \log T \int_{0}^{1/2} \left(\tau - \frac{1}{2}\right)^{-s} ds$$

$$(4.16) + C \log T \left[\int_{1/2}^{T} \frac{|r(s-\tau)|}{s^{n+\gamma-1}} ds + \int_{1/2}^{T} \frac{|r(s-\tau)|}{s} ds + \int_{1/2}^{T} \frac{|r(s-\tau)|}{s^{\gamma}} ds \right].$$

Taking into account that r is bounded $(|r(t)| \le r(0) = E|Y(t)|^2 < \infty, t \in \mathbb{R})$, and using similar arguments as in (4.7), from (3.10) we obtain that for any $\eta > 0$

$$\int_{1/2}^{T} \frac{|r(t-\tau)|}{t^{\eta}} dt \leq C \log T \left(T^{1-\alpha-\eta} + \tau^{1-\alpha-\eta} + \tau^{-\alpha} + \tau^{-\eta} \right)$$
$$\leq C \log T \left(1 + T^{1-\alpha-\eta} \right), \qquad 1 < \tau < T.$$

Applying this inequality for $\eta = \beta + \gamma - 1$, $\eta = \beta$ and $\eta = \gamma$, from (4.16) we obtain

(4.17)
$$q(\tau) \le C \log^2 T \left(1 + T^{2-\alpha-\beta-\tau} + T^{1-\alpha-\beta} \right), \quad 1 < \tau < T.$$

since $\alpha + \gamma > 1$. On the other hand, by (4.11) and (4.12) for T > 2 and $0 < \tau < 1$, we have

(4.18)
$$q(\tau) \le C \left[\log T \int_0^2 |\tau(s-\tau)| \, ds + \int_2^T \frac{|\nu(s)|}{(s-1)^{\alpha}} \, ds \right]$$
$$\le C \log T \left(1 + T^{2-\alpha-\beta-\gamma} + T^{1-\alpha-\beta} \right) \le C \log T \left(1 + T^{1-\beta} \right),$$

 $0 < \tau < 1$, since $\alpha + \gamma > 1$ and $\alpha > 0$.

Next, we denote

(4.19)
$$J(T) = \int_0^T |\nu(\tau)| q(\tau) d\tau = \int_0^1 + \int_1^T =: J_1(T) + J_2(T),$$

and estimate $J_1(T)$ and $J_2(T)$. By (4.12) and (4.18), for $J_1(T)$ we have

(4.20)
$$J_1(T) \le C \log^2 T \left(1 + T^{1-\beta} \right) = o(T) \quad \text{as} \quad T \to \infty,$$

since $\beta > 0$.

To estimate $J_2(T)$ we consider three cases, and use conditions (3.8). (3.10), (4.10) and inequalities (4.11), (4.17).

Case 1. If $\beta \geq 1$, then we have

$$|
u(\tau)| \leq C \log T \left(\tau^{-\beta} + \tau^{-\gamma} \right), \quad q(\tau) \leq C \log^2 T, \qquad 1 < \tau < T,$$

and hence

$$(4.21) J_2(T) \le C \log^3 T \left(1 + T^{1-\beta} + T^{1-\gamma}\right) = o(T) \quad \text{as} \quad T \to \infty.$$

Case 2. If $\beta < 1 < \gamma$, then we have

$$|
u(\tau)| \le C \log T \cdot \tau^{-eta}, \qquad q(\tau) \le C \log^2 T \left(1 + T^{1-\alpha-eta}\right) \qquad 1 < \tau < T.$$

and hence

$$J_{2}(T) \leq C \log^{3} T \left(1 + T^{1-\beta}\right) \left(1 + T^{1-\alpha-\beta}\right)$$

$$(4.22) \leq C \log^{3} T \left(1 + T^{1-\alpha-\beta} + T^{1-\beta} + T^{2-\alpha-2\beta}\right) = o(T) \text{ as } T \to \infty$$

since in this case by assumption $\alpha + 2\beta > 1$.

Case 3. If $\beta < 1$ and $\gamma \leq 1$, then we have

$$|\nu(\tau)| \le C \log T \cdot \tau^{1-\beta-\gamma}, \qquad q(\tau) \le C \log^2 T \left(1 + T^{2-\alpha-\beta-\gamma}\right), \qquad 1 < \tau < T,$$

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and hence

$$\begin{aligned} (4.23) & J_2(T) \leq C \log^3 T \left(1 + T^{2-\beta-\gamma} \right) \left(1 + T^{2-\alpha-\beta-\gamma} \right) \leq \\ & \leq C \log^3 T \left(1 + T^{2-\alpha-\beta-\gamma} + T^{2-\beta-\gamma} + T^{4-\alpha-2\beta-2\gamma} \right) = o(T) \end{aligned}$$

as $T \to \infty$, since $\beta + \gamma > 1$ and $\alpha + 2\beta + 2\gamma = (2\beta + \gamma) + (\alpha + \gamma) > 3$.

From (4.19) (4.23) we obtain J(T) = o(T) as $T \to \infty$. Thus, the relation (4.13) and hence (4.2) are proved.

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