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THE RICCI FLOW AS A GEODESIC ON THE MANIFOLD OF RIEMANNIAN METRICS

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Abstract. The Ricci flow is an evolution equation in the space of Riemannian metrics. A solution for this equation is a curve on the manifold of Riemannian metrics. In this paper we introduce a metric on the manifold of Riemannian metrics such that the Ricci flow becomes a geodesic. We show that the Ricci solitons introduce a special slice on the manifold of Riemannian metrics.

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1. Introduction

The collection $\mathfrak{M}$ of all smooth Riemannian metrics on a compact smooth n-dimensional manifold M is an infinite dimensional Fréchet manifold. Geometry of this space has been studied at first by D. Ebin [8], where he proved the existence of a slice in the space of Riemannian metrics. The basic facts about the manifold of Riemannian metrics $\mathfrak{M}$ can be found in [8, 10, 12]. In [4, 5], B. Clarke proved that the geodesic distance for the natural metric is a positive topological metric on $\mathfrak{M}$, and determined the metric completion of $\mathfrak{M}$. The existence of a vanishing geodesic distance for some infinite dimensional manifolds has been established in [3, 16, 17].

The Ricci flow is a valuable geometric flow introduced by R. Hamilton in the early 1980 [14]. Following the paper by J. Eells and J. Sampson [9], he introduced an evolution equation for a family of Riemannian metrics as follows:

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t)), \quad g(0) = g_0,$$

where $Ric(g(t))$ denotes the Ricci curvature of the metric $g(t)$.
The short time existence of solutions of the above evolution equation has been proved by R. Hamilton [13, 14], by using Nash and Moser implicit function theorem. Later D. DeTurck [7] gave a shorter proof based on linearization of differential operators.

In [11], the authors have proved this result by considering geometry of the manifold of Riemannian metrics $\mathfrak{M}$.

Ricci solitons are special solutions of the Ricci flow (see [1]). Namely, a solution $g(t)$ of the Ricci flow on M is a Ricci soliton (or self-similar solution) if there exist a positive time-dependent function $\sigma(t)$ with $\sigma(0) = 1$, and an 1−parameter family of time-dependent diffeomorphisms $\varphi_t : M \longrightarrow M$ with $\varphi_0 = id$, such that

$$g(t) = \sigma(t)\varphi(t)^* g(0).$$

It is a very useful tool in the sdudy of the differential geometry and physics (see, e.g., [6, 18, 19, 21]). Observe that the solution of Ricci flow is a curve in the space of Riemannian metrics. In this paper, guided by the results of [2], we show that the Ricci flow can be considered as a geodesic of a Riemannian metric on $\mathfrak{M}$. Also, we show that the Ricci solitons are applicable to give a special slice on $\mathfrak{M}$.

The paper is organized as follows. In Section 2, we present the necessary notation and some preliminary facts. In Section 3, we recall some results of D. Ebin [8] on the manifold of Riemannian metrics, and prove a useful lemma concerning Levi-Civita connection. In Section 4, we prove the main results of the paper (Theorems 4.1 and 4.2), giving a Riemannian metric on $\mathfrak{M}$ such that the Ricci flow is a geodesic on $\mathfrak{M}$. In Section 5, the relation between Ricci solitons and slices on $\mathfrak{M}$ is described. We show that Ricci soliton is equivalent to existence of a finite dimensional slice for $\mathfrak{M}$.

2. Notation

2.1. A metric on tensor spaces. A Riemannian metric $g : TM \times_M TM \to \mathbb{R}$ will equivalently be interpreted as musical isomorphisms:

$$\flat = g : TM \to T^*M \qquad \sharp = g^{-1} : T^*M \to TM$$

The metric g can be extended to the cotangent bundle $T^*M = T_1^0 M$ by setting

$$g^{-1}(\alpha, \beta) = g_1^0(\alpha, \beta) = \alpha(\sharp\beta)$$

for $\alpha, \beta \in T^*M$, and the product metric

$$g_s^r = \bigotimes^r g \otimes \bigotimes^s g^{-1}$$

extends g to all tensor spaces $T_s^r M$. A useful formula is

$$g_2^0(h, k) = Tr(g^{-1}hg^{-1}k)$$

for symmetric $h, k \in T_2^0 M$.

2.2. **A metric on tensor fields.** A metric on the space of tensor fields can be defined by integrating the appropriate metric on the tensor space with respect to the volume density:

$$\tilde{g}^r_s(h,k) = \int_M g^r_s(h(x), k(x))vol(g)(x)$$

for $h, k \in \Gamma(T^r_s M)$, where $vol(g)$ is the volume density $\sqrt{det(g_{ij})}dx^1 \wedge ... \wedge dx^n$ in local coordinates $\{x^i\}$ for M. According to Section 2.1, if h and k are tensor fields of type $\binom{0}{2}$ and h or k is symmetric, then we have

$$\tilde{g}^0_2(h,k) = \int_M Tr(g^{-1}h(x)g^{-1}k(x))vol(g)(x).$$

2.3. **Directional derivatives of functions.** We use the following ways to denote directional derivatives of functions, in particular in infinite dimensions. Given a function $F(x,y)$, for instance, we will write:

$$D_{(x,h)}F \quad or \quad dF(x)(h) \quad \text{as shorcut for} \quad \partial_t|_0 F(x+th, y).$$

Here (x,h) in the subscript denotes the tangent vector with foot point x and direction h. Here the calculus in infinite dimensions as explained in [15] has been applied.

3. The manifold of Riemannian metrics

In this section we recall some fundamentals on the manifold of Riemannian metrics and the natural L^2 metric. The manifold of Riemannian metrics $\mathfrak{M}$ is the subset of all sections in S^2T^*M of symmetric rank-2 covariant tensor fields that are positive definite on each T^*_pM for $p \in M$, and $\mathfrak{M}$ is an open convex positive cone in $\Gamma(S^2T^*M)$, which is an infinite-dimensional Fréchet manifold (see [13]).

We first recall some results of Ebin [8]. Let $\mathfrak{D}$ be the group of smooth functions on M, and let

$$\Psi : \mathfrak{M} \times \mathfrak{D} \to \mathfrak{M}, \qquad (g,f) \mapsto f^*g$$

denote the usual "pull-back"action of $\mathfrak{D}$ on $\mathfrak{M}$. For $g \in \mathfrak{M}$, let

$$\Psi_g : \mathfrak{D} \to \mathfrak{M}, \qquad f \mapsto f^*g$$

denote the orbit map at g. Then Ψ_g is a smooth map with derivative at the identity $e \in \mathfrak{D}$ given by

$$\alpha_g = T_e\Psi_g : \mathfrak{X}(M) \to S_2(M), \qquad X \mapsto L_X g,$$

where L_X is the Lie derivative with respect to the vector field X. We can describe the canonical splitting of $S_2(M)$. Let $O_g = \{f^*g | f \in \mathfrak{D}\} \subseteq \Psi_g(\mathfrak{D}) \subseteq \mathfrak{M}$ be the orbit through g. Then O_g is a smooth closed sub-manifold of $\mathfrak{M}$, with tangent space at g given by T_gO_g =rangeα_g. Observe that there exists, orthogonal to O_g, a slice $S_g \subseteq \mathfrak{M}$, which is also a smooth closed manifold of $\mathfrak{M}$ with tangent space at g given by $T_gS_g = S_2^0(g)$. Here $S_2^0(g) = \{h \in S_2(M) | \delta_g h = 0\}$ is the space of C^∞ divergence free two-covariant symmetric tensor fields on M. Thus, the canonical splitting of $S_2(M)$ can be written as follows:

$$T_g\mathfrak{M} = T_gS_g \oplus T_gO_g.$$

The curvature and the geodesic spaces in $\mathfrak{M}$ relative to the canonical metric were studied in [10, 12]. Other weak Riemannian metrics on $\mathfrak{M}$ have been introduced in [20], and formulas for covariant derivative, curvature tensor, sectional curvature and geodesics have been obtained. Metrics on $\mathfrak{M}$ that are stronger than L^2-metric has recently been described in [2]. Using a pseudo-differential operator they introduced the following general metrics:

$$G_g^p(h,k) = \int_M g_2^0(P_g h, k) vol(g) = \int_M Tr(g^{-1}.P_g(h).g^{-1}.k) vol(g),$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ is a positive, symmetric, bijective pseudo-differential operator of order $2p, p \geq 0$, depending smoothly on the metric g. They obtained a geodesic equation for the general metric and all particular cases, and among other results, they showed that under certain conditions on the operator P_g, the geodesic equation is well-posed.

The next lemma will be used in Section 4, in the proofs of the main results of the paper.

Lemma 3.1. *The Levi-Civita connection induced by the Sobolev metric G_g^p on the manifold of Riemannian metrics is given by the following formula:*

$$\begin{aligned}\nabla_h k &= \frac{1}{2}P_g^{-1}[-hg^{-1}P_g k - P_g k g^{-1}h + D_{(g,h)}P_g k + D_{(g,k)}P_g h - (D_{(g,.)}P_g h)^*(k))] \\ &+ \frac{1}{4}[Tr(g^{-1}h)k + Tr(g^{-1}k)h - Tr(g^{-1}P_g h g^{-1}k)P_g^{-1}g].\end{aligned}$$

Proof. The Levi-civita connection on any Riemannian manifold is determined by the following six terms formula:

$$\begin{aligned}2G_g^p(\nabla_h k, m) &= hG_g^p(k,m) + kG_g^p(h,m) - mG_g^p(h,k) \\ &\quad -G_g^p(h,[k,m]) - G_g^p(k,[m,h]) + G_g^p(m,[h,k]).\end{aligned}$$

It suffices to look at constant vector fields h and k satisfying $[h,k]=0$. So, we can write

$$\begin{aligned}
2G_g^P(\nabla_h k, m) &= hG_g^P(k,m) + kG_g^P(h,m) - mG_g^P(h,k) \\
&= \int_M [-Tr(g^{-1}hg^{-1}P_g kg^{-1}m) - Tr(g^{-1}kg^{-1}P_g hg^{-1}m) \\
&\quad -Tr(g^{-1}mg^{-1}P_g hg^{-1}k) - Tr(g^{-1}P_g kg^{-1}hg^{-1}m) \\
&\quad -Tr(g^{-1}P_g hg^{-1}kg^{-1}m) + Tr(g^{-1}P_g hg^{-1}mg^{-1}k) \\
&\quad +Tr(g^{-1}D_{(g,h)}(P_g k)g^{-1}m) + Tr(g^{-1}D_{(g,k)}(P_g h)g^{-1}m) \\
&\quad -Tr(g^{-1}D_{(g,m)}(P_g h)g^{-1}k) + \frac{1}{2}Tr(g^{-1}P_g kg^{-1}m)Tr(g^{-1}h) \\
&\quad +\frac{1}{2}Tr(g^{-1}P_g hg^{-1}m)Tr(g^{-1}k) - \frac{1}{2}Tr(g^{-1}P_g hg^{-1}k)Tr(g^{-1}m)]vol(g).
\end{aligned}$$

Notice that some terms in the last formula cancel out because for symmetric h,k,m one has

$$Tr(hkm) = Tr((hkm)^T) = Tr(m^T k^T h^T) = Tr(h^T m^T k^T) = Tr(hmk).$$

Therefore, we have

$$\begin{aligned}
2G_g^P(\nabla_h k, m) &= \int_M [Tr(g^{-1}hg^{-1}P_g kg^{-1}m) - Tr(g^{-1}P_g kg^{-1}hg^{-1}m) \\
&\quad +Tr(g^{-1}D_{(g,h)}(P_g k)g^{-1}m) + Tr(g^{-1}D_{(g,k)}(P_g h)g^{-1}m) \\
&\quad -Tr(g^{-1}D_{(g,m)}(P_g h)g^{-1}k) + \frac{1}{2}Tr(g^{-1}P_g kg^{-1}m)Tr(g^{-1}h) \\
&\quad +\frac{1}{2}Tr(g^{-1}P_g hg^{-1}m)Tr(g^{-1}k) - \frac{1}{2}Tr(g^{-1}P_g hg^{-1}k)Tr(g^{-1}m)]vol(g) \\
&= -G_g^P(P_g^{-1}(hg^{-1}P_g k), m) - G_g^P(P_g^{-1}(P_g kg^{-1}h), m) \\
&\quad +G_g^P(P_g^{-1}(D_{(g,h)}(P_g k)), m) + G_g^P(P_g^{-1}(D_{(g,k)}(P_g h)), m) \\
&\quad -\int_M Tr(g^{-1}D_{(g,m)}(P_g h)g^{-1}k)vol(g) + \frac{1}{2}[G_g^P(Tr(g^{-1}h)k, m) \\
&\quad +G_g^P(Tr(g^{-1}k)h, m) - G_g^P(Tr(g^{-1}P_g hg^{-1}k)P_g^{-1}g, m)].
\end{aligned}$$

We assume that there exists an adjoint in the following sense

$$\int_M g_2^0((D_{(g,m)}P)h, k)vol(g) = \int_M g_2^0(m, (D_{(g,.)}Ph)^*(k))vol(g),$$

which is smooth in (g,h,k) and is bilinear in (h,k).

Thus, we can write

$$\begin{aligned}2G_g^p(\nabla_h k, m) &= -G_g^P(P_g^{-1}(hg^{-1}P_g k), m) - G_g^P(P_g^{-1}(P_g k g^{-1}h), m)\\ &+G_g^P(P_g^{-1}(D_{(g,h)}(P_g k)), m) + G_g^P(P_g^{-1}(D_{(g,k)}(P_g h)), m)\\ &-G_g^P((D_{(g,.)}P_g h)^*(k), m) + \frac{1}{2}[G_g^P(Tr(g^{-1}h)k, m)\\ &+G_g^P(Tr(g^{-1}k)h, m) - G_g^P(Tr(g^{-1}P_g h g^{-1}k)P_g^{-1}g, m)].\end{aligned}$$

Finally, we have

$$\begin{aligned}\nabla_h k &= \frac{1}{2}P_g^{-1}[-hg^{-1}P_g k - P_g k g^{-1}h + D_{(g,h)}P_g k + D_{(g,k)}P_g h - (D_{(g,.)}P_g h)^*(k))]\\ &+ \frac{1}{4}[Tr(g^{-1}h)k + Tr(g^{-1}k)h - Tr(g^{-1}P_g h g^{-1}k)P_g^{-1}g].\end{aligned}$$

Lemma 3.1 is proved. □

Remark 3.1. The above formula, applied to the geodesic equation $\nabla_{g'}g' = g''$ yields:

$$\begin{aligned}g'' &= P_g^{-1}[-\frac{1}{2}(D_{(g,.)}P_g g')^*(g')) - \frac{1}{2}g'g^{-1}P_g g' - \frac{1}{2}P_g g' g^{-1}g'\\ &+\frac{1}{2}Tr(g^{-1}g')P_g g' - \frac{1}{4}Tr(g^{-1}P_g g' g^{-1}g')g + (D_{(g,g')}P_g)g'],\end{aligned}$$

which coincides with the geodesic equation obtained in [2] using minimizing energy function.

4. A variant of Riemannian metric using a pseudo-differential operator

Let g_0 be a fixed Riemannian metric. The formula

$$\begin{aligned}G_{g_0}^p(h,k) &= \int_M g_2^0(P_g h, k)vol(g_0)\\ &= \int_M Tr(g^{-1}.P_g(h).g^{-1}.k)vol(g_0),\end{aligned} \tag{4.1}$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ is a positive, symmetric and bijective pseudo-differential operator of order $2p, p \geq 0$ depending smoothly on the metric g, defines a Riemannian metric on the manifold of Riemannian metrics.

Theorem 4.1. *The geodesic equation for $G_{g_0}^p$ metrics defined on the manifold of Riemannian metrics $\mathfrak{M}$ is given by the following formula:*

$$\begin{aligned}g'' = P_g^{-1}[&-\frac{1}{2}(D_{(g,.)}P_g g')^*(g')) - \frac{1}{2}g'g^{-1}P_g g'\\ &-\frac{1}{2}P_g g' g^{-1}g' + (D_{(g,g')}P_g)g']\end{aligned} \tag{4.2}$$

Proof. In view of Lemma 3.1 and Remark ??, for $G^p_{g_0}$ we have

$$\begin{aligned}
2G^p_{g_0}(\nabla_h k, m) &= hG^p_{g_0}(k,m) + kG^p_{g_0}(h,m) - mG^p_{g_0}(h,k) \\
&= \int_M [-Tr(g^{-1}hg^{-1}P_g kg^{-1}m) - Tr(g^{-1}kg^{-1}P_g hg^{-1}m) \\
&\quad -Tr(g^{-1}mg^{-1}P_g hg^{-1}k) - Tr(g^{-1}P_g kg^{-1}hg^{-1}m) \\
&\quad -Tr(g^{-1}P_g hg^{-1}kg^{-1}m) + Tr(g^{-1}P_g hg^{-1}mg^{-1}k) \\
&\quad +Tr(g^{-1}D_{(g,h)}(P_g k)g^{-1}m) + Tr(g^{-1}D_{(g,k)}(P_g h)g^{-1}m) \\
&\quad -Tr(g^{-1}D_{(g,m)}(P_g h)g^{-1}k)]vol(g_0).
\end{aligned}$$

Some terms in the last formula cancel out because $vol(g_0)$ is fixed for $G^p_{g_0}$ metric. The rest of the proof is similar to that of Lemma 3.1, and so is omitted. □

In the following, as (non-linear) mappings at the base point g, we assume that $P_g h, (Pg)^{-1}h, (D_{(g,.)}Ph)^*(m)$ are compositions of operators of the following type (see [2]):

(a) non-linear differential operators of order $l \le 2p$, that is,

$$A(g)(x) = A(x, g(x), (\bar{\nabla} g)(x), ..., (\bar{\nabla}^l g)(x)),$$

(b) linear pseudo-differential operators of order $\le 2p$, such that the total (top) order of the composition is $\le 2p$.

Now consider P_g as a pseudo-differential operator defined on $\Gamma(S^2T^*M)$ such that it has the following forms for special tensors.

$$P_g(Ric) := e^g Ric \tag{4.3}$$

$$P_g(g^{pq}\nabla^2_{q,i}R_{jp}) := -2Ricg^{-1}e^g Ric \tag{4.4}$$

$$P_g(\nabla^2_{i,j}R) := -4(Ric)e^g Ric \tag{4.5}$$

$$P_g(\Delta R_{ij}) := ge^g Ricg^{-1}Ric \tag{4.6}$$

Now we can state the following result.

Theorem 4.2. *There is a pseudo-differential operator on $\Gamma(S^2T^*M)$ such that the Ricci flow is a geodesic of $G^p_{g_0}$ metric on the manifold of Riemannian metrics.*

Proof. The result can easily be deduced from equation (4.2) with $g' = -2Ric$ and formulas (4.3)-(4.6), since

$$P_g(\frac{\partial Ric}{\partial t}) = -(D_{(g,.)}P_g(Ric))^*(Ric))$$

$$-Ricg^{-1}P_g(Ric) - P_g(Ric)g^{-1}Ric - 2(D_{(g,Ric)}P_g)Ric,$$

and for the adjoint operator $P_g(Ric)$ we have

$$\int_M g_2^0((D_{(g,m)}P)Ric, Ric)vol(g_0) = \int_M g_2^0(me^q Ric, Ric)vol(g_0)$$

$$= \int_M Tr(G^{-1}me^q Ricg^{-1}Ric)vol(g_0) = \int_M (m, ge^q Ricg^{-1}Ric)vol(g_0)$$

$$= \int_M g_2^0(m, (D_{(g,.)}P_g(Ric))^*(Ric))vol(g_0).$$

Theorem 4.2 is proved. □

Other Riemannian metrics. The Ricci flow as a curve is not a geodesic of Riemannian metrics on $\mathfrak{M}$ defined in [8, 20]. In [11], we have shown that the Ricci flow is not a geodesic of the known Riemannian metric on $\mathfrak{M}$. Let g_0 be a fixed Riemannian metric on M. In fact we have the following.

(1) For the metric defined as follows:

$$< h, k >_g: \ = \ \int_M Tr(g_0^{-1}hg_0^{-1}k)vol(g_0)$$

for $h, k \in T_g\mathfrak{M}$, the geodesics with initial conditions $(\bar{g}, a)$ are of the form $g(t) = \bar{g} + ta$ (see [20]). It is obvious that the Ricci flow is not a geodesic of this metric. Since the velocity vector for geodesic is constant, that is, $\frac{\partial g}{\partial t} = a$. For more general metric defined by

$$< h, k >_0^\alpha: \ = \ \int_M Tr(g_0^{-1}hg_0^{-1}k)vol(g_0) + \alpha\int_M Tr(g_0^{-1}h)Tr(g_0^{-1}k)vol(g_0),$$

where $\alpha > -\frac{1}{n}$, the geodesics are the same as above (see [20]). Therefore the Ricci flow is not a geodesic of this general metric, too.

(2) Consider the following Riemannian metric on $\mathfrak{M}$:

$$< h, k >_g \ = \ \int_M Tr(g_0^{-1}hg_0^{-1}k)vol(g).$$

The geodesics are solutions of the following second-order equation (see [20]):

$$\frac{d}{dt}\left(\rho(g)\frac{dg}{dt}\right) \ = \ kd\psi,$$

where $vol(g) = \rho(g)vol(g_o)$, $k = k(x)$ is a positive function on M and $\psi = ln\rho(g)$. It can be shown that the Ricci flow is not a geodesic of this metric, too. Indeed, since

$$\begin{aligned}\frac{d\rho}{dt}\frac{\partial g}{\partial t}+\rho(g)\frac{\partial^2 g}{\partial t^2} &= \frac{1}{2}k(x)g_0g^{-1}g_0\\ 2scal\rho(g)Ric+\rho(g)\frac{\partial Ric}{\partial t} &= \frac{1}{2}k(x)g_0g^{-1}g_0,\end{aligned}$$

we have

$$\frac{\partial Ric}{\partial t} = \frac{1}{\rho(g)}\frac{1}{2}k(x)g_0g^{-1}g_0-2RRic$$

But under the Ricci flow we have (see [1]):

$$\frac{\partial}{\partial t}R_{ik} = g^{pq}(-\nabla^2_{q,k}R_{ip}+\nabla^2_{i,k}R_{pq}-\nabla^2_{q,i}h_{kp}+\nabla^2_{q,p}R_{ik}).$$

Therefore the Ricci flow is not a geodesic on $\mathfrak{M}$.

(3) For the metric defined by

$$<h,k>_g: = \int_M Tr(g^{-1}hg^{-1}k)vol(g_0),$$

the geodesics with initial condition $(\bar{g},a)$ have the form $g(t)=\bar{g}e^{tA}$, where $A=\bar{g}^{-1}a$ (see [20]). The velocity vector of geodesic is

$$\frac{\partial g}{\partial t}=A\bar{g}e^{tA}=-2\bar{g}^{-1}\bar{Ric}\bar{g}e^{tA},$$

which does not coincide with the Ricci flow $\frac{\partial g}{\partial t}=-2Ric$. For more general metric defined on $\mathfrak{M}$ by

$$<h,k>_{g,\alpha}: = \int_M Tr(g^{-1}hg^{-1}k)vol(g_0)+\alpha\int_M Tr(g^{-1}h)Tr(g^{-1}k)vol(g_0)$$

the geodesics are exactly the same as above, that is, $g(t)=\bar{g}e^{tA}$. Thus, the Ricci flow can not be a geodesic.

(4) Consider the following metric on $\mathfrak{M}$

$$<h,k>^{\alpha}_g: = \int_M Tr(g^{-1}hg^{-1}k)vol(g)+\alpha\int_M Tr(g^{-1}h)Tr(g^{-1}k)vol(g).$$

The geodesics of this metric coincide with the geodesics of $<h,k>^0_g$, that is with those of the canonical metric on $\mathfrak{M}$ (see [20]). In [11], we have shown that the Ricci flow is not a geodesic of the canonical metric on $\mathfrak{M}$.

Remark 4.1. The Ricci flow is not a geodesic of the three special metrics defined by pseudo-differential operators in [2].

5. Slice and Ricci solitons

The existence of a slice for the manifold of Riemannian metrics at first have been studied by Ebin in [8]. He proved that the manifold of Riemannian metrics has a slice such that it is infinite dimensional sub-manifold of $\mathfrak{M}$. Observing that the Ricci flow and Ricci solitons are curves on $\mathfrak{M}$, we show that for every manifold M with Riemannian metric g_0 which has Ricci solitons, the manifold of Riemannian metrics has a slice such that it is a finite dimensional sub-manifold of $\mathfrak{M}$.

Indeed, as we know the Ricci solitons $g(t) = \sigma(t)\varphi(t)^*g(0)$ with initial metric g_0 are equivalent to existence of a vector field X and a scalar λ, such that

$$-2Ric = \lambda g_0 - 2L_X g_0$$

On the other hand, according to canonical splitting around g_0, there exist a slice $S_{g_0} \subsetneq \mathfrak{M}$ such that

$$T_{g_0}\mathfrak{M} = T_{g_0}S_{g_0} \oplus T_{g_0}O_{g_0}.$$

Combining the above equations and discussion in Section 3, we obtain the following result.

Theorem 5.1. *Ricci soliton is equivalent to existence of a finite dimensional slice for $\mathfrak{M}$ and then the tangent space at initial metric is λg_0, where λ is a real scalar.*

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