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MAGNETIC BI-HARMONIC DIFFERENTIAL OPERATORS ON RIEMANNIAN MANIFOLDS AND THE SEPARATION PROBLEM

H. A. ATIA

King Abdulaziz University, Saudi Arabia;
Zagazig University, Zagazig, Egypt.
E-mail: h_a_atia@hotmail.com

Abstract. In this paper we obtain sufficient conditions for the bi-harmonic differential operator $A = \Delta_E^2 + q$ to be separated in the space $L^2(M)$ on a complete Riemannian manifold (M, g) with metric g, where Δ_E is the magnetic Laplacian on M and $q > 0$ is a locally square integrable function on M. Recall that, in the terminology of Everitt and Giertz, the differential operator A is said to be separated in $L^2(M)$ if for all $u \in L^2(M)$ such that $Au \in L^2(M)$ we have $\Delta_E^2 u \in L^2(M)$ and $qu \in L^2(M)$.

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1. Introduction

The separation problem for differential operators was first introduced in 1971 by Everitt and Giertz [10], then this problem for different differential expressions was studied by many authors, such as Boimatove [4], Brown [6-7], Mohamed and Atia [16-17], Zayed et al [19], Atia et al [1-3], etc. In [14], Milatovic has studied the separation property for Schrodinger operators on the Riemannian manifolds. Recently Milatovic [15], has introduced the magnetic Schrodinger operator of the form $L = \Delta_E + q$ on a complete Riemannian manifold (M, g) with metric g, where Δ_E is the magnetic Laplacian on M and $q \geq 0$ is a locally square integrable function on M. Sufficient conditions for the operator L to be separated in $L^2(M)$ were obtained in [15]. Atia et al [2] have studied the separation property for the bi-harmonic differential expression of the form $\Delta_E^2 + q$ with $E = 0$.

In this paper we generalize the results of [2] to the magnetic bi-harmonic differential expression of the form $A = \Delta_E^2 + q$, where $E \neq 0$. Let (M, g) be a Riemannian manifold without boundary (that is, M is a C^∞-manifold without boundary and

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(g_{jk}) is a Riemannian metric on M), and let $dim M = n$. We will assume that M is connected.

Throughout the paper we use the following notation. By $d\mu$ we will denote the Riemannian volume element of M, by $L^2(M)$ we denote the space of complex-valued square integrable functions on M with the inner product:

$$(u,v) = \int_M (u\overline{v})\, d\mu, \tag{1.1}$$

and $\|\cdot\|$ denotes the norm in $L^2(M)$ corresponding to the inner product (1.1). We will use the notation $L^2(\Lambda^1 T^* M)$ for the space of complex-valued square integrable 1-forms on M with the inner product:

$$(W,\Psi)_{L^2(\Lambda^1 T^* M)} = \int_M \langle W, \overline{\Psi}\rangle\, d\mu, \tag{1.2}$$

where for 1-forms $W = W_j dx^j$ and $\Psi = \Psi_k dx^k$, we define $\langle W, \Psi\rangle = g^{jk} W_j \Psi_k$, where (g^{jk}) stands for the inverse of the matrix (g_{jk}), and $\overline{\Psi} = \overline{\Psi_k} dx^k$. (We use the standard Einstein summation convention). By $\|\cdot\|_{L^2(\Lambda^1 T^* M)}$ we denote the norm in $L^2(\Lambda^1 T^* M)$ corresponding to the inner product (1.2). By $C^\infty(M)$ we denote the space of smooth functions on M, by $C_c^\infty(M)$ – the space of smooth compactly supported functions on M, by $\Omega^1(M)$ – the space of smooth 1-forms on M and by $\Omega_c^1(M)$ – the space of smooth compactly supported 1-forms on M.

Recall that a magnetic potential is a real valued 1-form $E \in \Omega^1(M)$, and note that in any local coordinates $x^1, x^2, \ldots, x^n$, the form E can be written as $E = E_j dx^j$, where $E_j = E_j(x)$ are real valued C^∞–functions of the local coordinates. The operator $d : C^\infty(M) \to \Omega^1(M)$ stands for the usual differential and by $d_E : C^\infty(M) \to \Omega^1(M)$ we denote the deformed differential defined by $d_E(u) = du + iuE$, for every $u \in C^\infty(M)$, where $i = \sqrt{-1}$. We denote the formal adjoint of d_E by $d_E^* : \Omega^1(M) \to C^\infty(M)$ which is defined by the identity: $(d_E u, w)_{L^2(\Lambda^1 T^* M)} = (u, d_E^* w)$, $\forall\ u \in C_c^\infty(M)$, $w \in \Omega^1(M)$. By $\Delta_E = d_E^* d_E : C^\infty(M) \to C^\infty(M)$ we denote the magnetic Laplacian on M, with magnetic potential E. In this paper we consider the bi-harmonic differential expression:

$$A = \Delta_E^2 + q, \tag{1.3}$$

where $q \in L^2_{loc}(M)$ is a real-valued function, called the electric potential. Also, we use the notation

$$D_1 = \{u \in L^2(M) : Au \in L^2(M)\}. \tag{1.4}$$

Definition 1.1. Using the terminology of Everitt and Giertz [11], we say that the differential expression $A = \Delta_L^2 + q$ is separated in the space $L^2(M)$ if for all $u \in D_1$ we have $\Delta_L^2 u \in L^2(M)$ and $qu \in L^2(M)$.

Definition 1.2. Let A be as in (1.3). We define the minimal operator S in $L^2(M)$ associated with A by the formula $Su = Au$ with domain $Dom(S) = C_c^\infty(M)$.

Remark 1.1. Since S is a symmetric operator, it follows that S is closable (see [12], Section V.3.3). In what follows, we will denote by $\bar{S}$ and S^* the closure and the adjoint of the operator S in $L^2(M)$, respectively.

Lemma 1.1. *If (M, g) is a complete Riemannian manifold with a metric g and a positive smooth measure $d\mu$ and if $0 \le q \in L^2_{loc}(M)$, then the operator S is essentially self-adjoint in $L^2(M)$ (see [5, 8, 13, 18]). Moreover, in this case we have $\bar{S} = S^*$ (see [9, 12]).*

Definition 1.3. The set of admissible parameters $P \subset R^3$ is defined to be the set of parameters $(\alpha, \beta, \gamma) \in R^3$ satisfying the following three conditions: (1) $\gamma > 0$; (2) $\beta > 0$; (3) $0 < \alpha\beta < 1$ or $\alpha = \beta = 1$.

2. The main result

The main result of the present paper is the following theorem.

Theorem 2.1. *Let (M, g) be a complete and connected C^∞-Riemannian manifold without boundary, with a positive smooth measure $d\mu$ and a metric g satisfying the following conditions:*

$$0 \le q(x) \in L^2_{loc}(M), \quad dq(x),\ d^2q(x) \in L^2_{loc}(M), \tag{2.1}$$

$$\left\| d^2q(x)\ u(x) \right\| \le C_1 \left\| q^{\frac{3}{2}}(x)\ u(x) \right\|, \tag{2.2}$$

$$\left\| dq(x)\ du(x) \right\| \le C_2 \left\| q^{\frac{3}{2}}(x)\ u(x) \right\| \tag{2.3}$$

for every $x \in M$ and $u \in C_c^\infty(M)$, where $C_1 \ge 0$ and $C_2 \ge 0$ are constants with $C_1 + 2C_2 \in [0, 2)$. Then the differential expression A defined by (1.3) is separated in the space $L^2(M)$.

Proof. Let $(\alpha,\beta,\gamma) \in P$ and $u \in C_c^\infty(M)$. By the definition (1.3) of the expression A, for every $u \in C_c^\infty(M)$, we have

$$
\begin{aligned}
\|Au\|^2 &= (\Delta_E^2 u + qu, \Delta_E^2 u + qu) \\
&= \|\Delta_E^2 u\|^2 + 2\mathrm{Re}\,(\Delta_E^2 u, qu) + \|qu\|^2 \\
&= \|\Delta_E^2 u\|^2 + 2\mathrm{Re}\,(\Delta_E^2 u, Au - \Delta_E^2 u) + \|qu\|^2 \\
&= -\|\Delta_E^2 u\|^2 + 2\mathrm{Re}\,(\Delta_E^2 u, Au) + \|qu\|^2 \\
&= (2\beta^{-1} - 1)\|\Delta_E^2 u\|^2 - 2\beta^{-1}(\Delta_E^2 u, \Delta_E^2 u) + 2\mathrm{Re}\,(\Delta_E^2 u, Au) + \|qu\|^2 \\
&= (2\beta^{-1} - 1)\|\Delta_E^2 u\|^2 - 2\beta^{-1}\mathrm{Re}\,(\Delta_E^2 u, Au - qu) + 2\mathrm{Re}\,(\Delta_E^2 u, Au) + \|qu\|^2 \\
&= (2\beta^{-1} - 1)\|\Delta_E^2 u\|^2 + 2\beta^{-1}\mathrm{Re}\,(\Delta_E^2 u, qu) \\
&\quad + 2(1-\beta^{-1})\mathrm{Re}\,(\Delta_E^2 u, Au) + \|qu\|^2 . \qquad (2.4)
\end{aligned}
$$

For any imaginary number z, we have

$$
-|z| \le \mathrm{Re} z \le |z| . \qquad (2.5)
$$

Also, for any two positive real numbers a and b, we have

$$
ab \le \frac{k}{2}a^2 + \frac{1}{2k}b^2, \qquad (2.6)
$$

where k is a positive real number. Hence, applying the inequalities (2.6), (2.7) and the Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
2 \quad & (1-\beta^{-1})\mathrm{Re}\,(\Delta_E^2 u, Au) \ge -2\left|(1-\beta^{-1})(\Delta_E^2 u, Au)\right| \\
&= -2\left|1-\beta^{-1}\right| \left|(\Delta_E^2 u, Au)\right| \ge -2\left|1-\beta^{-1}\right| \|\Delta_E^2 u\| \, \|Au\| \\
&\ge -\left|1-\beta^{-1}\right| \left(k\|\Delta_E^2 u\|^2 + k^{-1}\|Au\|^2\right). \qquad (2.7)
\end{aligned}
$$

Since $q \ge 0$, by the definitions of d_E and d_E^*, we obtain

$$
\begin{aligned}
\mathrm{Re}\,(\Delta_E^2 u, qu) &= \mathrm{Re}\,(qu, \Delta_E^2 u) = \mathrm{Re}\,(\Delta_E(qu), \Delta_E u) \\
&= \mathrm{Re}\,((d^2 q)u + 2(dq)(du) + q(\Delta_E u), \Delta_E u) \\
&= \mathrm{Re}\,((d^2 q)u, \Delta_E u) + 2\mathrm{Re}\,((dq)(du), \Delta_E u) + \mathrm{Re}\,(q(\Delta_E u), \Delta_E u) \\
&= \mathrm{Re}\,((d^2 q)u, \Delta_E u) + 2\mathrm{Re}\,((dq)(du), \Delta_E u) + \left\|q^{\frac{1}{2}}\Delta_E u\right\|^2 . \qquad (2.8)
\end{aligned}
$$

Next, by using the Cauchy-Schwartz inequality, the inequalities (2.5) and (2.6) with $k = \gamma$, and the conditions (2.2) and (2.3), we can write

$$\begin{aligned} \mathrm{Re}\left((d^2q)u, \triangle_E u\right) &\geq -\left|\left((d^2q)u, \triangle_E u\right)\right| \geq -\left\|(d^2q)u\right\| \left\|\triangle_E u\right\| \\ &\geq -C_1 \left\|q^{\frac{3}{2}} u\right\| \left\|\triangle_E u\right\| = -C_1 \left\|qu\right\| \left\|q^{\frac{1}{2}} \triangle_E u\right\| \\ &\geq -C_1 \frac{\gamma}{2} \left\|qu\right\|^2 - C_1 \frac{\gamma^{-1}}{2} \left\|q^{\frac{1}{2}} \triangle_E u\right\|^2, \qquad (2.9) \end{aligned}$$

and

$$\begin{aligned} 2\mathrm{Re}\left((dq)(du), \triangle_E u\right) &\geq -2\left|(dqdu, \triangle_E u)\right| \geq -2\left\|dqdu\right\| \left\|\triangle_E u\right\| \\ &\geq -2C_2 \left\|q^{\frac{3}{2}} u\right\| \left\|\triangle_E u\right\| = -2C_2 \left\|qu\right\| \left\|q^{\frac{1}{2}} \triangle_E u\right\| \\ &\geq -C_2\gamma \left\|qu\right\|^2 - C_2\gamma^{-1} \left\|q^{\frac{1}{2}} \triangle_E u\right\|^2. \qquad (2.10) \end{aligned}$$

Taking into account (2.9) and (2.10), from (2.8) we obtain

$$\mathrm{Re}\left(\triangle_E^2 u, qu\right) \geq -\left(C_1 \frac{\gamma}{2} + C_2\gamma\right) \left\|qu\right\|^2 + \left(1 - C_1 \frac{\gamma^{-1}}{2} - C_2\gamma^{-1}\right) \left\|q^{\frac{1}{2}} \triangle_E u\right\|^2,$$

implying that

$$\begin{aligned} 2\beta^{-1}\mathrm{Re}\left(\triangle_E^2 u, qu\right) &\geq -\beta^{-1}\gamma\left(C_1 + 2C_2\right) \left\|qu\right\|^2 \\ &\quad +\beta^{-1}\left(2 - C_1\gamma^{-1} - 2C_2\gamma^{-1}\right) \left\|q^{\frac{1}{2}} \triangle_E u\right\|^2. \qquad (2.11) \end{aligned}$$

Taking into account (2.7) and (2.11), from (2.4) we get

$$(2.12) \qquad \left(1 + \left|1 - \beta^{-1}\right| k^{-1}\right) \left\|Au\right\|^2 \geq \left(1 - \beta^{-1}C_1\gamma - 2\beta^{-1}C_2\gamma\right) \left\|qu\right\|^2$$

$$+\beta^{-1}\left(2 - C_1\gamma^{-1} - 2C_2\gamma^{-1}\right) \left\|q^{\frac{1}{2}} \triangle_E u\right\|^2 + \left(2\beta^{-1} - 1 - \left|1 - \beta^{-1}\right| k\right) \left\|\triangle_E^2 u\right\|^2.$$

If $\alpha\beta < 1$, we choose $k > 0$ such that $1 + \left|1 - \beta^{-1}\right| k^{-1} = (\alpha\beta)^{-1}$, and multiply both sides of (2.12) by $\alpha\beta$ to obtain

$$\begin{aligned} \left\|Au\right\|^2 &\geq \alpha\left(\beta - C_1\gamma - 2C_2\gamma\right) \left\|qu\right\|^2 + \alpha\left(2 - C_1\gamma^{-1} - 2C_2\gamma^{-1}\right) \left\|q^{\frac{1}{2}} \triangle_E u\right\|^2 \\ &\quad +\alpha\beta\left(2\beta^{-1} - 1 - \left|1 - \beta^{-1}\right| k\right) \left\|\triangle_E^2 u\right\|^2. \qquad (2.13) \end{aligned}$$

Next, we show that the following equality holds:

$$(2.14) \qquad \alpha\beta\left(2\beta^{-1} - 1 - \left|1 - \beta^{-1}\right| k\right) = 1 - (1 - \alpha)^2 (1 - \alpha\beta)^{-1}.$$

Indeed, since $1 + |1-\beta^{-1}| k^{-1} = (\alpha\beta)^{-1}$, we have $k = \frac{\alpha\beta|1-\beta^{-1}|}{1-\alpha\beta}$. Hence, we can write

$$\begin{aligned}
\alpha\beta\left(2\beta^{-1}-1-|1-\beta^{-1}|k\right) &= \alpha\beta\left(2\beta^{-1}-1-\frac{\alpha\beta|1-\beta^{-1}|^2}{1-\alpha\beta}\right)\\
&= \alpha\beta\left(\frac{2\beta^{-1}-2\alpha-1+\alpha\beta-\alpha\beta|1-\beta^{-1}|^2}{1-\alpha\beta}\right)\\
&= \frac{2\alpha-2\alpha^2\beta-\alpha\beta+\alpha^2\beta^2-\alpha^2\beta^2\left(1-2\beta^{-1}+\beta^{-2}\right)}{1-\alpha\beta}\\
&= \frac{-\alpha\beta-\alpha^2+2\alpha}{1-\alpha\beta} = \frac{(1-\alpha\beta)-(1-\alpha)^2}{1-\alpha\beta} = 1-(1-\alpha)^2(1-\alpha\beta)^{-1}.
\end{aligned}$$

Now from (2.13) and (2.14), we get

$$\begin{aligned}
\|Au\|^2 \ \geq\ & \alpha\left(\beta - C_1\gamma - 2C_2\gamma\right)\|qu\|^2 + \alpha\left(2 - C_1\gamma^{-1} - 2C_2\gamma^{-1}\right)\left\|q^{\frac{1}{2}}\Delta_E u\right\|^2\\
& + \left(1-(1-\alpha)^2(1-\alpha\beta)^{-1}\right)\left\|\Delta_E^2 u\right\|^2. \qquad (2.15)
\end{aligned}$$

If $\alpha = \beta = 1$, we can take any $k > 0$ in (2.13) to obtain

$$\begin{aligned}
\|Au\|^2 \ \geq\ & \left(1 - C_1\gamma - 2C_2\gamma\right)\|qu\|^2 + \left\|\Delta_E^2 u\right\|^2\\
& + \left(2 - C_1\gamma^{-1} - 2C_2\gamma^{-1}\right)\left\|q^{\frac{1}{2}}\Delta_E u\right\|^2. \qquad (2.16)
\end{aligned}$$

From (2.15) and (2.16), we obtain

$$a\|qu\|^2 + b\left\|q^{\frac{1}{2}}\Delta_E u\right\|^2 + c\left\|\Delta_E^2 u\right\|^2 \leq \|Au\|^2, \qquad (2.17)$$

where $a = \alpha\left(\beta - (C_1 + 2C_2)\gamma\right)$, $b = \alpha\left(2 - (C_1 + 2C_2)\gamma^{-1}\right)$ and

$$c = \begin{cases} 1-(1-\alpha)^2(1-\alpha\beta)^{-1} & \text{if } \alpha\beta < 1\\ 1 & \text{if } \alpha = \beta = 1. \end{cases}$$

If $C_1 + 2C_2 \in (0,2)$, then there exists an admissible triple of parameters $(\alpha,\beta,\gamma) \in P$ satisfying the inequalities:

$$\beta \geq (C_1 + 2C_2)\gamma, \quad 2\gamma \geq C_1 + 2C_2, \quad \textit{and} \quad \alpha + \beta \leq 2, \qquad (2.18)$$

which implies that $a \geq 0$, $b \geq 0$, and $c \geq 0$. If $C_1 = C_2 = 0$, then from (7) and (8) we get $d^2q(x)\ u(x) = 0$ and $dq(x)\ du(x) = 0$, for every $x \in M$ and $u \in C_c^\infty(M)$. Consequently, we have $\Delta_E(qu) = q(\Delta_E u)$, and for every $u \in C_c^\infty(M)$, we can write

$$\|Au\|^2 = (Au, Au) = \left(\Delta_E^2 u + qu, \Delta_E^2 u + qu\right) \qquad (2.19)$$

$$
\begin{aligned}
&= \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(\Delta_E^2 u, qu\right) + \|qu\|^2 = \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(qu, \Delta_E^2 u\right) + \|qu\|^2 \\
&= \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(\Delta_E (qu), \Delta_E u\right) + \|qu\|^2 \\
&= \left\|\Delta_E^2 u\right\|^2 + 2\mathrm{Re}\left(q(\Delta_E u), \Delta_E u\right) + \|qu\|^2 \\
&= \left\|\Delta_E^2 u\right\|^2 + 2\left\|q^{\frac{1}{2}}\Delta_E u\right\|^2 + \|qu\|^2 .
\end{aligned}
$$

It follows from (2.17) – (2.19) that the inequality (2.17) holds for all $u \in C_c^{\infty}(M)$, with $a \geq 0$, $b \geq 0$, and $c \geq 0$ if $C_1 + 2C_2 \in [0, 2)$.

Now we proceed to prove that under the hypotheses of the theorem the inequality (2.17) holds for all $u \in D_1$. To this end, observe first that from the completeness of (M, g), it follows that the operator S is essentially self-adjoint and $Dom(\bar{S}) = D_1$. Let $u \in D_1$, then there exists a sequence $\{u_n\}$ in $C_c^{\infty}(M)$ such that $u_n \to u$ and $Au_n \to \bar{S}u$ in $L^2(M)$ as $n \to \infty$. Applying (2.17) with $u = u_n - u_m$, we conclude that the sequences $\{qu_n\}$, $\{\Delta_E^2 u_n\}$ and $\{q^{\frac{1}{2}}\Delta_E u_n\}$ are Cauchy sequences in $L^2(M)$. Since $\|\Delta_E u_n\|^2 = (\Delta_E u_n, \Delta_E u_n) = \left(\Delta_E^2 u_n, u_n\right) \leq \left\|\Delta_E^2 u_n\right\| \|u_n\|$, and $\{\Delta_E^2 u_n\}$ and $\{u_n\}$ are Cauchy sequences in $L^2(M)$, it follows that $\{\Delta_E u_n\}$ is also a Cauchy sequence in $L^2(M)$. Taking into account that the operator Δ_E is essentially self-adjoint on $C_c^{\infty}(M)$ (see [17]), we have

$$\Delta_E u_n \to \Delta_E u, \tag{2.20}$$

implying that

$$q^{\frac{1}{2}}\Delta_E u_n \to q^{\frac{1}{2}}\Delta_E u \quad \text{and} \quad \Delta_E^2 u_n \to \Delta_E^2 u, \tag{2.21}$$

in $L^2(M)$ as $n \to \infty$.

Finally, taking into account that $\{qu_n\}$ is a Cauchy sequence in $L^2(M)$ and $C_c^{\infty}(M)$ is dense in $L^2(M)$, it follows that

$$qu_n \to qu, \tag{2.22}$$

in $L^2(M)$ as $n \to \infty$. Now, replacing u by u_n in (2.17), passing to the limit as $n \to \infty$ in all terms, and using (2.20) – (2.22), we conclude that the inequality (2.17) holds for all $u \in D_1$. This means that the differential expression A defined by (1.3) is separated in the space $L^2(M)$. This completes the proof of the theorem. □

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