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UNIQUENESS THEOREMS FOR SOLUTIONS OF PAINLEVÉ TRANSCENDENTS

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Abstract. In this paper we deal with the uniqueness problems when two meromorphic functions f and g share three distinct values CM and f satisfies the first, second or fourth Painlevé transcendents.

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1. Introduction and main results

In this paper,[1] a meromorphic function will always mean meromorphic in the complex plane $\mathbb{C}$. We adopt the standard notations in the Nevanlinna value distribution theory of meromorphic functions such as $T(r, f)$, $m(r, f)$, $N(r, f)$ and $\bar{N}(r, f)$ as explained in [1, 16]. For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\big(T(r, f)\big)$, possibly outside a set of finite linear measure that is not necessarily the same at each occurrence.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane $\mathbb{C}$ and let a be a complex number. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicity). If $1/f$ and $1/g$ share the value 0 CM, we say that f and g share ∞ CM. By $N_0(r)$ we denote the counting function of the 0-points of $f - g$ that are not the 0-points of f, $f - 1$, and $1/f$.

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Consider the first, second and fourth Painlevé equations:

$$\begin{aligned}
\text{(I)} \quad \omega'' &= 6\omega^2 + z, \\
\text{(II)} \quad \omega'' &= 2\omega^3 + z\omega + \alpha, \\
\text{(IV)} \quad \omega'' &= \frac{(\omega')^2}{2\omega} + \frac{3}{2}\omega^3 + 4z\omega^2 + 2(z^2 - \alpha)\omega + \frac{\beta}{\omega},
\end{aligned}$$

where α and β are arbitrary complex constants. It has been proved by A. Hinkkanen, I. Laine, S. Shimomura and N. Steinmetz, independently, that all the solutions of the above three equations are meromorphic in the whole complex plane (see [3, 5, 14]). It is easy to see that every solution of equation (I) is transcendental. The equation (II) admits a unique rational solution if and only if $\alpha \in \mathbb{Z}$ (see [10]). The equation (IV) admits a rational solution as discussed by M. Mazzocco (see [9, 10]). Notice that the solutions of Painlevé equations are generally transcendental. The non-rational solutions of equations (I), (II) and (IV) are called the first, second and fourth Painlevé transcendents, respectively.

In 2007, W. C. Lin and K. Tohge [8] studied the share-value properties of Painlevé transcendents and proved the following result.

Theorem A. *Let $\omega(z)$ be an arbitrary non-constant solution of one of the equations (I), (II), (IV), and let $f(z)$ be a non-constant meromorphic function which shares four distinct values a_j $(j = 1, 2, 3, 4)$ IM with $\omega(z)$. Then $f(z) \equiv \omega(z)$.*

A question of great interest is to obtain unicity results for solutions of differential equations. There is little related research in this direction (see [15, 16]), except maybe the results on the growth order and the deficiency of Painlevé transcendents (see [3], [11] - [13]).

In this paper we study unicity of meromorphic solutions of Painlevé equations which share three distinct values. The main results of the paper are the following theorems.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions sharing three distinct values c_j $(j = 1, 2, 3)$ CM. Suppose that f satisfies the first Painlevé transcendents, then $f(z) \equiv g(z)$.*

Theorem 1.2. *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions sharing three distinct values c_j $(j = 1, 2, 3)$ CM. Suppose that f satisfies the second Painlevé* *transcendents and $\alpha \neq 0$, then $f(z) \equiv g(z)$.*

Remark 1.1. If in Theorem 1.2 we drop the condition $\alpha \neq 0$, then we need to impose the condition $c_j \neq 0$ (see Lemma 2.3 and the proof of Theorem 1.2).

Theorem 1.3. *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions sharing three distinct values c_j $(j = 1, 2, 3)$ CM. Suppose that f satisfies the fourth Painlevé transcendents and $\beta \neq 0$, then $f(z) \equiv g(z)$.*

Remark 1.2. If in Theorem 1.3 we drop the condition $\beta \neq 0$, then we need to impose the condition $c_j \neq 0$ (see Lemma 2.4 and the proof of Theorem 1.3).

2. Some lemmas

In this section we state a number of known lemmas from [3, 6, 7, 16, 18] that will be used in the proofs of our main results in Section 3. The first lemma, due to A. Mohon'ko and V. Mohon'ko (see [6]), plays an important role in proving the subsequent lemmas.

Lemma 2.1 ([6]). *Let $P(z, u_0, u_1, \ldots, u_n)$ be a polynomial in all of its arguments, and let f be a transcendental meromorphic solution of the following algebraic differential equation:*

$$P(z, f, f', \cdots, f^{(n)}) = 0. \tag{2.1}$$

If a finite complex number c does not solve (2.1), then

$$m\left(r, \frac{1}{f-c}\right) = S(r, f).$$

Lemma 2.2 ([3]). *Let ω be an arbitrary solution of equation (I), and let $c \in \hat{\mathbb{C}}$. Then we have $m(r, \frac{1}{\omega - c}) = O(\log r)$.*

Lemma 2.3 ([3]). *Let ω be an arbitrary solution of equation (II), and let $c \in \hat{\mathbb{C}}$. Then we have $m(r, \omega) = O(\log r)$, and*

i) if $\alpha \neq 0$, then $m(r, \frac{1}{\omega - c}) = O(\log r)$ for $c \in \mathbb{C}$,

ii) if $\alpha = 0$, then $m(r, \frac{1}{\omega - c}) = O(\log r)$ for $c \in \mathbb{C} \backslash \{0\}$, while

$$m(r, \frac{1}{\omega}) \leq \frac{1}{2} T(r, \omega) + O(\log r).$$

Lemma 2.4 ([3]). *Let ω be an arbitrary solution of equation (IV), and let $c \in \hat{\mathbb{C}}$. Then we have $m(r,\omega) = O(\log r)$, and*

i) if $\beta \neq 0$, then $m(r, \frac{1}{\omega - c}) = O(\log r)$ for $c \in \mathbb{C}$,

ii) if $\beta = 0$, then $m(r, \frac{1}{\omega - c}) = O(\log r)$ for $c \in \mathbb{C}\backslash\{0\}$; further if $\omega(z)$ satisfies the Riccati differential equation $\omega' = \mp(\omega^2 + 2z\omega)$, then $m(r, \frac{1}{\omega}) = T(r,\omega) + O(1)$, otherwise we have

$$m(r, \frac{1}{\omega}) \leq \frac{1}{2}T(r,\omega) + O(\log r).$$

Lemma 2.5 ([7, 18]). *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1, and ∞ CM. If*

$$0 < \limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r,f)} \leq \frac{1}{2},$$

where E is a set of r of finite linear measure, then f is not a fractional linear transformation of g, and f and g assume one of the following three relations:

(i) $f \equiv \dfrac{e^{s\gamma} - 1}{e^{(t+1)\gamma} - 1}, \quad g \equiv \dfrac{e^{-s\gamma} - 1}{e^{-(t+1)\gamma} - 1};$

(ii) $f \equiv \dfrac{e^{(t+1)\gamma} - 1}{e^{s\gamma} - 1}, \quad g \equiv \dfrac{e^{-(t+1)\gamma} - 1}{e^{-s\gamma} - 1};$

(iii) $f \equiv \dfrac{e^{s\gamma} - 1}{e^{-(t+1-s)\gamma} - 1}, \quad g \equiv \dfrac{e^{-s\gamma} - 1}{e^{(t+1-s)\gamma} - 1},$

where γ is a nonconstant entire function, s and $t(\geq 2)$ are positive integers such that s and $t+1$ are mutually prime and $1 \leq s \leq t$.

If

$$\limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r,f)} \geq \frac{1}{2},$$

then f is a fractional linear transformation of g.

Lemma 2.6 ([18]). *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1, and ∞ CM. If f is a fractional linear transformation of g, then f and g satisfy one of the following six relations:*

(i) $fg \equiv 1$;

(ii) $(f-1)(g-1) \equiv 1$;

(iii) $f + g = 1$;

(iv) $f \equiv cg$;

(v) $f - 1 = c(g-1)$;

(vi) $[(c-1)f + 1][(c-1)g - c] \equiv -c$,

where $c(\neq 0, 1)$ is a constant.

Lemma 2.7 ([7]). *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1, and ∞ CM. Then there exist two entire functions p and q such that*

$$f \equiv \frac{e^p - 1}{e^q - 1}, \qquad g \equiv \frac{e^{-p} - 1}{e^{-q} - 1}, \tag{2.2}$$

where $e^q \not\equiv 1$, $e^p \not\equiv 1$, $e^{q-p} \not\equiv 1$ and

$$T(r, g) + T(r, e^p) + T(r, e^q) = O(T(r, f)) \quad (r \notin E), \tag{2.3}$$

where E is a set of r of finite linear measure.

Lemma 2.8 ([16]). *Let $f_1(z), f_2(z), \cdots, f_n(z)$ $(n \geq 2)$ be meromorphic functions, and let $g_1(z), g_2(z), \cdots, g_n(z)$ be entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$,

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$,

(iii) *for* $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}$ $\quad (r \to \infty,\ r \notin E)$.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \cdots, n)$.

3. Proof of theorems

Proof of Theorem 1.1. We first assume that $c_1 = 0$, $c_2 = 1$, $c_3 = \infty$. Then by Lemma 2.2, we have

$$N(r, \frac{1}{f - c}) = T(r, f) + S(r, f), \quad c = 0, 1, \infty. \tag{3.1}$$

Suppose, to the contrary that $f \not\equiv g$, and consider the following two cases:

Case 1. Suppose that f is a fractional linear transformation of g. Then, by Lemma 2.6, f and g satisfy one of the six relations in Lemma 2.6. It is not difficult to check that at least one of the values 0, 1 and ∞ is the Picard value of f in all the six relations, which contradicts (3.1).

Case 2. Suppose that f is not a fractional linear transformation of g. In view of Lemma 2.5, we consider the following two subcases:

Subcase 2.1. Suppose that

$$0 < \limsup_{r \to \infty, r \notin E} \frac{N_0(r)}{T(r, f)} \leq \frac{1}{2},$$

then, by Lemma 2.5, f and g assume one of the three relations in Lemma 2.5.

Suppose first that f and g satisfy (i), and note that s and $t \geq 2$ are positive integers satisfying $(s, t+1) = 1$. Then we have $T(r, f) = tT(r, e^{\gamma}) + S(r, f)$ and $N(r, \frac{1}{f}) = (s-1)T(r, e^{\gamma}) + S(r, f)$, which contradicts (3.1).

Now let f and g satisfy (ii), then we have $T(r, f) = tT(r, e^{\gamma}) + S(r, f)$ and $N(r, f) = (s-1)T(r, e^{\gamma}) + S(r, f)$, which contradicts (3.1).

Finally, suppose that f and g satisfy (iii), and note that s and $t \geq 2$ are positive integers satisfying $(s, t+1) = 1$, which implies $(s, t+1-s) = 1$. Thus, we have $T(r, f) = tT(r, e^{\gamma}) + S(r, f)$ and $N(r, f) = (t-s)T(r, e^{\gamma}) + S(r, f)$, $N(r, \frac{1}{f}) = (s-1)T(r, e^{\gamma}) + S(r, f)$, which contradicts (3.1).

Subcase 2.2. Suppose that

$$\limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r, f)} = 0,$$

then we have

$$N_0(r) = S(r, f). \tag{3.2}$$

Since f and g share 0, 1, and ∞ CM, by Lemma 2.7, we get

$$f \equiv \frac{e^p - 1}{e^q - 1}, \qquad g \equiv \frac{e^{-p} - 1}{e^{-q} - 1}, \tag{3.3}$$

where p and q satisfy the conditions in Lemma 2.7. If $e^q \equiv C$ with a non-zero constant C, then by (3.3) we have

$$\frac{(f-1)g}{f(g-1)} \equiv C,$$

showing that f is a fractional linear transformation of g, which is a contradiction. Hence, e^q is not a constant. Similarly, we can prove that e^p and e^{p-q} are not constants. Meanwhile, we can get

$$f - g = \frac{(e^p - 1)(1 - e^{q-p})}{e^q - 1}. \tag{3.4}$$

Next, let $N_0^*(r)$ denote the counting function of the common zeros of $e^p - 1$ and $e^q - 1$. Then by (3.4) we have $N_0(r) = N_0^*(r) + S(r, f)$. From this and (3.2), we obtain

$$N_0^*(r) = S(r, f). \tag{3.5}$$

Combining (3.3)–(3.5), we can write

$$N(r, f) = N(r, \frac{1}{e^q - 1}) + S(r, f),$$

$$N(r, \frac{1}{f}) = N(r, \frac{1}{e^p - 1}) + S(r, f),$$

$$N(r, \frac{1}{f-1}) = N(r, \frac{1}{e^{p-q}-1}) + S(r,f).$$

Therefore, taking into account (3.1), we obtain

$$T(r,e^q) = T(r,e^p) + S(r,f) = T(r,e^{p-q}) + S(r,f) = T(r,f) + S(r,f). \tag{3.6}$$

Suppose that f is a transcendental meromorphic solution of equation (I), then we can substitute (3.3) into (I) to obtain

$$\begin{aligned} &ze^{3q} + [(3q'^2 + q'') - (p'^2 + p'') - 2p'q']e^{p+2q} + 6e^{2p+q} - (3z + 3q'^2 + q'')e^{2q} \\ &\quad + (2p'q' + 2p'' - q'^2 - q'' - 12)e^{p+q} + (12 + p'^2 - p'')e^p \\ &\quad + (q'' + q'^2 + 3z + 6)e^q - (z+6) = 0. \end{aligned} \tag{3.7}$$

We rewrite (3.7) as follows

$$\begin{aligned} \rho_0 = \rho_1 e^{3q} + \rho_2 e^{p+2q} + \rho_3 e^{2p+q} + \rho_4 e^{2q} + \rho_5 e^{p+q} + \rho_6 e^p \\ + \rho_7 e^q = \sum_{j=1}^{7} \rho_j e^{g_j}, \end{aligned} \tag{3.8}$$

where $T(r,\rho_j) = S(r,f)$ $(j = 0, 1, \cdots, 7)$, $\rho_0 = z + 6 \not\equiv 0$.

Note that $g_j, g_h - g_k = \nu p + \mu q$ for $j = 1, \cdots, 5$ and $h \neq k$, where μ, ν are integers and at least one of them is different from zero. Now we proceed to prove that

$$T(r, e^{\nu p + \mu q}) \geq T(r,f) + S(r,f). \tag{3.9}$$

By the first fundamental theorem, we need to consider two cases: $\nu \geq 0, \mu \geq 0$ and $\nu \leq 0, \mu \geq 0$. Suppose first that $\nu \geq 0, \mu \geq 0, \mu + \nu > 0$. If $\nu > \mu$, then from (3.6) we get

$$T(r, e^{\nu p + \mu q}) \geq (\nu - \mu)T(r,f) + S(r,f) \geq T(r,f) + S(r,f).$$

If $\nu < \mu$, then since $T(r,e^{2p}) \leq T(r,e^{p+q}) + T(r,e^{p-q})$, by (3.6) we have $T(r,e^{p+q}) \geq T(r,f) + S(r,f)$, and hence (3.9) holds. If $\mu = \nu$, then $T(r,e^{\nu p + \mu q}) = \nu T(r,f) + S(r,f)$.

Next, suppose that $\nu \leq 0, \mu \geq 0, \mu - \nu \neq 0$. If $-\nu = \mu$, then it is not difficult to check that (3.9) holds. If $-\nu < \mu$, we have

$$T(r, e^{\nu p + \mu q}) \geq \mu T(r,e^q) + \nu T(r,e^p),$$

and by (3.6) we obtain (3.9). Similarly, we can prove (3.9) in the case $-\nu > \mu$.

Finally, in view of Lemma 2.8, (3.8) and (3.9) we conclude that $\rho_k \equiv 0$ $(k = 0, 1, \cdots, 7)$, which contradicts $\rho_0 \not\equiv 0$. Therefore, we have $f(z) \equiv g(z)$.

This completes the proof of the theorem in the special case where $c_1 = 0$, $c_2 = 1$, $c_3 = \infty$. For the general case, we consider the transformations

$$F(z) = \frac{f - c_1}{f - c_3}\frac{c_2 - c_3}{c_2 - c_1}, \quad G(z) = \frac{g - c_1}{g - c_3}\frac{c_2 - c_3}{c_2 - c_1},$$

and observe that F and G share $0, 1, \infty$ CM. By Lemma 2.2, we can write F and G in the form, similar to (3.3). Hence we can get

$$f = \frac{c_1(c_2 - c_3)(e^q - 1) - c_3(c_2 - c_1)(e^p - 1)}{(c_2 - c_3)(e^q - 1) - (c_2 - c_1)(e^p - 1)}. \tag{3.10}$$

Substituting (3.10) into (II), and using arguments similar to that of applied in the special case $c_1 = 0$, $c_2 = 1$, $c_3 = \infty$, we can prove that $F \equiv G$. Hence, we have $f \equiv g$. Theorem 1.1 is proved. □

Proof of Theorem 1.2. Suppose that f is a transcendental meromorphic solution of the equation (II), then we can substitute (3.3) into (II) to obtain

$$\begin{aligned} &2e^{3p} + \alpha e^{3q} - 6e^{2p} + [z - (p'^2 + p'' + 2p'q') + 3q'^2 + q'']e^{p+2q} \\ &+ (2p'' - q'' - q'^2 - 2z)e^{p+q} - (3q'^2 + q'' + z + 3\alpha)e^{2q} \\ &+ (6 + z + p'^2 - p'')e^p + (q'' + q'^2 + 2z + 3\alpha)e^q - (z + \alpha + 2) = 0. \end{aligned} \tag{3.11}$$

We can rewrite (3.11) as follows

$$\begin{aligned} \rho_0 = \rho_1 e^{3p} &+ \rho_2 e^{3q} + \rho_3 e^{2q} + \rho_4 e^{p+2q} + \rho_5 e^{p+q} + \rho_6 e^{2q} \\ &+ \rho_7 e^p + \rho_8 e^q = \sum_{j=1}^{8} \rho_j e^{g_j}, \end{aligned} \tag{3.12}$$

where $T(r, \rho_j) = S(r, f)$ $(j = 0, 1, \cdots, 8)$, $\rho_0 = z + \alpha + 2 \not\equiv 0$.

The rest of the proof is similar to that of Theorem 1.1, and so is omitted. □

Proof of Theorem 1.3. Suppose that f is a transcendental meromorphic solution of the equation (IV), then we can substitute (3.3) into (IV) to obtain

(3.13)

$$\begin{aligned} &3e^{4p} + 2\beta e^{4q} + 8ze^{3p+q} + [4(z^2 - \alpha) + 7q'^2 - p'^2 + 2q'' - 2p'' - 6p'q']e^{2p+2q} \\ &+ [8(z^2 + \alpha) + 2p'^2 - 2p'q' + 2(q'' + q'^2 - 2p'')]e^{2p+q} - 8(z + \beta)e^{3q} - 12e^{3p} \\ &- 2[3p'q' - 7q'^2 - 2q'' + p'^2 + p'' - 4(z^2 - \alpha)]e^{p+2q} \\ &+ [4(z^2 - \alpha) + 24z + 18 + 2(p'^2 - p'')]e^{2p} + [7q'^2 + 2q'' + 4(z^2 - \alpha) + 12\beta]e^{2q} \\ &+ [24z - 2p'q' + 16(z^2 - \alpha) + 2(q'^2 + q'' - 2p'')]e^{p+q} \\ &- [24z + 12 + 8(z^2 - \alpha) + 2p'^2 - 2p'']e^p - [8z + 8\beta + 8(z^2 - \alpha) + 2q'' + 2q'^2]e^q \\ &+ 4(z^2 - \alpha) + 8z + 2\beta + 3 = 0. \end{aligned}$$

We can rewrite (3.13) as follows

$$\begin{aligned}\rho_0 &= \rho_1 e^{4p} + \rho_2 e^{4q} + \rho_3 e^{3p+q} + \rho_4 e^{2p+2q} + \rho_5 e^{2p+q} + \rho_6 e^{3p} + \rho_7 e^{3p} \\ &+ \rho_8 e^{p+2q} + \rho_9 e^{2q} + \rho_{10} e^{2p} + \rho_{11} e^{p+q} + \rho_{12} e^{p} + \rho_{13} e^{q} = \sum_{j=1}^{13} \rho_j e^{g_j},\end{aligned} \tag{3.14}$$

where $T(r,\rho_j) = S(r,f)$ $(j=0,1,\cdots,13)$, $\rho_0 = 4(z^2-\alpha)+8z+2\beta+3 \not\equiv 0$.

The rest of the proof is similar to that of Theorem 1.1, and so is omitted. Theorem 1.3 is proved.

4. Higher order analogues of the first Painlevé equation

It is known that the Kortewey-de Vries equation can be reduced to the following algebraic differential equation (see [2])

$$d^{n+1}(\omega) + 4z = 0, \tag{${}_{2n}P_1$}$$

where $d^{n+1}(\omega) = D^{-1}[(D^3 - 8\omega D - 4\omega')d^n(\omega)]$, $D = \frac{d}{dz}$ and D^{-1} denotes the inverse of D, that is, $D^{-1}(\cdot) = \int \cdot dz$ and $d^1(\omega) = -4\omega$.

For $n=1$, we have $d^2(\omega) + 4z = -4\omega'' + 24\omega^2 + 4z$. Therefore

$$\omega'' = 6\omega^2 + z, \tag{${}_2P_1$}$$

which is nothing but the first Painlevé equation, and the equation $({}_{2n}P_1)$ is called the higher order analogue of the first Painlevé equation of order n.

In [4], Y. Z. He has proved some value distribution properties of the meromorphic solutions of equation $({}_{2n}P_1)$. Observe that for equation $({}_{2n}P_1)$ rational solutions cannot exist. Below we investigate the equation $({}_{2n}P_1)$ and obtain a result, which is similar to that of the first Painlevé equation.

Theorem 4.1. *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions sharing three distinct values c_j $(j=1,2,3)$ CM. Suppose that f satisfies the higher order analogue of the first Painlevé transcendents $({}_{2n}P_1)$, then $f(z) \equiv g(z)$.*

In order to prove the theorem we need the following two lemmas (see [4]).

Lemma 4.1. *Let ω be an arbitrary solution of equation $({}_{2n}P_1)$, then we have $\delta(a,\omega) = 0$ for all $a \in \hat{C}$.*

Lemma 4.2. *The differential polynomial on the left hand side of* $({}_{2n}P_1)$ *has only one leading term of the form* $a_{n+1}\omega^{n+1}$, *where*

$$a_{n+1} = (-1)^{n+1} 6 \cdot 4^n \prod_{l=2}^{n} \frac{2l+1}{l+1}, \quad n \geq 2.$$

All coefficients in $({}_{2n}P_1)$ *are constants except for the term* $4z$.

Proof of Theorem 4.1. In view of Lemma 4.2, the equation $({}_{2n}P_1)$ can be written in the form $a_{n+1}\omega^{n+1} + 4z + \Omega_n(\omega, \cdots, \omega^{2n}) = 0$, where $\Omega_n(\omega, \cdots, \omega^{2n}) = \sum A_\lambda \omega^{\lambda_0} \cdots (\lambda^{(2n)})^{\lambda_{2n}}$ is a differential polynomial with constant coefficients and its total degree $\leq n$, for each term in Ω_n, and $\sum_{j=1}^{2n} \lambda_j \neq 0$. By Lemma 4.2 and (3.3) we have

$$a_{n+1}\left(\frac{e^p-1}{e^q-1}\right)^{n+1} + \Omega_n\left(\frac{e^p-1}{e^q-1}, \cdots, \left(\frac{e^p-1}{e^q-1}\right)^{2n}\right) + 4z = 0. \tag{4.1}$$

Rewrite (4.1) in the form $\rho_0 = \sum_{j=1}^{m} \rho_j e^{g_j}$, where $T(r, \rho_j) = S(r, f)$ $(j = 0, 1, \cdots, m)$, $\rho_0 = 4z + constant \not\equiv 0$. Note that $g_j, g_h - g_k = \nu p + \mu q$ for $j = 1, \cdots, m$ and $h \neq k$, where μ, ν are integers and at least one of them is different from zero. The rest of the proof is similar to that of Theorem 1.1, and so is omitted. □

Список литературы

[1] W. K. Hayman, Meromorphic functions, Clarendon Press, Oxford (1964).

[2] V. Gromak, "The first higher order Painlevé differential equations", Differential Equations, **35**, 37 – 41 (1999).

[3] V. Gromak, I. Laine, S. Shimomura, Painlevé differential equations in the complex plane, Walter de Gruyter Berlin, New York (2002).

[4] Y. Z. He, Value distribution of the higher order analogues of the first Painlevé equation, Value distribution theory and related topics, edited by G. Barsegian, I. Laine and C. C. Yang, Kluwer Academic Publishers, Boston/Dordrecht/New York/London.

[5] I. A. Hinkkanen and I. Laine, "Solutions of the first and second Painlevé equations are meromorphic", J. Anal. Math. **79**, 345 – 377 (1999).

[6] I. Laine, Nevanlinna theory and complex differential equations, Walter de Gruyter, Berlin-New York (1993).

[7] X. M. Li, H. X. Yi, "Meromorphic functions sharing three values", J. Math. Soc. Japan, **56**, 26 – 36 (2004).

[8] W. C. Lin and K. Tohge, "On shared-value properties of painleve transcendents", Computational Methods and Function Theory, no. 2, 477 – 499 (2007).

[9] M. Mazzocco, "Rational solutions of the Painlevé VI equation", J. Phys. A **34**, no. 11, 2281 – 2294 (2001).

[10] Y. Murata, "Rational solutions of the second and the fourth Painlevé equations", Funkcial. Ekvac., **28**, 1 – 32 (1985).

[11] Y. Sasaki, "Value distribution of the fifth Painlevé transcendents in sectorial domains", J. Math. Anal. Appl., **330**, 817 – 828 (2007).

[12] S. Shimomura, "Value distribution of Painlevé transcendenrs of the first and second kind", J. Anal. Math., **82**, 333 – 346 (2000).

[13] S. Shimomura, "On deficiencies of small functions for Painlevé transcendenrs of the fourth kind", Ann. Acad. Sci. Fenn. Math., **27**, 109 – 120 (2002).

[14] N. Steinmetz, "On Painlevé's Equations I, II and IV", J. Anal. Math., **82**, 363 – 377 (2000).
[15] J. Wang and H. P. Cai, "Uniqueness theorems for solutions of differential equations", J.Sys. Sci and Math. Scis. **26**, 21 – 30 (2006).
[16] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions. New York, Dordrecht, Boston, London (2003).
[17] H. X. Yi, Unicity theorems for meromorphic functions that share three values, Kodai Math. J., **18**, 300 – 314 (1995).
[18] Q. C. Zhang, "Meromorphic functions sharing three values", Indian J. Pure Appl. Math., **30**, 667 – 682 (1999).

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