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FUJIMOTO'S THEOREM - A FURTHER STUDY

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Abstract. In the paper we inprove Fujimoto's sufficient condition for a finite set to be a Unique Range Set under relaxed sharing hypothesis. We introduce a new sharing notion which directly improves one result of the paper [3]. In particular, we rectify the Application Part of [3], and extend and improve all results of [3]. Also, we pose an open question for future research.

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1. Introduction. Definitions and results

We will use the following notation: by $\mathbb{C}$ we denote the set of all complex numbers, by $\mathbb{N}$ the set of all positive integers, and $\overline{\mathbb{C}} := \mathbb{C} \cup \infty$, $\overline{\mathbb{N}} := 0 \cup \mathbb{N} \cup \infty$. Throughout the paper it is assumed without stating its explicitly that all the considered meromorphic functions are defined on $\mathbb{C}$ and that they are non-constant.

For such a function f and $a \in \overline{\mathbb{C}}$, each point z with $f(z) = a$ will be called an a-point of f. For a meromorphic function f and a set $S \subset \overline{\mathbb{C}}$ we define $E_f(S)$ (resp. $\overline{E}_f(S)$) as the set of all a-points of f, when $a \in S$, together with their multiplicities (resp. without their multiplicities). If $E_f(S) = E_g(S)$ (resp. $\overline{E}_f(S) = \overline{E}_g(S)$), then we say that f and g share S Counting Multiplicities or CM (resp. Ignoring Multiplicities or IM). More formally, we have the following definition.

Definition 1.1. *Let f be a meromorphic function and $S \subset \overline{\mathbb{C}}$. If $z_0 \in f^{-1}(S)$, then the value of $E_f(S)$ at the point z_0 is denoted by $E_f(S)(z_0) : f^{-1}(S) \to \mathbb{N}$ and is equal to the multiplicity of zero of the function $f(z) - f(z_0)$ at z_0, that is, the order of the pole of function $(f(z) - f(z_0))^{-1}$ at z_0 if $f(z_0) \in \mathbb{C}$ (resp. of function $f(z)$ if z_0 is a pole for f).*

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Lahiri [12, 13], has introduced the notion of weighted sharing of values and sets. It expedited new directions of research in the uniqueness theory. Below we define this notion in a slightly different way in context of Definition 1.1.

Definition 1.2. *For $k \in \overline{\mathbb{N}}$ and $z_0 \in f^{-1}(S)$ we put $E_f(S,k)(z_0) = \min\{E_f(S)(z_0), k+1\}$. Given $S \subset \overline{\mathbb{C}}$, we say that the meromorphic functions f and g share the set S up to multiplicity k (or share S with weight k, or simply, share (S,k)) if $f^{-1}(S) = g^{-1}(S)$, and for each $z_0 \in f^{-1}(S)$ we have $E_f(S,k)(z_0) = E_g(S,k)(z_0)$, which is represented by the notation $E_f(S,k) = E_g(S,k)$.*

It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \quad (r \longrightarrow \infty, r \notin E).$$

Also, we denote by $T(r,f;g)$ the maximum of $T(r,f)$ and $T(r,g)$, and by $S(r,f;g)$ any quantity satisfying $S(r,f;g) = o(T(r,f;g))$ as $r \longrightarrow \infty$, $r \notin E$.

We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [10]. For $a \in \mathbb{C} \cup \{\infty\}$, we define

$$\Theta(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

In 1926, R. Nevanlinna proved two fundamental results on shared values. His famous five value theorem gives an upper bound on the number of distinct values that two different meromorphic functions can share IM. Taking multiplicities into account, Nevanlinna proved that if two meromorphic functions share four distinct values CM, then either they coincide or one of them is the fractional linear transformation of the other. These results are in fact the gateway to the uniqueness theory of meromorphic functions.

In [8], F. Gross first considered the problem of determining an entire function uniquely by the single pre-image of a finite set S counting multiplicities. In 1982, F. Gross and C. C. Yang [9] proved the following theorem.

Theorem A. *Let $S = \{z \in \mathbb{C} : e^z + z = 0\}$. If two entire functions f and g satisfy $E_f(S) = E_g(S)$, then $f \equiv g$.*

Let $S \subset \mathbb{C}$, and let f and g be two non-constant meromorphic (entire) functions. If $E_f(S) = E_g(S)$ implies $f \equiv g$, then S is called a unique range set for meromorphic (entire) functions, or in short URSM (URSE). We will call any set $S \subset \mathbb{C}$ a unique range set for meromorphic (entire) functions ignoring multiplicity (URSM-IM) (URSE-IM) for which $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$ for any pair of non-constant meromorphic (entire) functions.

Note that since the range set S determined in Theorem A is an infinite set, the above result does not give a solution to the Gross' problem.

In 1994, H.X.Yi [18] exhibited a URSE with 15 elements, and in 1995, P.Li and C.C.Yang [16] exhibited a URSM with 15 elements and a URSE with 7 elements. Up-to-date the URSM with 11 elements is the smallest available URSM obtained by G. Frank and M.Reinders [7]. This result has been highlighted by a number of researchers. Still there is another type of URSM with the same minimum cardinality 11, which is discussed in the last section.

Li and Yang [16] were the first who elucidated the fact that the finite URSM's are the sets of distinct zeros of some polynomials. Consequently, studying these polynomials is of great importance.

A polynomial P in $\mathbb{C}$ is called a strong uniqueness polynomial for meromorphic (entire) functions if for any non-constant meromorphic (entire) functions f and g, the condition $P(f) \equiv cP(g)$ implies $f \equiv g$, where c is a suitable nonzero constant. We say P is SUPM (SUPE) in short. On the other hand, if for a polynomial P in $\mathbb{C}$, the condition $P(f) \equiv P(g)$ implies $f \equiv g$ for any non-constant meromorphic (entire) function f and g, then P is called a uniqueness polynomial for meromorphic (entire) functions. We say P is a UPM (UPE) in short.

Let P be a polynomial of degree n in $\mathbb{C}$ having only simple zeros, and let S be the set of all zeros of P. If S is a URSM (URSE), then from the definition it follows that P is UPM (UPE). The converse, in general, is not true as evidenced the following example, given in [4].

Example 1.1. Let $P(z) = az + b$ $(a \neq 0)$. Then it is clear that $P(z)$ is a UPM, but for $f = -\dfrac{b}{a}e^{z}$ and $g = -\dfrac{b}{a}e^{-z}$ we see that $E_f(S) = E_g(S)$, where $S = \{-\frac{b}{a}\}$ is the set of zeros of $P(z) = az + b$.

To find a condition under which the converse is true, H. Fujimoto [6] first invented a special property of a polynomial, which he called the property (H). Fujimoto's property (H) may be stated as follows: a polynomial P is said to satisfy the property (H) if $P(\alpha) \neq P(\beta)$ for any two distinct zeros α, β of the derivative P'.

Since the inner meaning of property (H) is that the polynomial P is injective on the set of distinct zeros of $P^{'}$, which are known as critical points of P, in [4] we have judiciously called property (H) as the critical injection property. However, this property can also be called as critical injectiveness property. Naturally, a polynomial with critical injectiveness property may be called a critically injective polynomial, and in the rest of the paper we use this terminology. Fujimoto [6] found a sufficient condition for a set of zeros S of a SUPM (SUPE) P to be a URSM (URSE) as follows.

Theorem B ([6]). *Let $P(z) = (z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_n)$ be a critically injective polynomial of degree n in $\mathbb{C}$ having only simple zeros. Let P' have k distinct zeros and either $k \geq 3$ or $k = 2$, and let P' have no simple zeros. Further, suppose that P is a* SUPM (SUPE). *If S is the set of zeros of P, then S is a* URSM (URSE) *whenever $n > 2k+6$ $(n > 2k+2)$, while S is a* URSM-IM (URSE-IM) *whenever $n > 2k+12$ $(n > 2k+5)$.*

Let us consider the following definition in connection with that of URSM (URSE).

Definition 1.3. [4] *A set $S \subset \mathbb{C} \cup \{\infty\}$ is called a unique range set for meromorphic (entire) functions with weight k if for any two non-constant meromorphic (entire) functions f and g, the condition $E_f(S,k) = E_g(S,k)$ implies $f \equiv g$. We write S is URSMk (URSEk) in short.*

Let k be a positive integer or infinity. We denote by $E_{k)}(a,f)$ the set of a-points of f whose multiplicities are not greater than k and each a-point is counted according to its multiplicity. For $S \subset \mathbb{C} \cup \{\infty\}$ we put $E_{k)}(S,f) = \bigcup_{a \in S} E_{k)}(a,f)$. The set S is called a $URSM_{k)}$ $(URSE_{k)})$ if for any two non-constant meromorphic (entire) functions f and g, the condition $E_{k)}(S,f) = E_{k)}(S,g)$ implies $f \equiv g$.

Note that when $k = 0$, the definition of URSMk (URSEk) coincides with that of URSM-IM (URSE-IM).

In 2009, X. Bai, Q. Han and A. Chen proved the following truncated sharing version of Theorem B.

Theorem C. [3] *In addition to the hypothesis of Theorem B we suppose that m is a positive integer or ∞. Let S be the set of zeros of P. If*

(i) $m \geq 3$ *or* ∞ *and* $n > 2k+6$ $(2k+2)$,

(ii) $m = 2$ *and* $n > 2k+7$ $(2k+2)$,

(iii) $m = 1$ *and* $n > 2k+10$ $(2k+4)$, *then* S *is a* URSM$_{m)}$ (URSE$_{m)}$).

To improve the URSM-IM version of Theorem B under smaller lower bound of n, in [3] the authors imposed an additional condition that the union of the sets of the double α_j points of f for $j = 1, 2, \ldots, n$ is the same as that of g.

Below we are stating elaborately their result (see [3], Theorem 1.6).

Theorem D. [3] *Let P as defined in Theorem B be a critically injective polynomial of degree n in $\mathbb{C}$ having only simple zeros whose zero set is denoted by S. Let P' have k distinct zeros and either $k \geq 3$ or $k = 2$ and P' have no simple zeros. Further, suppose that P is a* SUPM (SUPE) *and $E_f(S, 0) = E_g(S, 0)$. Also, let the union of the sets of the double α_j points of f for $j = 1, 2, \ldots, n$ is the same as that of g. Then S is a* URSM-IM (URSE-IM) *whenever $n > 2k + 9$ ($n > 2k + \frac{7}{2}$).*

The above theorem certainly gives a novel approach towards reducing the lower bound of n whenever f and g share the set S IM. However, we think that under the assumptions of Theorem D to expect that the set S is a URSM-IM (URSE-IM) is hardly tenable. Rather, one can use the terminology that the set S is a Restricted URSM-IM2 (URSE-IM2). Below we develop this idea.

Also, note that if in the statement of Theorem D we assume that the union of the sets of the simple α_j points of f for $j = 1, 2, \ldots, n$ is the same as that of g, then we simply will be in the case where f and g share the set $(S, 2)$.

Definition 1.4. *For $a \in \mathbb{C} \cup \{\infty\}$ let f and g share $(a, 0)$ and k be a positive integer. We say that f and g share the value a Restricted IM with weight k, denoted by Restricted IMk, if all the zeros of $f - a$ and $g - a$ with multiplicity exactly k coincide. If f and g share the value a Restricted IMp for all $1 \leq p \leq k$, then f and g share (a, k).*

We say f and g share a set $S \subset \mathbb{C} \cup \{\infty\}$ Restricted IM with weight k if for any a_i, $a_j \in S$ the totality of zeros of $f - a_i$ with exact multiplicity k coincides with that of $g - a_j$. Also, a set $S \subset \mathbb{C} \cup \{\infty\}$ is said to be a Restricted unique range set Ignoring Multiplicity with weight k for meromorphic (entire) functions if whenever two non-constant meromorphic (entire) functions f and g share S Restricted IM with weight k implies $f \equiv g$. We write S is Restricted URSM-IMk (Restricted URSE-IMk) in short.

The main purpose of the paper is to show that Theorem B can be improved in the context of weighted sharing of sets. In fact, in the application part of the paper we will show that our method is more effective and efficient than truncated sharing version improvement of Fujimoto's result. We also improve Theorem D by further reducing the lower bound of n. The following theorems are the main results of the paper.

Theorem 1.1. *In addition to the hypothesis of Theorem B we suppose that S is the set of zeros of P. If*

(i) $m = 2$ *and* $n > 2k + 6 - 2\Theta(\infty; f) - 2\Theta(\infty; g)\ (2k + 2)$,
(ii) $m = 1$ *and* $n > 2k + 7 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \frac{1}{2}\min\{\Theta(\infty; f), \Theta(\infty; g)\}$ $(2k + 2)$,
(iii) $m = 0$ *and* $n > 2k + 12 - 3\Theta(\infty; f) - 3\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}$ $(2k + 5)$,

then S is a URSMm (URSEm).

Theorem 1.2. *Under the same assumption of Theorem D if*

$$n > 2k + 8 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\ (2k + 3),$$

then S is a Restricted URSM-IM2 (URSE-IM2).

We now give some additional definitions which will be used in the rest of the paper.

Definition 1.5. [11] *For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)$ $(N(r, a; f \mid\geq m))$ the counting functions of those a-points of f whose multiplicities are not greater (less) than m, where each a-point is counted according to its multiplicity.*

The counting functions $\overline{N}(r, a; f \mid\leq m)(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also, the functions $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.6. [20] *Let f and g be two non-constant meromorphic functions such that f and g share $(a, 0)$. Let z_0 be an a-point of f with multiplicity p and an a-point of g with multiplicity q. Define $\overline{N}_L(r, a; f)$ to be the reduced counting function of those a-points of f and g with $p > q$, $N_E^{1)}(r, a; f)$ to be the counting function of those a-points of f and g with $p = q = 1$, and $\overline{N}_E^{(2}(r, a; f)$ to be the reduced counting function of those a-points of f and g with $p = q \geq 2$. In the same way can be defined the functions $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$, and also the functions $\overline{N}_L(r, a; f)$ and $\overline{N}_L(r, a; g)$ for $a \in \mathbb{C} \cup \{\infty\}$.*

Notice that when f and g share (a, m), $m \geq 1$, then $N_E^{1)}(r, a; f) = N(r, a; f \mid= 1)$.

Definition 1.7. [12, 13] *Let f and g share a value a IM. Define $\overline{N}_*(r, a; f, g)$ to be the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly, $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ *and* $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. We suppose that $P(z)$ is a polynomial of degree n and its derivative $P'(z)$ has distinct zeros $\beta_1, \beta_2, \ldots, \beta_k$ with respective multiplicities $q_1, q_2, \ldots, q_k$. So, $P'(z) = (z - \beta_1)^{q_1}(z - \beta_2)^{q_2} \ldots (z - \beta_k)^{q_k}$, where $q_1 + q_2 + \ldots + q_k = n - 1$.

Unless otherwise stated, F and G will stand for two non-constant meromorphic functions given by $F = P(f)$ and $G = P(g)$. Henceforth we shall denote by H the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G}\right). \tag{2.1}$$

Lemma 2.1. [15] *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum\limits_{k=0}^{n} a_k f^k}{\sum\limits_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.2. *If F and G are two non-constant meromorphic functions such that they share $(0,0)$ and $H \not\equiv 0$, then*

$$N_E^{1)}(r, 0; F \mid= 1) = N_E^{1)}(r, 0; G \mid= 1) \leq N(r, H) + S(r, f) + S(r, g).$$

Proof. By the lemma of logarithmic derivative we have

$$m(r, H) = S(r, f) + S(r, g)(:= S(r)).$$

Using Laurent expansion of H we can easily verify that each simple zero of F (and hence, of G) is a zero of H. Therefore

$$N_E^{1)}(r,0;F\mid=1)=N_E^{1)}(r,0;G\mid=1)\le N(r,0;H)$$
$$\le T(r,H)+O(1)=N(r,\infty;H)+S(r,f)+S(r,g).$$

Lemma 2.3. *Let S be the set of zeros of P. If for two non-constant meromorphic functions f and g, $E_f(S,0)=E_g(S,0)$ and $H\not\equiv 0$, then*

$$\overline{N}(r,\infty;H)\le\sum_{j=1}^{k}\{\overline{N}(r,\beta_j;f)+\overline{N}(r,\beta_j;g)\}+\overline{N}_*(r,0;F,G)$$
$$+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'),$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function of those zeros of f' which are not the zeros of $F\prod_{j=1}^{k}(f-\beta_j)$, and $\overline{N}_0(r,0;g')$ denotes the similar function corresponding to g.

Proof. Since $E_f(S,0)=E_g(S,0)$, it follows that F and G share $(0,0)$. Also, observe that $F'=(f-\beta_1)^{q_1}(f-\beta_2)^{q_2}\ldots(f-\beta_k)^{q_k}f'$. It can easily be verified that the possible poles of H can be: (i) the poles of f and g, (ii) those 0-points of F and G that have different multiplicities, (iii) the zeros of f' which are not zeros of $F\prod_{j=1}^{k}(f-\beta_j)$, (iv) the zeros of g' which are not zeros of $G\prod_{j=1}^{k}(g-\beta_j)$, (v) the β_j points ($j=1,2,\ldots,k$) of f and g.

Since H has only simple poles, the result follows. □

Lemma 2.4. [6] *Let the assumptions of Theorem B be fulfilled. Also, assume that there are two meromorphic functions f and g such that for any two constants $c_0\,(\neq 0)$ and c_1*

$$\frac{1}{P(f)}\equiv\frac{c_0}{P(g)}+c_1.$$

Then $c_1=0$ provided that $n\ge 5$.

Lemma 2.5. [14] *Let $N(r,0;f^{(k)}\mid f\neq 0)$ be the counting function of those zeros of $f^{(k)}$ which are not zeros of f, where the zeros of $f^{(k)}$ are counted according to their multiplicities. Then*

$$N(r,0;f^{(k)}\mid f\neq 0)\le k\overline{N}(r,\infty;f)+N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\ge k)+S(r,f).$$

3. Proofs of the theorems

Proof of Theorem 1.1. Since $E_f(S,m) = E_g(S,m)$, it follows that the functions F and G share $(0,m)$. We consider two possible cases: $H \not\equiv 0$ and $H \equiv 0$.

Case 1. Assume that $H \not\equiv 0$.

Subcase 1.1. Let $m \geq 1$. If $m \geq 2$, then using Lemma 2.5 we obtain

$$\begin{aligned} &\overline{N}_0(r,0;g') + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,0;F,G) \\ \leq\ & \overline{N}_0(r,0;g') + \overline{N}(r,0;G \mid\geq 2) + \overline{N}(r,0;G \mid\geq 3) \\ \leq\ & N(r,0;g' \mid g \neq 0) + S(r,g) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g). \end{aligned} \tag{3.1}$$

Hence using (3.1) and Lemmas 2.1 - 2.3, from second fundamental theorem for $\varepsilon > 0$ we get

$$\begin{aligned} (n+k-1)T(r,f) &\leq \overline{N}(r,\infty;f) + \sum_{j=1}^{k} \overline{N}(r,\beta_j;f) + N(r,0;F \mid= 1) + \\ &\overline{N}(r,0;F \mid\geq 2) - N_0(r,0;f') + S(r,f) \leq \sum_{j=1}^{k} \{2\overline{N}(r,\beta_j;f) + \overline{N}(r,\beta_j;g)\} \\ &+2\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_0(r,0;g') + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,0;F,G) + S(r,f;g) \\ &\leq 2kT(r,f) + kT(r,g) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) + \overline{N}(r,0;g) + S(r,f;g). \\ &\leq (3k+5-2\Theta(\infty;f) - 2\Theta(\infty;g) + \varepsilon)\, T(r,f;g) + S(r,f;g). \end{aligned} \tag{3.2}$$

In a similar way we can obtain

$$\begin{aligned} &(n+k-1)T(r,g) \leq \\ &(3k+5-2\Theta(\infty;f) - 2\Theta(\infty;g) + \varepsilon)\, T(r,f;g) + S(r,f;g). \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) we see that

$$(n-2k-6+2\Theta(\infty;f) + 2\Theta(\infty;g) - \varepsilon)\, T(r,f;g) \leq S(r,f;g). \tag{3.4}$$

Since $\varepsilon > 0$, the relation (3.4) leads to a contradiction.

If $m = 1$, then using Lemma 2.5, similar to (3.1) we obtain

$$\begin{aligned} &\overline{N}_0(r,0;g') + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,0;F,G) \\ \leq\ & \overline{N}_0(r,0;g') + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_L(r,0;G) + \overline{N}(r,0;F \mid\geq 3) \\ \leq\ & N(r,0;g' \mid g \neq 0) + \frac{1}{2}\sum_{j=1}^{n}\{N(r,\alpha_j;f) - \overline{N}(r,\alpha_j;f)\} \\ \leq\ & \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \frac{1}{2}\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\} + S(r,f) + S(r,g). \end{aligned} \tag{3.5}$$

So, using (3.5), Lemmas 2.2, 2.3 and proceeding as in (3.2), from second fundamental theorem for $\varepsilon > 0$ we get

$$\begin{aligned}(3.6)\qquad & (n+k-1)T(r,f)\\ \leq\ & (2k+\frac{1}{2})T(r,f)+(k+1)T(r,g)+\frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+S(r,f;g)\\ \leq\ & \left(3k+6-\frac{5}{2}\Theta(\infty;f)-2\Theta(\infty;g)+\varepsilon\right)T(r,f;g)+S(r,f;g).\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}(3.7)\qquad & (n+k-1)T(r,g)\\ \leq\ & \left(3k+6-2\Theta(\infty;f)-\frac{5}{2}\Theta(\infty;g)+\varepsilon\right)T(r,f;g)+S(r,f;g).\end{aligned}$$

Combining (3.6) and (3.7) we see that

$$\begin{aligned}(3.8)\qquad & (n-2k-7+2\Theta(\infty;f)+2\Theta(\infty;g)+\\ & +\frac{1}{2}\min\{\Theta(\infty;f),\Theta(\infty;g)\}-\varepsilon\Big)\ T(r,f;g)\leq S(r,f;g).\end{aligned}$$

Since $\varepsilon > 0$, the relation (3.8) leads to a contradiction.

Subcase 1.2. Let $m = 0$. Using Lemma 2.5 we can write

$$\begin{aligned}(3.9)\qquad & \overline{N}_0(r,0;g')+\overline{N}_E^{(2}(r,0;F)+2\overline{N}_L(r,0;G)+2\overline{N}_L(r,0;F)\\ \leq\ & \overline{N}_0(r,0;g')+\overline{N}_E^{(2}(r,0;G)+\overline{N}_L(r,0;G)+\overline{N}_L(r,0;G)+2\overline{N}_L(r,0;F)\\ \leq\ & \overline{N}_0(r,0;g')+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_L(r,0;G)+2\overline{N}_L(r,0;F)\\ \leq\ & N(r,0;g'\mid g\neq 0)+\overline{N}(r,0;G\mid\geq 2)+2\overline{N}(r,0;F\mid\geq 2)\\ \leq\ & \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\\ & +2\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\ \leq\ & 2\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\}+S(r,f)+S(r,g).\end{aligned}$$

Hence, using (3.9) and Lemmas 2.1 - 2.3, from second fundamental theorem for $\varepsilon > 0$ we get

$$(3.10)\quad (n+k-1)T(r,f)\leq\overline{N}(r,\infty;f)+\sum_{j=1}^{k}\overline{N}(r,\beta_j;f)+N_E^{1)}(r,0;F)+$$

$$\begin{aligned}(3.11)\quad & +\overline{N}_L(r,0;F)+\overline{N}_L(r,0;G)+\overline{N}_E^{(2}(r,0;F)-N_0(r,0;f')+S(r,f)\leq\\ & \leq\sum_{j=1}^{k}\{2\overline{N}(r,\beta_j;f)+\overline{N}(r,\beta_j;g)\}+2\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_E^{(2}(r,0;F)+\\ & +2\overline{N}_L(r,0;G)+2\overline{N}_L(r,0;F)+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\end{aligned}$$

$$\leq (2k+2)T(r,f) + (k+2)T(r,g) + 4\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + S(r,f) + S(r,g)$$
$$\leq (3k+11-4\Theta(\infty;f)-3\Theta(\infty;g)+\varepsilon)\,T(r,f;g) + S(r,f;g).$$

In a similar manner we can obtain

$$(3.12)\qquad (n+k-1)T(r,g)$$
$$\leq \quad (3k+11-3\Theta(\infty;f)-4\Theta(\infty;g)+\varepsilon)\,T(r,f;g) + S(r,f;g).$$

Combining (3.10) and (3.12) we see that

$$(3.13)\ (n-2k-12+3\Theta(\infty;f)+3\Theta(\infty;g)+\min\{\Theta(\infty;f),\Theta(\infty;g)\}-\varepsilon)\,T(r,f;g)$$
$$\leq \quad S(r,f;g).$$

Since $\varepsilon > 0$, the relation (3.13) leads to a contradiction.

Case 2. Now assume that $H \equiv 0$. By integration we get from (2.1) $\frac{1}{F} \equiv \frac{c_0}{G} + c_1$, where c_0 and c_1 are constants and $c_0 \neq 0$. So, using Lemma 2.4 we get

$$P(f) \equiv \frac{1}{c_0}P(g).$$

Now noting that P is a SUPM we have $f \equiv g$, implying that S is a URSMm. □

Proof of Theorem 1.2. Let $H \not\equiv 0$. From the definition of Restricted $IM2$ sharing, we observe that here $\overline{N}_L(r,0;F) \geq \overline{N}(r,0;F \mid\geq 3)$ and so (3.9) becomes

$$(3.14)\qquad \overline{N}_0(r,0;g') + \overline{N}_E^{(2}(r,0;F) + 2\overline{N}_L(r,0;G) + 2\overline{N}_L(r,0;F)$$
$$\leq \overline{N}_0(r,0;g') + \overline{N}_E^{(2}(r,0;G) + 2\overline{N}(r,0;G \mid\geq 3) + 2.\frac{1}{2}\sum_{j=1}^{n}\{N(r,\alpha_j;f) - \overline{N}(r,\alpha_j;f)\}$$
$$\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + N(r,0;g) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g)$$

Hence in view of (3.14) and Lemmas 2.1 - 2.3 for $\varepsilon > 0$ the relation (3.10) becomes

$$(3.15)\qquad (n+k-1)T(r,f) \leq$$
$$(3k+7-3\Theta(\infty;f)-2\Theta(\infty;g)+\varepsilon)\,T(r,f;g) + S(r,f;g).$$

Similarly, we get

$$(3.16)\qquad (n+k-1)T(r,g) \leq$$
$$(3k+7-2\Theta(\infty;f)-3\Theta(\infty;g)+\varepsilon)\,T(r,f;g) + S(r,f;g).$$

From (3.15) we obtain

$$(3.17)\ (n-2k-8+2\Theta(\infty;f)+2\Theta(\infty;g)+\min\{\Theta(\infty;f),\Theta(\infty;g)\}-\varepsilon)\,T(r,f;g)$$
$$\leq \quad S(r,f;g).$$

Since $\varepsilon > 0$, the relation (3.17) leads to a contradiction.
The rest of the proof is similar to that of Theorem 1.1, and so the details are omitted. Theorem 1.2 is proved. □

Remark. To the best of our knowledge, so far two types of critically injective Strong Uniqueness Polynomials have been invented such that their zero sets are Unique Range Sets.

The first type of URS is considered by Frank and Reinders in [7] which is the zero set of the polynomial:

$$P_{FR}(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c \quad (c \neq 0, 1).$$

From the results of [7], it is clear that here $k = 2$ and P_{FR} is a UPM if $n > 5$. Also, from [[7], p. 191, Case 2], it follows that whenever $n > 7$, $P_{FR}(f) \equiv cP_{FR}(g)$ implies $P_{FR}(f) \equiv P_{FR}(g)$. Hence, if we denote the zero set of $P_{FR}(z)$ by S_{FR}, then S_{FR} becomes a URSMm (URSEm) for the cases $m = 2$, $m = 1$ and $m = 0$, when it contains 11, 12 and 17 elements (respectively 7, 7 and 10 elements).

The second type of URS is demonstrated by Yi in [19], which is the zero set of the polynomial:

$$P_Y(z) = z^n + az^{n-r} + b,$$

where n and r are two positive integers having no common factors, $r \geq 2$ and a and b are chosen so that P has n distinct zeros. Here $k = r + 1$ and P_Y is a UPM if $n > 6$ (see [19], p.79 Case 3, last part). Also, from [[19], p.79, Case 3, first part], it follows that whenever $n > 2r + 4$, $P_Y(f) \equiv cP_Y(g)$ implies $P_Y(f) \equiv P_Y(g)$. Hence, if we denote the zero set of $P_Y(z)$ by S_Y, then S_Y becomes a URSMm (URSEm) for the cases $m = 2$, $m = 1$ and $m = 0$, when it contains $2r+9$, $2r+10$ and $2r+15$ elements (respectively $2r+5$, $2r+5$ and $2r+8$ elements).

4. Applications

The following theorem was proved in the application part of [3].

Theorem E. *In addition to the hypothesis of Theorem B, we suppose that m is a positive integer or ∞. Let S be the set of zeros of P. If*

(i) $m \geq 3$ *or* ∞ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 3 + k - \frac{n}{2}$,

(ii) $m = 2$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{28+8k-4n}{9}$,

(iii) $m = 1$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{20+4k-2n}{6}$,

then S is a URSM$_{m)}$.

Notice that the proof of Theorem E given in [3] is not flawless. For example, it is not difficult to observe that in the proof of Theorem 6.1 of [3], the relation on the third line after formula (6.1) (see [3], p. 641), has been obtained on the basis of the assumption that

$$T(r) \leq 2T(r,f;g) \Rightarrow (2k-n+2)T(r)(=T(r,f)+T(r,g)) \leq 2(2k-n+2)T(r,f;g),$$

which is true only when $2k+2 \geq n$. Taking into account that for all three cases $m \geq 3$, $m=2$ and $m=1$, we have $\Theta(\infty;f)+\Theta(\infty;g) > 2$, we conclude that Theorem 6.1 in [3] is not correct.

The next result is a corrected version of Theorem E.

Theorem 4.1. *In addition to the hypothesis of* Theorem B, *we suppose that m is a positive integer or ∞. Let S be the set of zeros of P. If*

(i) $m \geq 3$ *or* ∞ *and* $\min\{\Theta(\infty;f),\Theta(\infty;g)\} > \frac{6+2k-n}{4}$,
(ii) $m=2$ *and* $\min\{\Theta(\infty;f),\Theta(\infty;g)\} > \frac{14+4k-2n}{9}$,
(iii) $m=1$ *and* $\min\{\Theta(\infty;f),\Theta(\infty;g)\} > \frac{10+2k-n}{6}$,

then S is a $\mathrm{URSM}_{m)}$.

It follows from Theorem 4.1 that there exists an $\mathrm{URSM}_{3)}$ ($\mathrm{URSM}_{2)}$), say S_{FR}, consisting of 7 elements with the assumption

$$\min\{\Theta(\infty;f),\Theta(\infty;g)\} > \frac{3}{4} \quad (\min\{\Theta(\infty;f),\Theta(\infty;g)\} > \frac{8}{9}),$$

and there exists an $\mathrm{URSM}_{1)}$, say S_{FR}, consisting of 9 elements with the assumption $\min\{\Theta(\infty;f),\Theta(\infty;g)\} > \frac{5}{6}$.

Notice that Theorem 6.2 in [3] also is not correct, because the proof contains the same inaccuracy mentioned above.

Using the arguments of the proof of Theorem 1.1, one can prove the following result.

Theorem 4.2. *In addition to the hypothesis of Theorem B, we suppose that S is the set of zeros of P. If*

(i) $m=2$ *and* $\Theta(\infty;f)+\Theta(\infty;g) > 3+k-\frac{n}{2}$,
(ii) $m=1$ *and* $\Theta(\infty;f)+\Theta(\infty;g)+\frac{1}{4}\min\{\Theta(\infty;f),\Theta(\infty;g)\} > \frac{7}{2}+k-\frac{n}{2}$,
(iii) $m=0$ *and* $\Theta(\infty;f)+\Theta(\infty;g)+\frac{1}{3}\min\{\Theta(\infty;f),\Theta(\infty;g)\} > 4+\frac{2k}{3}-\frac{n}{3}$,

then S is a $\mathrm{URSM}_{m)}$.

It follows from Theorem 4.2 that there exists an URSM2 (URSM1), say S_{FR}, consisting of 7 elements with the assumption

$$\Theta(\infty; f) + \Theta(\infty; g) > \frac{3}{2} \quad (\Theta(\infty; f) + \Theta(\infty; g) + \frac{1}{4}\min\{\Theta(\infty; f), \Theta(\infty; g)\} > 2)$$

and there exists an UESM-IM, say S_{FR}, consisting of 10 elements with the assumption $\Theta(\infty; f) + \Theta(\infty; g) + \frac{1}{3}\min\{\Theta(\infty; f), \Theta(\infty; g)\} > 2$.

Note that the last result corresponding to URSM-IM improves the result of S. Bartels [5], and at the same time, it improves and rectifies Theorem 6.2 of [3].

Meanwhile, we note that the above discussion already rectifies the *Concluding Remark* by Bai, Han and Chen (see [3], p. 642). Actually, the main lacuna was started in the paper by Y. Xu [17], where the author first started to reduce the cardinality of S_{FR} under some restriction on the deficiency conditions. Analyzing the proof of Theorem 1 from [17] (see pp. 1492-1493), we easily find that the relation (3.12) in the proof can be deduced from (3.11) only when $\min\{\Theta(\infty; f), \Theta(\infty; g)\} > \frac{3}{4}$. In the *Application Part* of [3] the lacuna has just been carried forward.

5. Concluding Remark and an Open Question

Suppose that the polynomial $P(z)$ is defined by (see [1]):

$$P(z) = az^n - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2, \tag{5.1}$$

where $n \geq 3$ is an integer and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. We have from (5.1) that

$$P'(z) = \frac{n}{z}[az^n - 2(n-1)z^2 + 2(n-2)bz]. \tag{5.2}$$

Note that $P'(0) \neq 0$, and so in view of (5.2), $P'(z) = 0$ implies

$$az^n - 2(n-1)z^2 + 2(n-2)bz = 0.$$

Now at each root of $P'(z) = 0$ we have

$$\begin{aligned} P(z) &= 2(n-1)z^2 - 2(n-2)bz - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2 \\ &= -(n-1)(n-2)(z-b)^2. \end{aligned}$$

So at a root of $P'(z) = 0$, $P(z)$ will be zero if $P'(b) = 0$. But $P'(b) = nb(ab^{n-2} - 2) \neq 0$, which implies that a zero of $P'(z)$ is not a zero of $P(z)$. In other words, each zero of $P(z)$ is simple. Also, we have $P'' = n(n-1)az^{n-2} - 2n(n-1)$. So $P''(z) = 0$

implies $az^{n-2} = 2$. Now at a root of $P''(z) = 0$ we get

$$\begin{aligned} P'(z) &= nzaz^{n-2} - 2n(n-1)z + 2n(n-2)b \\ &= 2nz - 2n(n-1)z + 2n(n-2)b \\ &= 2n\{z - (n-1)z + (n-2)b\} \\ &= 2n\{-(n-2)z + (n-2)b\} = -2n(n-2)(z-b). \end{aligned}$$

We see that $P''(b) \neq 0$, and hence a zero of $P''(z)$ is not a zero of $P'(z)$. This implies that each zero of $P'(z)$ is simple. Therefore, $P'(z)$ has $k = n-1$ zeros. With the help of Theorem 1 of [2], we conclude that $P(z)$ is a SUPM when $n \geq 6$. Following the procedure as adopted by T. C. Alzahary in [1], one can easily see that $P(z)$ produces a URSM when $n \geq 11$.

If α, β are two distinct zeros of $P'(z)$, then $P(\alpha) \neq P(\beta)$ implies $(\alpha + \beta - 2b)(\alpha - \beta) \neq 0$, and it is satisfied only when $\alpha + \beta \neq 2b$. So, if $\alpha + \beta \neq 2b$, then $P(z)$ is critically injective.

But till date this fact can not be ascertained. Hence there is a doubt about whether $P(z)$ is critically injective polynomial or not, though its zero set is producing a URSM. So there remain an open question:-

What is the general criterion for a SUPM of degree n having n zeros of the form $P(z) = a_n z^n + \sum_{j=0}^{m} a_j z^j$, where $n > m$ and none of a_j $(j = 0, 1, \ldots, m)$ vanish, the zero set of which is a URSM ?

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