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REGULARITY IN ORLICZ SPACES FOR NON-DIVERGENCE DEGENERATE ELLIPTIC EQUATIONS ON HOMOGENEOUS GROUPS

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Abstract. Let G be a homogeneous group, and let $X_1, X_2, \cdots, X_{p_0}$ be left-invariant real vector fields on G that are homogeneous of degree one with respect to the dilation group of G and satisfy Hormander's condition. We establish a regularity result in the Orlicz spaces for the following equation: $Lu(x) = \sum_{j=1}^{p_0} a_{ij}(x)X_iX_ju(x) = f(x)$, where $a_{ij}(x)$ are real valued, bounded measurable functions defined on G, satisfying the uniform ellipticity condition, and belonging to the space $VMO(G)$ (Vanishing Mean Oscillation) with respect to the subelliptic metric induced by the vector fields $X_1, X_2, \cdots, X_{p_0}$.

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1. Introduction and main results

Let[1] G be a homogeneous group, and let $X_1, X_2, \cdots, X_{p_0}$ form a system of C^∞ real vector fields defined on $\mathbb{R}^N$ ($p_0 \le N$), which are left invariant with respect to the left translations on G and are homogeneous of degree one with respect to the dilation group of G. Also, assume that they satisfy the finite rank condition at every point of $\mathbb{R}^N$, that is,

$$rank\mathcal{L}(X_1, \dots, X_{p_0})(x) = N, \ \ x \in \mathbb{R}^N,$$

where $\mathcal{L}(X_1, \dots, X_{p_0})$ denotes the Lie algebra generated by the fields $X_1, \dots, X_{p_0}$.

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Our aim is to establish regularity results in the Orlicz spaces for solutions of the following equation

$$Lu(x) = \sum_{i,j=1}^{p_0} a_{ij}(x) X_i X_j u(x) = f(x), \quad x \in G, \tag{1.1}$$

where $p_0 < N$, $A = (a_{ij}(x))$ are real valued, bounded measurable functions defined in G, satisfying very weak regularity conditions, namely, they belong to the class $VMO(G)$ defined with respect to the homogeneous distance. Also, the matrix $(a_{ij}(x))$ is assumed to satisfy the condition:

$$a_{ij} = a_{ji}, \quad \nu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{p_0} a_{ij}(x)\xi_i\xi_j \leq \nu|\xi|^2,$$

for every $i, j = 1, \ldots, p_0$, $\xi \in \mathbb{R}^{p_0}$, $\nu > 0$ and a.e. $x \in G$.

In 1967, Hörmander [12] investigated the operator $L_1 = \sum_{i=1}^{m} X_i^2 + X_0$, and pointed out that the finite rank condition implies the hypoellipticity of L_1. In [8], Folland proved that homogeneous hypoelliptic operators on nilpotent groups have homogeneous fundamental solutions. Later, Bramanti and Brandolini [4] have obtained L^p estimates for the operator L on homogeneous groups. The Orlicz spaces originally introduced by Orlicz [17] as generalizations of L^p spaces in Euclidean groups, have been extensively studied in the literature (see [1, 13, 14, 22] and references therein). The theory of Orlicz spaces plays a major role in a wide range of areas (see [18]). A number of papers are devoted to regularity theory of elliptic equations in the Orlicz spaces (see [2, 13, 21]. Criteria of weighted inequalities in Orlicz classes for maximal functions defined on homogeneous type spaces were obtained by Gogatishvili and Kokilashvili in [10].

Definition 1.1. *For a measurable function $f \in L^1_{loc}(G)$, denote*

$$\eta_f(\mathcal{R}) = \sup_{B_r \subset G} \frac{1}{|B_r|} \int_{B_r} |f(y) - f_{B_r}|\, dy, \quad \mathcal{R} > 0, \tag{1.2}$$

where f_{B_r} is the average of f over B_r. A function f is said to belong to the class $BMO(G)$ (Bounded Mean Oscillation on G), if $\|f\|_* := \sup_{\mathcal{R}} \eta_f(\mathcal{R}) < +\infty$, while we say that $f \in VMO(G)$ (Vanishing Mean Oscillation), if $\lim_{\mathcal{R}\to 0} \eta_f(\mathcal{R}) = 0$.

The class of all functions $\phi : [0, \infty) \to [0, \infty)$ which are increasing and convex we denote by Φ.

Definition 1.2. *A function $\phi \in \Phi$ is said to be a Young function if*

$$\lim_{t\to 0^+} \frac{\phi(t)}{t} = \lim_{t\to+\infty} \frac{t}{\phi(t)} = 0.$$

Definition 1.3. *A Young function $\phi \in \Phi$ is said to satisfy the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if there exists a number $a > 1$ such that $\phi(t) \leq \frac{\phi(at)}{2a}$ for every $t > 0$.*

Definition 1.4. *A Young function $\phi \in \Phi$ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if there exists a positive constant K such that for every $t > 0$*

$$\phi(2t) \leq K\phi(t). \tag{1.3}$$

Lemma 1.1 ([6]). *Let ϕ be a Young function. Then $\phi \in \nabla_2 \cap \Delta_2$ if and only if there exist constants $A_2 \geq A_1 > 0$ and $\alpha_1 \geq \alpha_2 > 1$ such that for any $0 < s \leq t$*

$$A_1 \left(\frac{s}{t}\right)^{\alpha_1} \leq \frac{\phi(s)}{\phi(t)} \leq A_2 \left(\frac{s}{t}\right)^{\alpha_2}. \tag{1.4}$$

Moreover, the condition (1.2) implies that for $0 < \theta_1 \leq 1 \leq \theta_2 < \infty$

$$\phi(\theta_1 t) \leq A_2 \theta_1^{\alpha_2} \phi(t) \text{ and } \phi(\theta_2 t) \leq A_1^{-1} \theta_2^{\alpha_1} \phi(t). \tag{1.5}$$

A simple example of functions $\phi(t)$ satisfying the $\Delta_2 \cap \nabla_2$ condition is the power function $\phi(t) = t^p$ with $p > 1$. Moreover, we remark that the $\Delta_2 \cap \nabla_2$ condition makes the function grow moderately. For instance, $\phi(t) = |t|^p(1 + |\log |t||) \in \Delta_2 \cap \nabla_2$ for $p > 1$.

Definition 1.5. *Let ϕ be a Young function. The Orlicz class $K^\phi(G)$ is defined to be the set of measurable functions g satisfying the condition:*

$$\int_G \phi(|g|)dx < \infty,$$

and the Orlicz space $L^\phi(G)$ is defined to be the linear hull of $K^\phi(G)$.

In this class we consider the following analog of the Luxemburg norm:

$$\|u\|_\phi = \inf\{k > 0 : \int_G \phi\left(\frac{|u(x)|}{k}\right) dx \leq 1\}. \tag{1.6}$$

Observe that, in general, $K^\phi \subset L^\phi$. However, if ϕ satisfies the global Δ_2 condition, then we have $K^\phi = L^\phi$. Moreover, if $g \in L^\phi(G)$, then $\int_G \phi(|g|)dx$ can be written in the form (see [21]):

$$\int_G \phi(|g|)dx = \int_0^\infty |\{x \in G : |g| > \lambda\}| d[\phi(\lambda)]. \tag{1.7}$$

Lemma 1.2 ([6]). *Let U be a bounded domain in G and $\phi \in \nabla_2 \cap \Delta_2$. Then*

$$L^{\alpha_1}(U) \subset L^\phi(U) \subset L^{\alpha_2}(U) \subset L^1(U),$$

where $\alpha_1 \geq \alpha_2 > 1$ are as in Lemma 1.1.

For $p \in [1, \infty]$ we define

$$\|Du\|_{L^p(G)} = \sum_{j=1}^{p_0} \|X_j u\|_{L^p(G)}, \quad \|D^2 u\|_{L^p(G)} = \sum_{i,j=1}^{p_0} \|X_i X_j u\|_{L^p(G)}.$$

Similarly can be defined $\|D^m u\|_\phi$ for $m = 1, 2$.

Definition 1.6. *Let ϕ be a Young function. The Orlicz-Sobolev space $S^{2,\phi}(G)$ is defined to be the set of those functions $u \in L^\phi(G)$ whose derivatives $D^h u$ also belong to $L^\phi(G)$ for all $0 < h \leq 2$ such that $\|u\|_{S^{2,\phi}(G)} = \sum_{h=0}^{2} \|D^h u\|_\phi$ is finite.*

As in the case of ordinary Sobolev spaces, $S_0^{2,\phi}(G)$ is defined to be the closure of $C_0^\infty(G)$ in $S^{2,\phi}(G)$.

In this paper, by using the same techniques as in [20, 21], an approximation argument and the reverse Hölder inequality, we obtain Orlicz estimates for solutions of equation (1.1). The main result of the paper is the following theorem.

Theorem 1.1. *Assume that $\phi \in \Phi$ is a Young function and $\phi \in \Delta_2 \cap \nabla_2$. If $f \in L^\phi(G)$ and $u \in S^{2,\phi}(G)$ is a solution of equation*

$$Lu - \mu u = f \text{ in } G,$$

then there exist positive constants μ_0 and c such that for any $\mu > \mu_0$, we have

$$\int_G \phi(|D^2 u|) + \phi(|Du|) + \phi(|u|)dx \leq c \int_G \phi(|f|)dx, \tag{1.8}$$

where the constant c depends only on G, ν, μ_0 and ϕ.

The paper is organized as follows. In Section 2, we introduce the notion of homogenous groups. In Section 3 we derive several lemmas, which are used to prove the main result. Section 4 is devoted to the proof of the main result - Theorem 1.1.

2. Homogenous groups

Given a pair of smooth mappings:

$$[(x, y) \mapsto x \circ y] : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N; \quad [x \mapsto x^{-1}] : \mathbb{R}^N \to \mathbb{R}^N.$$

The space $\mathbb{R}^N$ together with these mappings forms a group with the identity being the origin. Next, assume that there exist $0 < \omega_1 \leq \omega_2 \leq \ldots \leq \omega_N$, such that the dilation $D(\varrho) : (x_1, \ldots, x_N) \mapsto (\varrho^{\omega_1} x_1, \ldots \varrho^{\omega_N} x_N)$ is a group automorphism, for all $\varrho > 0$. The space $\mathbb{R}^N$ with this structure is called a homogeneous group and is denoted by G. For more details on the subject we refer to [4, 19].

A homogeneous norm $\|\cdot\|$ on G is defined as follows. For any $x \in G\backslash\{0\}$ we set

$$\|x\| = \rho \Leftrightarrow |D(1/\rho)x| = 1,$$

where $|\cdot|$ denotes the Euclidean norm, and also let $\|0\| = 0$. Then we have

(i) $\|D(\varrho)x\| = \varrho\|x\|$ for every $x \in G$, $\varrho > 0$;

(ii) there exist constants $c_1, c_2 \geq 1$ such that for every $x, y \in G$,

$$\|x^{-1}\| \leq c_1\|x\|;$$

$$\|x \circ y\| \leq c_2(\|x\| + \|y\|).$$

In view of the above properties, it is natural to define the quasidistance $d(\cdot,\cdot)$ by

$$d(x, y) = \|y^{-1} \circ x\|. \tag{2.1}$$

We denote the ball with respect to d by

$$B_r(x) = \{y \in G : d(x, y) < r\}. \tag{2.2}$$

Note that $B(0, r) = D(r)B(0, 1)$ and

$$|B(x, r)| = r^Q|B(0, 1)|, \tag{2.3}$$

where $x \in G$, $r > 0$, and

$$Q = \omega_1 + \cdots + \omega_N, \tag{2.4}$$

which is called the homogeneous dimension of G. By (2.3), the doubling is valid, that is, $|B(x, 2r)| \leq c|B(x, r)|$, $x \in G$, $r > 0$, and therefore (G, dx, d) is a space of homogenous type.

In what follows, we will define another homogenous group S, whose underlying manifold is $\mathbb{R}^{N+1}$, endowed with the composition law

$$(x, t) \odot (y, \tau) = (x \circ y, t + \tau), \quad (x, t)^{-1} = (x^{-1}, -t)$$

for any $(x, t), (y, \tau) \in \mathbb{R}^{N+1}$. The dilation on $\mathbb{R}^{N+1}$ is defined by $\mathcal{D}(\varrho) : (x, t) \mapsto (D(\varrho), \varrho t)$ for all $\varrho > 0$.

Example 2.1. *Consider the Heisenberg group $G(\mathbb{R}^3, \circ, D(\lambda))$, where*

$$(x_1, y_1, t_1) \circ (x_1, y_1, t_1) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_2y_1 - x_1y_2)),$$

$$D(\lambda)(x, y, t) = (\lambda x, \lambda y, \lambda^2 t),$$

$$X_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t};\ X_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t};\ [X_1, X_2] = -4\frac{\partial}{\partial t}.$$

It is easy to check that X_1, X_2 are left invariant with respect to the left translations on G and are homogeneous of degree one with respect to the dilation group of G. Moreover, they satisfy the finite rank condition at every point of $\mathbb{R}^3$, that is,

$$rank\mathcal{L}(X_1, X_2)(x) = 3, \; x \in \mathbb{R}^3.$$

3. Some lemmas

For convenience, in this section we assume that $u \in C_0^\infty(B_{R_0})$ with some constant $R_0 > 0$ is a solution of equation (1.1). Let $p = (1+\alpha_2)/2 > 1$. In fact, in the subsequent proof we can choose any constant p with $1 < p < \alpha_2$. Now we define

$$\lambda_0^p = \int_G |D^2u|^p dx + M^p \int_G |f|^p dx,$$

where $M > 1$ is a large enough constant to be determined later. For any $\lambda > 0$ we set

$$u_\lambda = u/(\lambda_0\lambda), \; f_\lambda = f/(\lambda_0\lambda), \tag{3.1}$$

and observe that u_λ still will be a solution of equation (1.1) with f_λ instead of f. Next, for any ball B in G we define

$$J_\lambda[B] = \frac{1}{|B|}\left(\int_B |D^2u_\lambda|^p dx + M^p \int_B |f_\lambda|^p dx\right)$$

and $E_\lambda(1) = \{x \in G : |D^2u_\lambda| > 1\}$. Since $|D^2u_\lambda| \leq 1$ for $x \in G\backslash E_\lambda(1)$, we focus on the level set $E_\lambda(1)$.

By using methods similar to that of applied in [21], we can prove the following three lemmas.

Lemma 3.1. *For any $\lambda > 0$ there exists a family of disjoint balls $\{B_{\rho_i}(x_i)\}_{i\geq 1}$ with $x_i \in E_\lambda(1)$ and $\rho_i = \rho(x_i, \lambda) > 0$ such that*

$$J_\lambda[B_{\rho_i}(x_i)] = 1, \; J_\lambda[B_\rho(x_i)] < 1 \text{ for any } \rho > \rho_i;$$

and

$$E_\lambda(1) \subset \bigcup_{i\geq 1} B_{5\rho_i}(x_i) \cup \text{negligible set}.$$

Lemma 3.2. *Under the conditions of Lemma 3.1 we have*

$$\begin{aligned} |B_{\rho_i}(x_i)| \;&\leq\; \frac{2^{p-1}}{2^{p-1}-1}\Big(\int_{\{x\in B_{\rho_i}(x_i):|D^2u_\lambda|>1/2\}} |D^2u_\lambda|^p dx \\ &+\; M^p \int_{\{x\in B_{\rho_i}(x_i):|f_\lambda|>1/(2M)\}} |f_\lambda|^p dx\Big). \end{aligned}$$

Lemma 3.3. *If $\phi \in \Phi$ satisfies the global $\Delta_2 \cap \nabla_2$ condition, then for any $b_1, b_2 > 0$ we have*

$$\int_0^\infty \frac{1}{\mu^p}\left(\int_{\{x\in G:|g|>b_1\mu\}} |g|^p dx\right) d[\phi(b_2\mu)] \le C(b_1, b_2, \phi)\int_G \phi(|g|)dx.$$

Lemma 3.4 ([9]). *Let $g \in L^q(B_{2R})$ for some $R > 0$ and $q > 1$, and let $f \in L^r(B_{2R})$ for $r > q$. Assume that the following estimate holds*

$$\frac{1}{|B_1|}\int_{B_1} g^q dx \le c\left(\frac{1}{|B_2|}\int_{B_2} g dx\right)^q + \frac{1}{|B_2|}\int_{B_2} f^q dx + \theta\frac{1}{|B_2|}\int_{B_2} g^q dx,$$

where $c > 1$ and $0 \le \theta < 1$. Then there exist $C = C(G, c, q, r, \theta)$ and $\varepsilon = \varepsilon(G, c, q, r, \theta)$ such that $g \in L^p(B_1)$ for $p \in [q, q+\varepsilon)$ and

$$\left(\frac{1}{|B_1|}\int_{B_1} g^p dx\right)^{1/p} \le C\left\{\left(\frac{1}{|B_2|}\int_{B_2} g^q dx\right)^{1/q} + \left(\frac{1}{|B_2|}\int_{B_2} f^p dx\right)^{1/p}\right\}.$$

Lemma 3.5 ([4]). *Let Ω be a bounded domain in G and $\Omega' \subset\subset \Omega$. If $u \in S^{2,p}(\Omega)$ and $Lu \in S^{k,s}(\Omega)$ for some positive integer k, $1 < p < \infty$ and $s > Q/2$, then*

$$\|u\|_{\Lambda^{k,\alpha}(\Omega')} \le c\{\|Lu\|_{S^{k,r}(\Omega)} + \|u\|_{L^p(\Omega)}\},$$

where $r = \max(p, s)$, $\alpha \in (0, 1)$, c is a positive constant and Q is as in (2.4).

Lemma 3.6 ([7]). *Let $p \ge 1$ and $u \in S^{2,p}(B_r)$, and let $\mathcal{P}_2$ be the class of polynomials of homogeneous degree less that 2. Then there exists a polynomial $P \in \mathcal{P}_2$ such that*

$$\left(\frac{1}{|B_r|}\int_{B_r} |(u-P)(x)|^q dx\right)^{1/q} \le cr^2\left(\frac{1}{|B_r|}\int_{B_r} |D^2u(x)|^p dx\right)^{1/p} \tag{3.2}$$

for all $1 \le p < \frac{Q}{2}$ and $q = \frac{pQ}{Q-2p}$, where Q is as in (2.4) and c is a constant independent of B_r and u.

Lemma 3.7. *For any $\varepsilon > 0$ there exists a small enough number $\delta = \delta(\varepsilon) \in (0, 1)$ such that if $u \in C_0^\infty(B_{R_0})$ is a solution of equation (1.1) in G, and*

$$\frac{1}{|B_4|}\int_{B_4} |A - \bar{A}_{B_4}|dx \le \delta, \tag{3.3}$$

$$\frac{1}{|B_4|}\int_{B_4} |D^2u|^p dx \le 1, \quad \frac{1}{|B_4|}\int_{B_4} |f|^p dx \le \delta^p,$$

then there exists $N_1 > 1$ such that

$$\int_{B_2} |D^2(u-v)|^p dx \le \varepsilon \tag{3.4}$$

and

$$\sup_{B_1} |D^2 v| \le N_1, \tag{3.5}$$

where v is a solution of equation $\sum_{i,j=1}^{p_0}(a_{ij})_{B_4}X_iX_jv=0$ in B_4.

Proof. We first consider the following Dirichlet problem

(3.6)
$$\begin{cases} \sum_{i,j=1}^{p_0}(a_{ij})_{B_4}X_iX_jw(x)=-f+\sum_{i,j=1}^{p_0}(a_{ij}(x)-(a_{ij})_{B_4})X_iX_ju, & \text{if } x\in B_4,\\ w(x)=0, & \text{if } x\in\partial B_4. \end{cases}$$

In the paper [3], J. Bony proved that there exists a solution for the above problem. Then $v=u+w$ satisfies

$$\begin{cases} \sum_{i,j=1}^{p_0}(a_{ij})_{B_4}X_iX_jv(x)=0, & \text{if } x\in B_4,\\ v(x)=u(x), & \text{if } x\in\partial B_4. \end{cases} \tag{3.7}$$

Applying the L^p estimates given in [4] to (3.6), we can write

$$\begin{aligned} \int_{B_2}|D^2w|^pdx &\le \int_{B_4}\left|-f+\sum_{i,j=1}^{p_0}(a_{ij}(x)-(a_{ij})_{B_2})X_iX_ju\right|^pdx\\ &\le c\left(\int_{B_4}|f|^pdx+\sum_{i,j=1}^{p_0}\int_{B_4}|(a_{ij}(x)-(a_{ij})_{B_2})X_iX_ju|^pdx\right)\\ &\le c\left(\delta^p+\int_{B_4}|D^2u|^pdx\right)\le c. \end{aligned} \tag{3.8}$$

Let $P\in\mathcal{P}_2$. By using the local L^p estimates given in [4], we obtain

$$\begin{aligned} \frac{1}{|B_2|}\int_{B_2}|D^2u|^pdx &\le c\left(\frac{1}{|B_4|}\int_{B_4}|u-P|^pdx+\frac{1}{|B_4|}\int_{B_4}|f|^p\right)\\ &\le c\frac{1}{|B_4|}\int_{B_4}|u-P|^pdx+c\delta^p. \end{aligned} \tag{3.9}$$

By Lemma 3.6, for $1<p\le\frac{Q}{Q-2}$, we get

$$\left(\frac{1}{|B_4|}\int_{B_4}|u-P|^pdx\right)^{1/p}\le c\frac{1}{|B_4|}\int_{B_4}|D^2u|dx, \tag{3.10}$$

while for $p>\frac{Q}{Q-2}$, we have

$$\left(\frac{1}{|B_4|}\int_{B_4}|u-P|^pdx\right)^{1/p}\le c\left(\frac{1}{|B_4|}\int_{B_4}|D^2u|^{\frac{pQ}{Q+2p}}dx\right)^{\frac{Q+2p}{pQ}}. \tag{3.11}$$

Hence we obtain the weak reverse Hölder inequality

$$\left(\frac{1}{|B_2|}\int_{B_2}|D^2u|^pdx\right)^{1/p}\le c\left(\frac{1}{|B_4|}\int_{B_4}|D^2u|^{\frac{pQ}{Q+2p}}dx\right)^{\frac{Q+2p}{pQ}}. \tag{3.12}$$

In view of (3.9)-(3.12) and Lemma 3.4 we conclude that there exist positive constants ε_0 and C such that

$$\left(\frac{1}{|B_2|}\int_{B_2}|D^2u|^{p+\varepsilon_0}dx\right)^{\frac{1}{p+\varepsilon_0}}\le c\left(\frac{1}{|B_4|}\int_{B_4}|D^2u|^pdx\right)^{1/p}+c\delta^p\le C. \tag{3.13}$$

Next, it follows from (1.2) and (3.3) that

$$\begin{aligned}&\left(\int_{B_2} |a_{ij}(x) - (a_{ij})_{B_2}|^{\frac{p(p+\varepsilon_0)}{\varepsilon_0}} dx\right)^{\frac{\varepsilon_0}{p+\varepsilon_0}}\\ &\le (2\nu)^{\frac{p(p+\varepsilon_0)-\varepsilon_0}{p+\varepsilon_0}} \left(\int_{B_2} |a_{ij}(x) - (a_{ij})_{B_2}|dx\right)^{\frac{\varepsilon_0}{p+\varepsilon_0}} \le \delta^{\frac{\varepsilon_0}{p+\varepsilon_0}}.\end{aligned} \tag{3.14}$$

Hence by (3.8) and (3.14) we can write

$$\int_{B_2} |D^2 w|^p dx \le c\left(\int_{B_4} |f|^p dx + \sum_{i,j=1}^{m_1} \int_{B_4} |(a_{ij}(x) - (a_{ij})_{B_4})X_iX_ju|^p dx\right) \le$$

$$\le c\left\{\delta^p + \left(\sum_{i,j=1}^{m_1} \int_{B_4} |a_{ij}(x) - (a_{ij})_{B_4}|^{\frac{p(p+\varepsilon_0)}{\varepsilon_0}} dx\right)^{\frac{\varepsilon_0}{p+\varepsilon_0}} \left(\int_{B_4} |D^2u|^{p+\varepsilon_0} dx\right)^{\frac{p}{p+\varepsilon_0}}\right\}$$

$$\le c\left(\delta^p + \delta^{\frac{\varepsilon_0}{p+\varepsilon_0}}\right),$$

which implies that (3.4) holds by choosing $c\left(\delta^p + \delta^{\frac{\varepsilon_0}{p+\varepsilon_0}}\right) < \varepsilon$.

Next, we show that (3.5) is valid. Indeed, let $P \in \mathcal{P}_2$ be chosen so that (3.2) holds for $\vartheta = v - P$. Then ϑ satisfies the equation $\sum_{i,j=1}^{m_1}(a_{ij})_{B_4} X_iX_j v = 0$ in B_4. Note that $\vartheta \in C^\infty(B_4)$, and using Lemma 3.5 for $k = 2$ and $B_1 = \Omega' \subset\subset \Omega = B_2$, we obtain

$$\|\vartheta\|_{\Lambda^{2,\alpha}(B_1)} \le c\|\vartheta\|_{L^p(B_2)}, \tag{3.15}$$

By (3.15) and Lemma 3.6, we get

$$\begin{aligned}\|D^2 v\|_{L^\infty(B_1)} &\le \|\vartheta\|_{\Lambda^{2,\alpha}(B_1)} \le c\|\vartheta\|_{L^p(B_2)}\\ &\le c\left(\frac{1}{|B_2|}\int_{B_2} |D^2 v(x)|^p dx\right)^{1/p}\end{aligned} \tag{3.16}$$

with a constant c independent of v. Finally, it follows from (3.4), (3.16) that (3.5) holds, and N_1 is independent of v. □

By applying the scaling method on homogenous groups, from Lemma 3.7 we can deduce the following result.

Lemma 3.8. *For any $\varepsilon > 0$ there exists a small enough number $\delta = \delta(\varepsilon) \in (0, 1)$ such that if $u \in C_0^\infty(B_{R_0})$ is a solution of equation (1.1) in G, and*

$$\frac{1}{|B_{20\rho_i}(x_i)|}\int_{B_{20\rho_i}(x_i)} |A - \bar{A}_{B_{20\rho_i}(x_i)}| dx \le \delta, \tag{3.17}$$

$$\frac{1}{|B_{20\rho_i}(x_i)|}\int_{B_{20\rho_i}(x_i)} |D^2 u_\lambda^i|^p dx \le 1, \quad \frac{1}{|B_{20\rho_i}(x_i)|}\int_{B_{20\rho_i}(x_i)} |f_\lambda^i|^p dx \le \delta^p. \tag{3.18}$$

then there exists $N_1 > 1$ such that

$$\int_{B_{10\rho_i}(x_i)} |D^2(u_\lambda^i - v_\lambda^i)|^p dx \le \varepsilon, \quad \sup_{B_{5\rho_i}(x_i)} |D^2 v_\lambda^i| \le N_1, \tag{3.19}$$

where $v_\lambda^i \in S^{2,p}(B_{20\rho_i}(x_i))$ is a solution of equation $\sum_{i,j=1}^{m}(a_{ij})_{B_{20\rho_i}} X_i X_j v = 0$ in $B_{20\rho_i}(x_i)$.

Proof. Denoting

$$u_\lambda^i(x) = \frac{16}{(20\rho_i)^2} u_\lambda\left(D\left(\frac{20\rho_i}{4}\right)x\right), \quad f_\lambda^i(x) = f_\lambda\left(D\left(\frac{20\rho_i}{4}\right)x\right), \quad A^i(x) = A\left(D\left(\frac{20\rho_i}{4}\right)\right),$$

we can use the arguments of the proof of Lemma 3.7 to complete the proof.

Lemma 3.9. *Let $\phi \in \Delta_2 \cap \nabla_2$ and $f \in L^\phi(G)$. Assume that $u \in C_0^\infty(B_{R_0})$ with some constant $R_0 > 0$ is a solution of equation (1.1). Then there exists a positive constant c such that*

$$\int_G \phi(|D^2 u|)dx \le c \int_G \phi(|f|)dx.$$

Proof. Since $a_{ij} \in VMO(G)$, we can choose ρ_i small enough such that (3.17) holds. By Lemma 3.2, it is easy to see that, (3.18) is valid. It follows from (3.1), (3.19) that for any $\lambda > 0$

$$\begin{aligned}
&|\{x \in B_{5\rho_i}(x_i) : |D^2 u| > 2N_1\lambda\lambda_0\}| = |\{x \in B_{5\rho_i}(x_i) : |D^2 u_\lambda| > 2N_1\}| \\
\le\ & |\{x \in B_{5\rho_i}(x_i) : |D^2(u_\lambda - v_\lambda^i)| > N_1\}| + |\{x \in B_{5\rho_i}(x_i) : |D^2 v| > N_1\}| \\
=\ & |\{x \in B_{5\rho_i}(x_i) : |D^2(u_\lambda - v_\lambda^i)| > N_1\}| \le \frac{1}{N_1^p}\int_{B_{10\rho_i}(x_i)} |D^2(u_\lambda - v_\lambda^i)|^p dx \\
\le\ & c\varepsilon |B_{\rho_i}(x_i)|.
\end{aligned}$$

Setting $\mu = \lambda\lambda_0$, we can use Lemma 3.2 and (3.1) to obtain

$$\begin{aligned}
&|\{x \in B_{5\rho_i}(x_i) : |D^2 u| > 2N_1\mu\}| \\
\le\ & \frac{c\varepsilon}{\mu^p}\left(\int_{\{x\in B_{\rho_i}(x_i):|D^2 u|>\mu/2\}} |D^2 u|^p dx + M^p \int_{\{x\in B_{\rho_i}(x_i):|f|>\mu/(2M)\}} |f|^p dx\right).
\end{aligned}$$

Then recalling the fact that the balls $B_{5\rho_i}(x_i)$ are disjoint,

$$\bigcup_{i\ge 1} B_{5\rho_i}(x_i) \cup \text{negligible set} \supset E_\lambda(1) = \{x \in G : |D^2 u_\lambda| > 1\},$$

and that $E_\lambda(N) \subset E_\lambda(1)$ for any $N > 1$, we obtain

$$\begin{aligned}
&|\{x \in G : |D^2 u| > 2N_1\mu\}| \le \sum_i |\{x \in B_{5\rho_i}(x_i) : |D^2 u| > 2N_1\mu\}| \\
\le\ & \frac{c\varepsilon}{\mu^p}\left(\int_{\{x\in G:|D^2 u|>\mu/2\}} |D^2 u|^p dx + M^p \int_{\{x\in G:|f|>\mu/(2M)\}} |f|^p dx\right).
\end{aligned}$$

Furthermore, recalling (1.7) and Lemma 3.2, we can write

$$\begin{aligned}\int_G \phi(|D^2u|)dx &= \int_0^\infty |\{x\in G : |D^2u| > 2N_1\mu\}| d[\phi(2N_1\mu)] \\ &\le c\varepsilon \int_0^\infty \frac{1}{\mu^p}\left(\int_{x\in G:|D^2u|>\mu/2} |D^2u|^p dx\right) d[\phi(2N_1\mu)] \\ &\quad + cM^p \int_0^\infty \frac{1}{\mu^p}\left(\int_{x\in G:|f|>\mu/(2M)} |f|^p dx\right) d[\phi(2N_1\mu)] \\ &\le c_1\varepsilon \int_G \phi(|D^2u|)dx + c_2\int_G \phi(|f|)dx,\end{aligned}$$

where $c_1 = c_1(G,\phi)$ and $c_1 = c_2(G,\phi,\varepsilon,M)$.

Finally, choosing a suitable ε such that $c_1\varepsilon < 1/2$, we obtain

$$\int_G \phi(|D^2u|)dx \le c\int_G \phi(|f|)dx.$$

4. Proof of the main result

In order to prove our main result, we first establish a lemma by using the method applied in [15, 16].

Lemma 4.1. *Let the functions ϕ and f be as in Theorem 1.1, and let $u \in C_0^\infty(B_{R_0/2})$ be a solution of equation $Lu - \mu u = f$ in G. Then there exist positive constants μ_0 and c, depending only on G, ϕ, ν, R_0, such that*

$$\begin{aligned}&\mu^{\alpha_2}\int_G \phi(|u|)dx + \mu^{\alpha_2/2}\int_G \phi(|\nabla u|)dx + \int_G \phi(|D^2u|)dx \\ &\le c\int_G \phi(|Lu-\mu u|)dx = c\int_G \phi(|f|)dx\end{aligned} \tag{4.1}$$

for any $\mu \ge \mu_0$, where α_2 is as in (1.4).

Proof. Define $\tilde u(z) = \tilde u(x,t) = u(x)\varphi(t)cos(\sqrt{\mu}t)$, $\tilde L\tilde u(z) = Lu(x) + (\tilde u_t)_t$, where $\varphi \in C_0^\infty(-R_0/2, R_0/2)$ is a cut-off function. It is easy to check that the coefficients matrix of the operator $\tilde L$

$$\tilde A_{(n+1)\times(n+1)} = \begin{pmatrix} A_{n\times n} & 0 \\ 0 & 1\end{pmatrix}$$

satisfies (1.2) and the VMO condition. Moreover, we have $\tilde L\tilde u(z) = \tilde f$, where

$$\tilde f = \varphi(t)cos(\sqrt{\mu}t)(L-\mu) + u(x)\varphi''(t)cos(\sqrt{\mu}t) - 2\sqrt{\mu}u(x)\varphi'(t)sin(\sqrt{\mu}t).$$

For convenience, we denote $D^2_{zz}\tilde u(x,t) = \{D^2\tilde u(x), (Xu)t, \tilde u_{tt}\}$, where

$$D^2\tilde u = D^2_{xx}\tilde u = \{XiXju\}_{i,j=1}^N,\ (Xu)t = \{(Xiu)t\}_{i=1}^N.$$

It follows from Lemma 3.9 that

$$\int_S \phi(|D^2_{zz}\tilde{u}|)dz \le C\int_S \phi(|f|)dz, \tag{4.2}$$

where $dz = dxdt$. According to (1.5), we get

$$\phi(|D^2u(x)|) \le K(|\varphi(t)cos(\sqrt{\mu}t)|)^{-\alpha_1}\phi(|\varphi(t)cos(\sqrt{\mu}t)D^2u(x)|).$$

Since $XiXju = \varphi(t)cos(\sqrt{\mu}t)XiXju(x)$ and $cost$ is a periodic function, we have

(4.3)

$$\begin{aligned}&\int_G \phi(|D^2u(x)|)dx\\&= \left(\frac{1}{K}\int_{\mathbb{R}}(|\varphi(t)cos(\sqrt{\mu}t)|)^{\alpha_1}dt\right)^{-1}\int_S \frac{1}{K}(|\varphi(t)cos(\sqrt{\mu}t)|)^{\alpha_1}\phi(|D^2u(x)|)dxdt\\&\le C\int_S \phi(|\varphi(t)cos(\sqrt{\mu}t)D^2u(x)|)dxdt = C\int_S \phi(|D^2_{xx}\tilde{u}|)dz \le C\int_S \phi(|D^2_{zz}\tilde{u}|)dz,\end{aligned}$$

where the constant C depends only on N, ϕ. Similarly, we can obtain

$$\begin{aligned}\int_G \phi(|Du(x)|)dx &\le C\int_S \phi(|\varphi(t)cos(\sqrt{\mu}t)Du(x)|)dxdt\\&\le C\sum_{i=1}^{N}\int_S \phi\left(\frac{1}{\sqrt{\mu}}|Xiu)t(z) - Xiu\varphi'(t)cos(\sqrt{\mu}t)|\right)dxdt\\&\le \frac{C}{\mu^{\alpha_2/2}}\left(\int_S \phi(|\tilde{u}_{xt}(z)|)dz + \int_G \phi(|D_xu|)dx\right),\end{aligned}$$

which implies that

$$\mu^{\alpha_2/2}\int_G \phi(|Du(x)|)dx \le C\int_S \phi(|(Xu)t(z)|)dz \le C\int_S \phi(|D^2_{zz}\tilde{u}(z)|)dz. \tag{4.4}$$

Since

$$\begin{aligned}&\int_G \phi(|u(x)|)dx\\&\le C\int_S \phi(\frac{1}{\sqrt{\mu}}|\tilde{u}_{tt}(z) - u(x)(\varphi''(t)cos(\sqrt{\mu}t) - 2\sqrt{\mu}\varphi'(t)sin(\sqrt{\mu}t)|)dxdt,\end{aligned}$$

then by choosing $\mu > \mu_0$ large enough we obtain

$$\mu^{\alpha_2}\int_G \phi(|Du(x)|)dx \le C\int_S \phi(|D^2_{zz}\tilde{u}(z)|)dz. \tag{4.5}$$

Combining (4.2)-(4.5) and taking $\mu \ge \mu_0 > 0$ large enough, we conclude that

$$\begin{aligned}&\mu^{\alpha_2}\int_G \phi(|u|)dx + \mu^{\alpha_2/2}\int_G \phi(|\nabla u|)dx + \int_G \phi(|D^2u|)dx\\&\le C\int_S \phi(|D^2_{zz}\tilde{u}(z)|)dz \le C\int_S \phi(|f|)dz.\end{aligned}$$

Moreover, noting that

$$-\sqrt{\mu}\varphi'(t)sin(\sqrt{\mu}t) = u(x)((\varphi'(t)cos(\sqrt{\mu}t))_t - \varphi''(t)cos(\sqrt{\mu}t)),$$

we have

$$\int_S \phi(|f|)dz \leq C\left(\int_G \phi(|Lu - \mu u|)dx + \int_G \phi(|u|)dx\right).$$

Finally, combining the last two inequalities, and taking $\mu \geq \mu_0 > 0$ large enough, we complete the proof of Lemma 4.1.

To prove Theorem 1.1, we also need the following result from [5].

Lemma 4.2 ([5]). *Let (X, d, μ) be space of homogenous type. Then for every $r_0 > 0$ and $\mathcal{K} > 1$ there exist $\varrho \in (0, r_0)$ a positive integer $\mathcal{M}$ and a sequence of points $\{x_i\}_{i=1}^{\infty} \subset X$ such that*

$$\bigcup_{i=1}^{\infty} B(x_i, \varrho) = X; \quad \sum_{i=1}^{\infty} \chi_{B(x_i, \mathcal{K}\varrho)}(z) \leq \mathcal{M}, \ \forall\, x \in X.$$

Proof of Theorem 1.1. For $x_0 \in G$ let $\rho \in C_0^\infty(B_{R_0/2}(x_0))$. Denote

$$v(x) = u(x)\rho(x).$$

It follows that

$$Lv(x) - \mu v(x) = f\rho + 2a_{ij}XiuXi + a_{ij}uXiXj \equiv g.$$

Assume that $\mu \geq \mu_0 > 0$. It follows from Lemma 4.1 that

$$\mu^{\alpha_2}\int_G \phi(|v|)dx + \mu^{\alpha_2/2}\int_G \phi(|\nabla v|)dx + \int_G \phi(|D^2v|)dx \leq C\int_G \phi(|g|)dx$$
$$\leq C(\int_G \varphi(|f\chi_{B_{R_0/2}(x_0)}|)dx + \int_G \phi(|u\chi_{B_{R_0/2}(x_0)}|)dx + \int_G \phi(|Du\chi_{B_{R_0/2}(x_0)}|)dx).$$

Taking into account that

$$\int_G \phi(|\rho Du|)dx \leq \int_G \phi(|Dv|)dx + \int_G \phi(|uD\rho|)dx,$$

$$\int_G \phi(|\rho D^2u|)dx \leq C\left(\int_G \phi(|D^2v|)dx + \int_G \phi(|DuD\rho|)dx + \int_G \phi(|uD^2\rho|)dx\right),$$

we obtain

$$\mu^{\alpha_2}\int_G \phi(|v|)dx + \mu^{\alpha_2/2}\int_G \phi(|\rho Du|)dx + \int_G \phi(|\rho D^2u|)dx$$
$$\leq C(\int_G \varphi(|f\chi_{B_{R_0/2}(x_0)}|)dx + \mu^{\alpha_2/2}\int_G \phi(|u\chi_{B_{R_0/2}(x_0)}|)dx + \int_G \phi(|Du\chi_{B_{R_0/2}(x_0)}|)dx).$$

Hence, using Lemma 4.2, we can write

$$\begin{aligned}
&\mu^{\alpha_2}\int_G \phi(|u|)dx + \mu^{\alpha_2/2}\int_G \phi(|Du|)dx + \int_G \phi(|D^2u|)dx \\
&= \mu^{\alpha_2}\int_{\bigcup_{i=1}^{\infty} B(x_i,R_0/2)} \phi(|u|)dx + \mu^{\alpha_2/2}\int_{\bigcup_{i=1}^{\infty} B(x_i,R_0/2)} \phi(|Du|)dx \\
&+ \int_{\bigcup_{i=1}^{\infty} B(x_i,R_0/2)} \phi(|D^2u|)dx \\
&\le \sum_{i=1}^{\infty}\mu^{\alpha_2}\int_{B(x_i,R_0/2)} \phi(|u|)dx + \sum_{i=1}^{\infty}\mu^{\alpha_2/2}\int_{B(x_i,R_0/2)} \phi(|Du|)dx \\
&+ \sum_{i=1}^{\infty}\int_{B(x_i,R_0/2)} \phi(|D^2u|)dx \\
&\le C\sum_{i=1}^{\infty}\Big(\int_{B(x_i,R_0)} \phi(|f|)dx + \mu^{\alpha_2/2}\int_{B(x_i,R_0)} \phi(|u|)dx + \int_{B(x_i,R_0)} \phi(|Du|)dx\Big) \\
&\le CM\Big(\int_G \phi(|f|)dx + \mu^{\alpha_2/2}\int_G \phi(|u|)dx + \int_G \phi(|Du|)dx\Big).
\end{aligned}$$

By choosing $\mu \ge \mu_0 > 0$ large enough, we conclude that (1.8) is valid. □

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