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GENERALIZATIONS OF KÖTHE-TOEPLITZ DUALS AND NULL DUALS OF NEW DIFFERENCE SEQUENCE SPACES

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Abstract. The main purpose of the paper is to generalize the notions of the Köthe-Toeplitz duals and Null duals of sequence spaces by introducing the concepts of αEF-, βEF-, γEF-duals and NEF-duals, where $E = (E_n)$ and $F = (F_n)$ are two partitions of finite subsets of the positive integers. These duals are computed for the classical sequence spaces l_∞, c and c_0. The other purpose of the paper is to introduce the sequence spaces $X(E,\Delta) = \left\{x = (x_k) : \left(\sum_{i\in E_k} x_i - \sum_{i\in E_{k-1}} x_i\right)_{k=1}^{\infty} \in X\right\}$, where $X \in \{l_\infty, c, c_0\}$ We investigate the topological properties of these spaces, establish some inclusion relations between them, and compute the NEF-(or Null) duals for these spaces.

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1. Introduction

Let ω denote the space of all real-valued sequences. Any vector subspace of ω is called a sequence space. Let l_∞, c and c_0 be the spaces of bounded, convergent and null sequences $x = (x_k)$, respectively, endowed by the norm $\|x\|_\infty = \sup_{k\geq 1}|x_k|$. We write bs and cs for the spaces of all bounded and convergent series, respectively. Kizmaz [6] defined the difference sequence space

$$X(\Delta) = \{x = (x_k) \mid \Delta x \in X\},$$

for $X \in \{l_\infty, c, c_0\}$, where $\Delta x = (x_k - x_{k-1})_{k=1}^{\infty}$ and $x_0 = 0$. Observe that $X(\Delta)$ is a Banach space with the norm $\|x\|_\Delta = \sup_{k\geq 1}|x_k - x_{k-1}|$. For a sequence space X, the matrix domain X_A of an infinite matrix A is defined by

$$X_A = \{x = (x_n) \in \omega \ : \ Ax \in X\}, \tag{1.1}$$

which is a sequence space. The new sequence space X_A generated by the limitation matrix A from a sequence space X can be the extension or the contraction or the overlap of the original space X. A matrix $A = (a_{nk})$ is said to be a triangle matrix if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. If A is a triangle matrix, then one can easily observe that the sequence spaces X_A and X are linearly isomorphic, that is, $X_A \cong X$.

In the summability theory, the β-dual of a sequence space is very important in connection with inclusion theorems. The notion of a dual sequence space was introduced by Köthe and Toeplitz [8], and was extended to vector-valued sequence spaces by Maddox [9]. For the sequence spaces X and Y, the set $M(X,Y)$ defined by

$$M(X,Y) = \{a = (a_k) \in \omega \ : \ (a_k x_k)_{k=1}^{\infty} \in Y \quad \forall x = (x_k) \in X\}$$

is called the multiplier space of X and Y.

With the above notation, the α-, β- γ and N-duals of a sequence space X, denoted by X^α, X^β, X^γ and X^N, respectively, are defined as follows:

$$X^\alpha = M(X,l_1), \quad X^\beta = M(X,cs), \quad X^\gamma = M(X,bs), \quad X^N = M(X,c_0).$$

Let $E = (E_n)$ be a partition of finite subsets of the positive integers such that

$$\max E_n < \min E_{n+1}, \quad n = 1, 2, \cdots. \tag{1.2}$$

For $X \in \{l_p, l_\infty, c, c_0\}$ with $1 \leq p < \infty$, we define the sequence space $X(E)$ by

$$X(E) = \left\{ x = (x_k) \in \omega \ : \ \left(\sum_{i \in E_k} x_i \right)_{k=1}^{\infty} \in X \right\}.$$

The seminorms $\|\!|\cdot|\!\|_{p,E}$ on the sequence space $l_p(E)$ $(1 \leq p < \infty)$, and $\|\!|\cdot|\!\|_{\infty,E}$ on the space $X(E)$ for $X \in \{l_\infty, c, c_0\}$ are defined by formulas:

$$\|\!|x|\!\|_{p,E} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p \right)^{1/p}, \quad 1 \leq p < \infty, \qquad \|\!|x|\!\|_{\infty,E} = \sup_{n \geq 1} \left| \sum_{j \in E_n} x_j \right|.$$

It is worthwhile to note that in the special case $E_n = \{n \ : \ n = 1, 2, \cdots\}$, we have $X(E) = X$. Recently the Köthe-Toeplitz duals for these spaces were computed by Erfanmanesh and Foroutannia. In the past, several authors have studied the Köthe-Toeplitz duals of sequence spaces that are the matrix domains of triangle matrices in the classical spaces l_p, l_∞, c and c_0. For instance, some matrix domains of the difference operator have been studied in [2, 3, 10, 12]. In these papers the matrix domains are obtained by triangle matrices, and hence these spaces are normed sequence spaces. For more details on the domains of triangle matrices in some sequence spaces, we refer the reader to Chapter 4 of [1]. Note that the matrix domains considered in this paper are specified by a certain non-triangle matrix, so we should not expect that the related spaces are normed sequence spaces.

In this paper, the concepts of the Köthe-Toeplitz duals and Null duals are generalized and the αEF-, βEF-, γEF- and NEF-duals are determined for the classical sequence spaces l_∞, c and c_0. Also, the normed sequence space $X(\Delta)$ is extended to the semi-normed space $X(E,\Delta)$, where $X \in \{l_\infty, c, c_0\}$. We consider some topological properties of this space and derive inclusion relations. Moreover, we compute the NEF-(or Null) duals for the space $X(E,\Delta)$. The obtained results are generalizations of some results of Malkowsky and Rakocevic [11] and Kizmaz [7].

2. The αEF-, βEF-, γEF- and NEF-duals of sequence spaces

In this section, we generalize the concept of multiplier space to introduce new generalizations of Köthe-Toeplitz duals and Null duals of sequence spaces. Furthermore, we determine these duals for the sequence spaces l_∞, c and c_0.

Definition 2.1. *Let $E = (E_n)$ and $F = (F_n)$ be two partitions of finite subsets of the positive integers satisfying condition (1.2). For the sequence spaces X and Y, the set $M_{E,F}(X,Y)$ defined by*

$$M_{E,F}(X,Y) = \left\{ a = (a_k) \in \omega \ : \ \left(\sum_{i \in F_k} a_i \sum_{i \in E_k} x_i \right)_{k=1}^{\infty} \in Y \quad \forall x = (x_k) \in X \right\}$$

is called the generalized multiplier space of X and Y.

With the above notation, the αEF-, βEF-, γEF- and NEF-duals of a sequence space X, denoted by $X^{\alpha EF}$, $X^{\beta EF}$, $X^{\gamma EF}$ and X^{NEF}, respectively, are defined by

$$X^{\alpha EF} = M_{EF}(X, l_1), \ \ X^{\beta EF} = M_{EF}(X, cs),$$

$$X^{\gamma EF} = M_{EF}(X, bs), \ \ X^{NEF} = M_{EF}(X, c_0).$$

It should be noted that in the special case $E_n = F_n = \{n\}$ for all n, we have $M_{E,F}(X,Y) = M(X,Y)$, and hence

$$X^{\alpha EF} = X^{\alpha}, \qquad X^{\beta EF} = X^{\beta}, \qquad X^{\gamma EF} = X^{\gamma}, \qquad X^{NEF} = X^{N}.$$

Lemma 2.1. *Let $X, Y, Z \subset \omega$ and let $\{X_\delta : \delta \in A\}$ be any collection of subsets of ω. Then the following statements hold:*

(i) $X \subset Z$ implies $M_{E,F}(Z,Y) \subset M_{E,F}(X,Y)$,

(ii) $Y \subset Z$ implies $M_{E,F}(X,Y) \subset M_{E,F}(X,Z)$,

(iii) $X \subset M_{E,F}(M_{F,E}(X,Y),Y)$,

(iv) $M_{E,F}(X,Y) = M_{E,F}(M_{F,E}(M_{E,F}(X,Y),Y),Y)$,
(v) $M_{E,F}(\bigcup_{\delta\in A} X_\delta, Y) = \bigcap_{\delta\in A} M_{E,F}(X_\delta, Y)$.

Proof. The statements (i) and (ii) immediately follow from the definition of generalized multiplier space.
(iii) Let $x \in X$. We have $\left(\sum_{i\in E_k} a_i \sum_{i\in F_k} x_i\right)_{k=1}^{\infty} \in Y$ for all $a \in M_{F,E}(X,Y)$, and consequently $x \in M_{E,F}(M_{F,E}(X,Y),Y)$.
(iv) By applying (iii) with X replaced by $M_{F,E}(X,Y)$, we obtain

$$M_{E,F}(X,Y) \subset M_{E,F}(M_{F,E}(M_{E,F}(X,Y),Y),Y).$$

Conversely, due to (iii), we have $X \subset M_{F,E}(M_{E,F}(X,Y),Y)$. So, in view of part (i), we conclude that

$$M_{E,F}(M_{F,E}(M_{E,F}(X,Y),Y),Y) \subset M_{E,F}(X,Y).$$

(v) Observe first that in view of part (i), $X_\delta \subset \bigcup_{\delta\in A} X_\delta$ for all $\delta \in A$ implies

$$M_{E,F}(\bigcup_{\delta\in A} X_\delta, Y) \subset \bigcap_{\delta\in A} M_{E,F}(X_\delta, Y).$$

Conversely, if $a \in \bigcap_{\delta\in A} M_{E,F}(X_\delta, Y)$, then $a \in M_{E,F}(X_\delta, Y)$ for all $\delta \in A$. Hence

$$\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right)_{k=1}^{\infty} \in Y,$$

for all $\delta \in A$ and for all $x \in X_\delta$. This implies $\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right)_{k=1}^{\infty} \in Y$ for all $x \in \bigcup_{\delta\in A} X_\delta$, and hence $a \in M_{E,F}(\bigcup_{\delta\in A} X_\delta, Y)$. Thus $\bigcap_{\delta\in A} M_{E,F}(X_\delta, Y) \subset M_{E,F}(\bigcup_{\delta\in A} X_\delta, Y)$. □

Remark 2.1. *If $E_n = F_n = \{n\}$ for all n, then we have Lemma 1.25 from* [11].

Letting † to denote either of the symbols α, β, γ or N, from now on we will use the following notation

$$(X^{\dagger EF})^{\dagger EF} = X^{\dagger\dagger EF}.$$

Corollary 2.1. *Let $X, Y \subset \omega$ and $\{X_\delta : \delta \in A\}$ be any collection of subsets of ω, and let † denote either of the symbols α, β, γ or N. Then the following statements hold:*
(i) $X^{\alpha EF} \subset X^{\beta EF} \subset X^{\gamma EF} \subset \omega$; in particular, $X^{\dagger EF}$ is a sequence space.
(ii) $X \subset Z$ implies $Z^{\dagger EF} \subset X^{\dagger EF}$.
(iii) $X \subset X^{\dagger\dagger EE}$.

(iv) $X^{\dagger EE} = X^{\dagger\dagger\dagger EE}$.
(v) $\left(\bigcup_{\delta\in\Lambda} X_\delta\right)^{\dagger EF} = \bigcap_{\delta\in\Lambda} X_\delta^{\dagger EF}$.

Remark 2.2. *If $E_n = F_n = \{n\}$ for all n, then we have Corollary 1.26 from [11].*

Below, we determine the generalized multiplier space for some sequence spaces. To this end, we first recall the following result from [4]. Also, from now on, we denote the cardinal number of the set E_k by $|E_k|$.

Theorem 2.1 ([4], Corollary 2.5). *The following statements hold.*
(i) Let $\sup_n |E_n| < \infty$, then we have $X \subset X(E)$ for $X \in \{l_\infty, c_0\}$.
(ii) If $E_n = \{Nn - N + 1, Nn - N + 2, \cdots, Nn\}$ for all n, then $c \subset c(E)$.
(iii) If, in addition, $|E_n| > 1$ for an infinite number of n, then the inclusion relations in parts (i) and (ii) are strict.

Theorem 2.2. *If $\sup_k |E_k| < \infty$, then the following statements hold:*
(i) $M_{E,F}(c_0, X) = l_\infty(F)$, where $X \in \{l_\infty, c, c_0\}$.
(ii) $M_{E,F}(l_\infty, X) = c_0(F)$, where $X \in \{c, c_0\}$.
(iii) If, in addition, $E_n = \{Nn - N + 1, Nn - N + 2, \cdots, Nn\}$ for all n, then $M_{E,F}(c, c) = c(F)$.

Proof. (*i*) Since $c_0 \subset c \subset l_\infty$, by applying Lemma 2.1(ii), we obtain

$$M_{E,F}(c_0, c_0) \subset M_{E,F}(c_0, c) \subset M_{E,F}(c_0, l_\infty).$$

So, it is enough to verify the inclusions $l_\infty(F) \subset M_{E,F}(c_0, c_0)$ and $M_{E,F}(c_0, l_\infty) \subset l_\infty(F)$. Assume first that $a \in l_\infty(F)$ and $x \in c_0$. Then by Theorem 2.1 we have $x \in c_0(E)$, and hence $\lim_{k\to\infty} \left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right) = 0$, implying that $a \in M_{E,F}(c_0, c_0)$. Thus, we have $l_\infty(F) \subset M_{E,F}(c_0, c_0)$.

Now let $a \notin l_\infty(F)$. Then there is a subsequence $\left(\sum_{i\in F_{k_j}} a_i\right)_{j=1}^\infty$ of the sequence $\left(\sum_{i\in F_k} a_i\right)_{k=1}^\infty$ such that $\left|\sum_{i\in F_{k_j}} a_i\right| > j^2$ for $j = 1, 2, \cdots$. If the sequence $x = (x_i)$ is defined by

$$x_i = \begin{cases} \frac{(-1)^j j}{\sum_{i\in F_{k_j}} a_i} & if \ i = \min E_{k_j} \\ 0 & otherwise, \end{cases}$$

for $i = 1, 2, \cdots$, then we have $x \in c_0$ and $\sum_{i\in E_{k_j}} x_i \sum_{i\in F_{k_j}} a_i = (-1)^j j$, for all j. Hence

$$\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right)_{k=1}^\infty \notin l_\infty,$$

showing that $M_{E,F}(c_0, l_\infty) \subset l_\infty(F)$.

(ii) By Lemma 2.1(ii) we have

$$M_{E,F}(l_\infty, c_0) \subset M_{E,F}(l_\infty, c).$$

Hence, it is enough to verify the inclusions $c_0(F) \subset M_{E,F}(l_\infty, c_0)$ and $M_{E,F}(l_\infty, c) \subset c_0(F)$. Assume first that $a \in c_0(F)$. Then by Theorem 2.1 we have

$$\lim_{k\to\infty}\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right) = 0 \quad \text{for all} \quad x \in l_\infty,$$

that is, $a \in M_{E,F}(l_\infty, c_0)$. Thus $c_0(F) \subset M_{E,F}(l_\infty, c_0)$.

Now let $a \notin c_0(F)$. Then there are a real number $b > 0$ and a subsequence $\left(\sum_{i\in F_{k_j}} a_i\right)_{j=1}^{\infty}$ of the sequence $\left(\sum_{i\in F_k} a_i\right)_{k=1}^{\infty}$ such that $\left|\sum_{i\in F_{k_j}} a_i\right| > b$ for all for $j = 1, 2, \cdots$. Defining the sequence $x = (x_i)$ as in part (ii), we have $x \in l_\infty$ and $\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right)_{k=1}^{\infty} \notin c$, which implies $a \notin M_{E,F}(l_\infty, c)$ and shows that $M_{E,F}(l_\infty, c) \subset c_0(F)$.

(iii) Suppose that $a \in c(F)$. By applying Theorem 2.1, we conclude that

$$\lim_{k\to\infty}\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right) \quad \text{exists for all} \quad x \in c.$$

So $a \in M_{E,F}(c, c)$ and $c(F) \subset M_{E,F}(c, c)$.

Now we assume $a \notin c(F)$, and define the sequence x by $x = (\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \cdots)$. It is easy to see that $x \in c$ and $\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right)_{k=1}^{\infty} = \left(\sum_{i\in F_k} a_i\right)_{k=1}^{\infty} \notin c$. Thus, $a \notin M_{E,F}(c, c)$, showing that $M_{E,F}(c, c) \subset c(F)$. □

Remark 2.3. *If $E_n = F_n = \{n\}$ for all n, then we have Example 1.28 from* [11].

As an immediate consequence of Theorem 2.2, we have the following result.

Corollary 2.2. (i) *If $\sup_k |E_k| < \infty$, then $c_0^{NEF} = l_\infty(F)$ and $l_\infty^{NEF} = c_0(F)$.*
(ii) *If $E_n = \{Nn - N + 1, Nn - N + 2, \cdots, Nn\}$ for all n, then $c^{NEF} = c_0(F)$.*

Now we proceed to obtain the αEF-, βEF- and γEF-duals for the sequence spaces l_∞, c and c_0.

Theorem 2.3. *Suppose that $\sup_k |E_k| < \infty$, and let † denote one of the symbols α, β, γ. Then we have $c_0^{\dagger EF} = c^{\dagger EF} = l_\infty^{\dagger EF} = l_1(F)$.*

Proof. We only prove the statement for the case $\dagger = \beta$, the other cases can be proved similarly. By Corollary 2.1(ii) we have $l_\infty^{\beta EF} \subset c^{\beta EF} \subset c_0^{\beta EF}$. Hence, it is enough to show that $l_1(F) \subset l_\infty^{\beta EF}$ and $c_0^{\beta EF} \subset l_1(F)$. Let $a \in l_1(F)$ and $x \in l_\infty$ be given. Then by Theorem 2.1 we have $x \in l_\infty(F)$. Hence

$$\sum_{k=1}^{\infty}\left|\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right| \le \sup\left|\sum_{i\in E_k} x_i\right|\sum_{k=1}^{\infty}\left|\sum_{i\in F_k} a_i\right| < \infty.$$

showing that $\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right)_{k=1}^{\infty} \in cs$. Thus, we have $a \in l_\infty^{\beta EF}$ and $l_1(F) \subset l_\infty^{\beta EF}$. To prove the inclusion $c_0^{\beta EF} \subset l_1(F)$, it is enough to show that for a given $a \notin l_1(F)$ a sequence $x \in c_0$ can be found to satisfy $\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right)_{k=1}^{\infty} \notin cs$. To show the existence of a sequence $x \in c_0$ with the above property, observe first that since $a \notin l_1(F)$, we may choose an index subsequence (n_j) from $\mathbf{N}$ with $n_0 = 0$ and

$$\sum_{k=n_{j-1}}^{n_j-1}\left|\sum_{i\in F_k} a_i\right| > j, \quad j = 1, 2, \cdots.$$

Now we define the sequence $x \in c_0$ such that the first element of the set E_k is equal to $\frac{1}{j} sgn \sum_{i\in F_k} a_i$ and the remaining elements are zero, whenever $n_{j-1} \le k < n_j$. Then we have

$$\sum_{k=n_{j-1}}^{n_j-1}\left(\sum_{i\in F_k} a_i \sum_{i\in E_k} x_i\right) = \frac{1}{j}\sum_{k=n_{j-1}}^{n_j-1}\left|\sum_{i\in F_k} a_i\right| > 1,$$

for $j = 1, 2, \cdots$. Therefore $\left(\sum_{i\in E_k} x_i \sum_{i\in F_k} a_i\right)_{k=1}^{\infty} \notin cs$ and $a \notin c_0^{\beta EF}$, showing that $c_0^{\beta EF} \subset l_1(F)$. □

Remark 2.4. *If $E_n = F_n = \{n\}$ for all n, then we have Theorem 1.29 from [11].*

Definition 2.2. *A subset X of ω is said to be E-normal if $y \in X$ and $|\sum_{i\in E_k} x_i| \le |\sum_{i\in E_k} y_i|$, for $k = 1, 2, \cdots$, together imply $x \in X$. In the special case where $E_n = \{n\}$ for all n, the set X is called normal.*

Example 2.1. The sequence spaces c_0 and l_∞ are normal, but they are not E-normal. Indeed, taking $x = (1, -1, 2, -2, \cdots)$, $y = (1, \frac{1}{2}, \cdots)$ and $E_n = \{2n-1, 2n\}$ for all n, we have $|\sum_{i\in E_n} x_i| \le |\sum_{i\in E_n} y_i|$ and $y \in c_0, l_\infty$, while $x \notin c_0, l_\infty$.

Example 2.2. The sequence spaces $c_0(E)$ and $l_\infty(E)$ are E-normal, but they are not normal. Indeed, taking $x = (1, 1, 2, 2, \cdots)$, $y = (1, -1, 2, -2, \cdots)$ and $E_n = \{2n-1, 2n\}$ for every n, it is easy to see that $|x_i| \le |y_i|$ and $y \in c_0(E), l_\infty(E)$, while $x \notin c_0(E), l_\infty(E)$.

Example 2.3. The sequence spaces c and $c(E)$ are neither E-normal nor normal.

Theorem 2.4. *Let X be a E-normal subset of ω. Then we have*

$$X^{\alpha EF} = X^{\beta EF} = X^{\gamma EF}.$$

Proof. By Corollary 2.1(i) we have $X^{\alpha EF} \subset X^{\beta EF} \subset X^{\gamma EF}$. Hence, to prove the statement, it is enough to verify the inclusion $X^{\gamma EF} \subset X^{\alpha EF}$. Let $z \in X^{\gamma EF}$ and $x \in X$ be given. We define a sequence y such that $\sum_{i\in E_k} y_i = (sgn \sum_{i\in F_k} z_i) \left|\sum_{i\in E_k} x_i\right|$ for $k = 1, 2, \cdots$. It is clear that $\left|\sum_{i\in E_k} y_i\right| \le \left|\sum_{i\in E_k} x_i\right|$ for all k. Consequently $y \in X$, because X is E-normal. So, we obtain

$$\sup_n \left|\sum_{k=1}^{n} \left(\sum_{i\in F_k} z_i \sum_{i\in E_k} y_i\right)\right| < \infty.$$

Furthermore, by the definition of the sequence y, we have $\sum_{k=1}^{\infty} \left|\sum_{i\in F_k} z_i \sum_{i\in E_k} x_i\right| < \infty$. Taking into account that $x \in X$ is arbitrary, we conclude that $z \in X^{\alpha EF}$.

This completes the proof of the theorem. □

Remark 2.5. *If $E_n = F_n = \{n\}$ for all n, then we have Remark 1.27 from* [11].

3. Generalized difference sequence space

Suppose $E = (E_n)$ is a sequence of finite subsets of the positive integers that satisfy the condition (1.2). For every sequence space X, we define the generalized difference sequence space $X(E, \Delta)$ as follows:

$$X(E,\Delta) = \left\{ x = (x_k) \ : \ \left(\sum_{i\in E_k} x_i - \sum_{i\in E_{k-1}} x_i\right)_{k=1}^{\infty} \in X \right\},$$

where $X \in \{l_\infty, c, c_0\}$. The seminorm $\|\!|.\|\!|_{E,\Delta}$ on $X(E, \Delta)$ is defined by

$$\|\!|x\|\!|_{E,\Delta} = \sup_k \left|\sum_{i\in E_k} x_i - \sum_{i\in E_{k-1}} x_i\right|. \tag{3.1}$$

It should be noted that the function $\|\!|.\|\!|_{E,\Delta}$ cannot be a norm. Since if $x = (1, -1, 0, 0, \cdots)$ and $E_n = \{2n-1, 2n\}$ for all n, then $\|\!|x\|\!|_{E,\Delta} = 0$ while $x \neq 0$. It is also important to note that in the special case $E_n = \{n : \ n = 1, 2, \cdots\}$ we have $X(E, \Delta) = X(\Delta)$ and $\|\!|x\|\!|_{E,\Delta} = \|x\|_\Delta$.

If the infinite matrix $A = (a_{nk})$ is defined by

$$a_{nk} = \begin{cases} -1 & if\ k \in E_{n-1} \\ 1 & if\ k \in E_n \\ 0 & otherwise, \end{cases}$$

then with the notation of (1.1), we can redefine the spaces $l_\infty(E)$, $c(E)$ and $c_0(E)$ as follows:

$$l_\infty(E,\Delta) = (l_\infty)_A, \qquad c(E,\Delta) = (c)_A, \qquad c_0(E,\Delta) = (c_0)_A.$$

The purpose of this section is to consider some properties of the sequence spaces $X(E,\Delta)$ and to derive some inclusion relations for these spaces. Also, we characterize the N-duals of $X(E,\Delta)$, where $X \in \{l_\infty, c, c_0\}$. We begin with the following result which plays an essential role in our study of the spaces $X(E,\Delta)$.

Theorem 3.1. *For $X \in \{l_\infty, c, c_0\}$ the sequence spaces $X(E,\Delta)$ are complete semi-normed linear spaces with respect to the semi-norm defined by (3.1).*

Proof. The result can be obtained by a direct verification, and so we omit the details.

It can easily be checked that the absolute property does not hold on the space $X(E,\Delta)$, that is, $\|x\|_{E,\Delta} \neq \||x|\|_{E,\Delta}$ for at least one sequence in this space, where $|x| = (|x_k|)$. Thus, $X(E,\Delta)$ is a sequence space of non-absolute type.

Theorem 3.2. *Let $M = \{x = (x_n) \ : \ \sum_{j\in E_n} x_j = 0, \forall n\}$. For $X \in \{l_\infty, c, c_0\}$ the quotient space $X(E,\Delta)/M$ is linearly isomorphic to the space $X(\Delta)$.*

Proof. Consider the map

$$T : X(E,\Delta) \longrightarrow X(\Delta), \quad x \longrightarrow \left(\sum_{j\in E_n} x_j\right)_{n=1}^{\infty},$$

and observe that T is a linear and surjective map. So, the desired result follows from the first isomorphism theorem. □

Note that if $|E_n| > 1$ only for a finite number of n, then we have $X(\Delta) = X(E,\Delta)$.

In the following, we derive some inclusion relations for the spaces X, $X(E)$, $X(\Delta)$ and $X(E,\Delta)$, where $X \in \{l_\infty, c, c_0\}$.

Theorem 3.3. *The following statements hold.*

(i) If $\sup_n |E_n| < \infty$, then $X \subset X(E,\Delta)$ for $X \in \{l_\infty, c_0\}$.

(ii) If $E_n = \{Nn-N+1, Nn-N+2, \cdots, Nn\}$ for all n, then $c \subset c(E,\Delta)$.

(iii) If $|E_n| > 1$ for an infinite number of n, then the inclusion relations in parts (i) and (ii) are strict.

(iv) We have $X(E) \subset X(E,\Delta)$, where $X \in \{l_\infty, c, c_0\}$. Moreover, these inclusions are strict.

(v) If $E_n = \{Nn-N+1, Nn-N+2, \cdots, Nn\}$ for all n, then $X(\Delta) \subset X(E,\Delta)$, where $X \in \{l_\infty, c, c_0\}$. Moreover, these inclusions are strict when $N > 1$.

Proof. Parts (i) and (ii) can easily be obtained by applying Theorem 2.1.

(iii) Since by assumption $|E_n| > 1$ for an infinite number of n, one can choose a sequence (n_j) with $|E_{n_j}| > 1$ for $j = 1, 2, \cdots$. Define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} j, & \text{if} \quad k = \min E_{n_j} \\ -j, & \text{if} \quad k = \min E_{n_j} + 1 \\ 0, & \text{otherwise,} \end{cases} \qquad k = 1, 2, \cdots .$$

It is obvious that $\sum_{i \in E_n} x_i = 0$, hence $x \in X(E, \Delta)$, while $x \notin X$ for $X \in \{l_\infty, c, c_0\}$, showing that the inclusions in parts (i) and (ii) are strict.

(iv) Putting $E_n = \{n\}$ in parts (i) and (ii), it can be concluded that $X \subset X(\Delta)$. Let $x \in X(E)$ be given. It is easy to check that $\left(\sum_{i \in E_k} x_i\right)_{k=1}^{\infty} \in X$ and $\left(\sum_{i \in E_k} x_i\right)_{k=1}^{\infty} \in X(\Delta)$. Thus, $x \in X(E, \Delta)$, and hence $X(E) \subset X(E, \Delta)$. Moreover, if the sequences $x = (x_k)$ and $y = (y_k)$ are defined as follows:

$$x_k = \begin{cases} j, & \text{if } k = \min E_j \\ 0, & \text{otherwise,} \end{cases} \qquad y_k = \begin{cases} 1, & \text{if } k = \min E_j \\ 0, & \text{otherwise,} \end{cases}$$

then we have $x \in X(E, \Delta) - X(E)$ for $X \in \{l_\infty, c\}$ and $y \in c_0(E, \Delta) - c_0(E)$.

(v) Taking into account that

$$\begin{aligned} \sum_{i \in E_n} x_i - \sum_{i \in E_{n-1}} x_i &= (x_{nN} - x_{nN-1}) + 2(x_{nN-1} - x_{nN-2}) + \cdots + N(x_{nN-N+1} - x_{nN-N}) \\ &+ (N-1)(x_{nN-N} - x_{nN-N-1}) + \cdots + (x_{nN-2N+2} - x_{nN-2N+1}), \end{aligned}$$

it is clear that $x \in X(\Delta)$ implies $x \in X(E, \Delta)$. Moreover, if $N > 1$ then we define the sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} n, & \text{if} \quad k = nN - N + 1 \\ 1 - n, & \text{if} \quad k = nN - N + 2 \\ 0, & \text{otherwise,} \end{cases}$$

and observe that $x \in X(E, \Delta) - X(\Delta)$. □

Below, we compute the N-dual of the difference sequence spaces $X(E, \Delta)$, where $X \in \{l_\infty, c, c_0\}$. In order to do this, we first give a preliminary lemma.

Lemma 3.1. *The following statements hold.*

(i) *If $x \in l_\infty(\Delta)$, then $\sup_k \left|\frac{x_k}{k}\right| < \infty$.*

(ii) *If $x \in c(\Delta)$, then $\frac{x_k}{k} \to \xi$ $(k \to \infty)$, where $\Delta x_k \to \xi$ $(k \to \infty)$.*

(iii) *If $x \in c_0(\Delta)$, then $\frac{x_k}{k} \to 0$ $(k \to \infty)$.*

The proof is trivial and so is omitted.

Theorem 3.4. *The following equalities hold:*

$$c^{NEF}(E,\Delta) = l_\infty^{NEF}(E,\Delta) = \left\{ a=(a_k)\ :\ \left(k\sum_{i\in F_k} a_i\right)_{k=1}^\infty \in c_0 \right\} := d_1.$$

Proof. We first show that $c^{NEF}(E,\Delta) = d_1$. To this end, assume $a \in c^{NEF}(E,\Delta)$, and observe that

$$\lim_{k\to\infty} \sum_{i\in F_k} a_i \sum_{i\in E_k} x_i = 0,$$

for all $x \in c(E,\Delta)$. We choose the sequence x such that $\sum_{i\in E_k} x_i = k$ for all k, so $x \in c(E,\Delta)$ and hence $\lim_{k\to\infty} k\sum_{i\in F_k} a_i = 0$. Thus $c^{NEF}(E,\Delta) \subset d_1$. Now let $a \in d_1$. Since $\left(\sum_{i\in E_k} x_i\right)_{k=1}^\infty \in c(\Delta)$ for every $x \in c(E,\Delta)$, and there is a real number ξ such that

$$\lim_{k\to\infty} \left(\sum_{j\in E_k} x_j - \sum_{j\in E_{k-1}} x_j \right) = \xi,$$

by Lemma 3.1 we have

$$\lim_{k\to\infty} \sum_{i\in F_k} a_i \sum_{j\in E_k} x_j = \lim_{k\to\infty} k \sum_{i\in F_k} a_i \frac{\sum_{j\in E_k} x_j}{k} = 0.$$

Therefore $a \in c^{NEF}(E,\Delta)$, and hence $d_1 \subset c^{NEF}(E,\Delta)$.

Now we show that $l_\infty^{NEF}(E,\Delta) = d_1$. It is clear that $c(E,\Delta) \subset l_\infty(E,\Delta)$, implying that $l_\infty^{NEF}(E,\Delta) \subset c^{NEF}(E,\Delta) = d_1$. Let $a \in d_1$ and $x \in l_\infty(E,\Delta)$. Then we have $\left(\sum_{i\in E_k} x_i\right)_{k=1}^\infty \in l_\infty(\Delta)$ and $\sup_k \left| \frac{\sum_{i\in E_k} x_i}{k} \right| < \infty$ by Lemma 3.1. Therefore

$$\lim_{k\to\infty} \sum_{i\in F_k} a_i \sum_{i\in E_k} x_i = \lim_{k\to\infty} k \sum_{i\in F_k} a_i \frac{\sum_{j\in E_k} x_j}{k} = 0,$$

implying that $a \in l_\infty^{NEF}(E,\Delta)$. □

Corollary 3.1. *The following equalities hold:*

$$c^N(\Delta) = l_\infty^N(\Delta) = \{a=(a_k)\ :\ (ka_k) \in c_0\}.$$

Proof. The result follows from Theorem 3.4 with $E_n = F_n = \{n\}$ for all n. □

Let X and Y be two sequence spaces, and let $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n,k \in \mathbb{N} = \{1,2,\cdots\}$. We say that A defines a matrix mapping from X into Y, denoted by $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$ exists and is in Y, where $(Ax)_n = \sum_{k=1}^\infty a_{n,k}x_k$ for $n = 1,2,\cdots$. By (X,Y) we denote the class of all infinite matrices A such that $A : X \to Y$.

Theorem 3.5 ([11], Theorem 1.36). *We have $A \in (c_0, c_0)$ if and only if the following conditions hold:*

$$\lim_{n\to\infty} a_{nk} = 0 \quad (k = 1, 2, \cdots),$$

and

$$\sup_n \left(\sum_{k=1}^{\infty} |a_{nk}| \right) < \infty \quad .$$

Theorem 3.6. *We have*

$$c_0^{NEF}(E, \Delta) = \left\{ a = (a_k) \ : \ \left(k \sum_{i \in F_k} a_i \right)_{k=1}^{\infty} \in l_\infty \right\} := d_2.$$

Proof. Let $a \in d_2$. Since $\left(\sum_{i \in E_k} x_i\right)_{k=1}^{\infty} \in c_0(\Delta)$ for all $x \in c_0(E, \Delta)$, by Lemma 3.1 we have $\lim_{k\to\infty} \frac{\sum_{j \in E_k} x_j}{k} = 0$. Therefore

$$\lim_{k\to\infty} \sum_{i \in F_k} a_i \sum_{j \in E_k} x_j = \lim_{k\to\infty} k \sum_{i \in F_k} a_i \frac{\sum_{j \in E_k} x_j}{k} = 0,$$

implying that $a \in c_0^{NEF}(E, \Delta)$.

Now let $a \in c_0^{NEF}(E, \Delta)$ and $x \in (E, \Delta)$ be given. Then there exists only one sequence $y = (y_k) \in c_0$ such that $\sum_{j \subset E_k} x_j = \sum_{j=1}^{k} y_j$. Therefore

$$\lim_{k\to\infty} \sum_{j=1}^{k} \sum_{i \in F_k} a_i y_j = \lim_{k\to\infty} \sum_{i \in F_k} a_i \sum_{j \subset E_k} x_j = 0,$$

for all $y = (y_k) \in c_0$. Defining the matrix $A = (a_{kj})_{k=1}^{\infty}$ by

$$a_{kj} = \begin{cases} \sum_{i \in F_k} a_i & \text{if} \quad 1 \le j \le k \\ 0 & \text{if} \quad j > k, \end{cases}$$

we have $\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj} y_j = 0$ for all $y \in c_0$. Hence we can apply Theorem 3.5 to conclude that $A = (a_{kj}) \in (c_0, c_0)$ and

$$\sup_k \left| k \sum_{i \in F_k} a_i \right| = \sup_k \left| \sum_{j=1}^{k} \sum_{i \in F_k} a_i \right| = \sup_k \left| \sum_{j=1}^{\infty} a_{kj} \right| < \infty.$$

Corollary 3.2 ([7], Lemma 2). *We have $c_0^N(\Delta) = \{a = (a_k) \ : \ (ka_k) \in l_\infty\}$.*

Proof. The result follows from Theorem 3.6 with $E_n = F_n = \{n\}$ for all n. □

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