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EXISTENCE THEOREMS OF PERIODIC SOLUTIONS FOR SECOND-ORDER DIFFERENCE EQUATIONS CONTAINING BOTH ADVANCE AND RETARDATION

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Abstract. Using the critical point method, the existence of periodic solutions for second-order nonlinear difference equations containing both advance and retardation is established. The proof is based on the Saddle Point Theorem in combination with variational technique

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1. Introduction

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all naturals numbers, integers and real numbers, respectively. For any $a,\ b \in \mathbf{Z}$ with $a \leq b$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$ and $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$. The symbol $*$ will denote the transpose of a vector.

Recently, the theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields, such as computer science, economics, neural networks, ecology, cybernetics, etc. For the general background of difference equations, we refer to monograph [1]. The past twenty years, there has been much progress on the qualitative properties of difference equations, which includes results on stability and attractivity and results on oscillation and other topics (see, [2-8, 12, 13, 15, 17, 18]. Therefore, it is worthwhile to explore this topic.

The present paper considers the following second-order nonlinear difference equation containing both advance and retardation:

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + f(n, u_{n+M}, u_n, u_{n-M}) = 0, \quad n \in \mathbf{Z}. \tag{1.1}$$

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where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $\delta > 0$ is the ratio of odd positive integers, $\{p_n\}$ is a sequence of real numbers, M is a given nonnegative integer, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$, T is a given natural, $p_{n+T} = p_n > 0$, and $f(n+T, v_1, v_2, v_3) = f(n, v_1, v_2, v_3)$.

Note that the equation (1.1) can be considered as a discrete analogue of a special case of the following second-order nonlinear functional differential equation with retarded and advanced arguments

$$[p(t)\varphi(u')]' + f(t, u(t+M), u(t), u(t-M)) = 0, \ t \in \mathbf{R}. \tag{1.2}$$

The equation (1.2) includes the following equation

$$(p(t)\varphi(u'))' + f(t, u(t)) = 0, \ t \in \mathbf{R},$$

which appears in the study of fluid dynamics, combustion theory, gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, and adiabatic reactor (see, [9]).

Note also that equations similar in structure to (1.2) arise in the study of periodic solutions and homoclinic orbits of functional differential equations (see, [10, 11]).

Yu, Shi and Guo [18] have studied the question of existence of homoclinic orbits for the following second-order difference equation

$$Lu_n - \omega u_n = f(n, u_{n+M}, u_n, u_{n-M}) \tag{1.3}$$

containing both advance and retardation.

If $\delta = 1$ and $f(n, u_{n+M}, u_n, u_{n-M}) = q_n u_n$, the equation (1.1) becomes

$$\Delta\left(p_n \Delta u_{n-1}\right) + q_n u_n = 0, \tag{1.4}$$

which has been extensively investigated by many authors (see [1] and references therein), for results on oscillation, asymptotic behavior, boundary value problems, disconjugacy and disfocality.

If $f(n, u_{n+M}, u_n, u_{n-M}) = q_n u_n^\delta$, $n \in \mathbf{Z}(0)$, the equation (1.1) reduces to the following equation

$$\Delta\left(p_n (\Delta u_{n-1})^\delta\right) + q_n u_n^\delta = 0, \tag{1.5}$$

which has been studied in [1, 18] for results on oscillation, asymptotic behavior and the existence of positive solutions.

In the case where $f(n, u_{n+M}, u_n, u_{n-M}) = q_n g(u_n) + r_n$, the equation (1.1) has been considered in [15] for oscillatory properties of its all solutions.

Cai, Yu and Guo ([2], Theorem 3.1), assuming that $\beta > \delta + 1$, have obtained some sufficient conditions for the existence of periodic solutions of the following nonlinear difference equation

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + f(n, u_n) = 0, \quad n \in \mathbf{Z}. \tag{1.6}$$

Moreover, to our best knowledge, [2] is the only paper which deals with the problem of periodic solutions to second-order difference equation (1.6). When $\beta < \delta + 1$, can we still find the periodic solutions of (1.6)?

By using various methods and techniques, such as Schauder fixed point theorem, the cone theoretic fixed point theorem, the method of upper and lower solutions, coincidence degree theory, a number of existence results of nontrivial solutions for differential equations have been obtained in [11].

Another important tool that was used to deal with problems concerning differential equations is the critical point theory (see, [10, 14, 16]). Because of applications in many areas for difference equations (see. e.g., [1]), recently a few authors have gradually paid attention to apply the critical point theory to deal with periodic solutions of discrete systems (see [3, 12, 13, 17]).

For instance, in [12, 13] Guo and Yu have studied the existence of periodic solutions of second-order nonlinear difference equations by using the critical point theory. However, to the best of our knowledge, when $\delta \neq 1$ the results on periodic solutions of second-order nonlinear difference equation (1.1) are very scarce in the literature (see [2]), because there are only few known methods to establish the existence of periodic solutions of discrete systems. Furthermore, since f in equation (1.1) depends on u_{n+M} and u_{n-M}, the traditional methods used in [12, 13, 17] are inapplicable in our case.

The motivation for the present paper stems from the recent papers [3, 4, 11], and the main purpose is to give some sufficient conditions for the existence of periodic solutions for second-order nonlinear difference equations containing both advance and retardation. The basic approaches used in the paper are variational techniques and the Saddle Point Theorem. For basic knowledge of variational methods, the reader is referred to [14, 16].

The obtained results generalize and complement the existing results in the literature [2]. The details are given in Remark 1.4 below.

Let

$$\underline{p} = \min_{n \in \mathbf{Z}(1,T)} \{p_n\}, \ \bar{p} = \max_{n \in \mathbf{Z}(1,T)} \{p_n\}.$$

Now we are in position to state the main results of this paper.

Theorem 1.1. *Assume that the following conditions are satisfied:*

(F_1) *there exists a functional $F(n, v_1, v_2) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ such that*

$$F(n+T, v_1, v_2) = F(n, v_1, v_2),$$

$$\frac{\partial F(n-M, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3);$$

(F_2) *there exists a constant $M_0 > 0$ such that for all $(n, v_1, v_2) \in \boldsymbol{Z} \times \boldsymbol{R}^2$*

$$\left|\frac{\partial F(n, v_1, v_2)}{\partial v_1}\right| \le M_0, \qquad \left|\frac{\partial F(n, v_1, v_2)}{\partial v_2}\right| \le M_0;$$

(F_3) *$F(n, v_1, v_2) \to +\infty$ uniformly for $n \in \boldsymbol{Z}$ as $\sqrt{v_1^2 + v_2^2} \to +\infty$.*

Then for any natural integer m equation (1.1) has at least one mT-periodic solution.

Remark 1.1. The condition (F_2) implies that there exists a constant $M_1 > 0$ such that

(F_2') $\quad |F(n, v_1, v_2)| \le M_1 + M_0(|v_1| + |v_2|), \ \forall (n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2.$

Theorem 1.2. *Assume that (F_1) and the following conditions are satisfied:*

(F_4) *there exist constants $R_1 > 0$ and α, $1 < \alpha < 2$ such that for $n \in \boldsymbol{Z}$ and $\sqrt{v_1^2 + v_2^2} \ge R_1$,*

$$0 < \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 \le \frac{\alpha}{2}(\delta + 1) F(n, v_1, v_2);$$

(F_5) *there exist constants $a_1 > 0$, $a_2 > 0$ and γ, $1 < \gamma \le \alpha$ such that*

$$F(n, v_1, v_2) \ge a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^{\frac{\gamma}{2}(\delta+1)} - a_2, \quad (n, v_1, v_2) \in \boldsymbol{Z} \times \boldsymbol{R}^2.$$

Then for any given natural m equation (1.1) has at least one mT-periodic solution.

Remark 1.2. The condition (F_4) implies that for each $n \in \mathbf{Z}$ there exist constants $a_3 > 0$ and $a_4 > 0$ such that

(F_4') $F(n, v_1, v_2) \le a_3 \left(\sqrt{v_1^2 + v_2^2}\right)^{\frac{\alpha}{2}(\delta+1)} + a_4, \quad (n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2.$

Remark 1.3. The results of Theorems 1.1 and 1.2 ensure that equation (1.1) has at least one mT-periodic solution. However, in some cases, we are interested in the existence of nontrivial periodic solutions for (1.1).

The next two theorems contain sufficient conditions for existence of nontrivial periodic solutions for equation (1.1).

Theorem 1.3. *Assume that* (F_1) *and the following conditions are satisfied:*

(F_6) $F(n,0,0)=0$ *for all* $n \in \mathbf{Z}$;

(F_7) *there exists a constant* α, $1<\alpha<2$ *such that for* $n \in \mathbf{Z}$,

$$0<\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1+\frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2 \le \frac{\alpha}{2}(\delta+1)F(n,v_1,v_2), \qquad (v_1,v_2)\neq 0;$$

(F_8) *there exist constants* $a_5>0$ *and* γ, $1<\gamma\le\alpha$ *such that*

$$F(n,v_1,v_2)\ge a_5\left(\sqrt{v_1^2+v_2^2}\right)^{\frac{\gamma}{2}(\delta+1)}, \qquad (n,v_1,v_2)\in \mathbf{Z}\times\mathbf{R}^2.$$

Then for any given natural m *equation (1.1) has at least one nontrivial* mT*-periodic solution.*

Theorem 1.4. *Assume that the conditions* $(F_1)-(F_3)$ *and* (F_6) *hold, and also*

(F_9) *there exist constants* $a_6>0$ *and* θ, $0<\theta<2$ *such that*

$$F(n,v_1,v_2)\ge a_6\left(\sqrt{v_1^2+v_2^2}\right)^{\frac{\theta}{2}(\delta+1)}, \qquad (n,v_1,v_2)\in \mathbf{Z}\times\mathbf{R}^2.$$

Then for any given natural m *equation (1.1) has at least one nontrivial* mT*-periodic solution.*

Remark 1.4. For $M=0$, the equation (1.1) reduces to (1.6). In the case where $\beta>\delta+1$, Cai and Yu (see [2], Theorem 3.2), have obtained some criteria for the existence of periodic solutions of (1.6). When $\beta<\delta+1$, we still can find periodic solutions of (1.6), and hence, Theorems 1.3 and 1.4 generalize and complement the existing results.

The rest of the paper is organized as follows. In Section 2, we establish the variational framework associated with equation (1.1) and transfer the problem of existence of periodic solutions of (1.1) into that of existence of critical points of the corresponding functional. Some related fundamental results are also recalled. Section 3 contains the proofs of the main results by using the critical point method. Finally, in Section 4, we give two examples to illustrate the main results.

2. Variational structure and some lemmas

In order to apply the critical point theory, we first establish the corresponding variational framework for equation (1.1) and give some lemmas, which will be used to prove our main results. We start by some basic notation.

Let S be the set of sequences $u = (\cdots, u_{-n}, \cdots, u_{-1}, u_0, u_1, \cdots, u_n, \cdots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is,

$$S = \{\{u_n\} | u_n \in \mathbf{R},\ n \in \mathbf{Z}\}.$$

For any $u, v \in S$ and $a, b \in \mathbf{R}$, $au + bv$ is defined to be

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then S becomes a vector space.

For any given naturals m and T, by E_{mT} we denote the subspace of S defined by

$$E_{mT} = \{u \in S | u_{n+mT} = u_n, \text{ for any } n \in \mathbf{Z}\}.$$

It is clear that E_{mT} is isomorphic to $\mathbf{R}^{mT}$. The subspace E_{mT} can be equipped with the inner product $(u, v) = \sum_{j=1}^{mT} u_j v_j$, $u, v \in E_{mT}$, which defines the norm $\|u\| = \left(\sum_{j=1}^{mT} u_j^2\right)^{\frac{1}{2}}$, $u \in E_{mT}$.
It is obvious that E_{mT} is a finite dimensional Hilbert space and is linearly homeomorphic to $\mathbf{R}^{mT}$. On the other hand, we define the norm $\|\cdot\|_s$ on E_{mT} as follows:

$$\|u\|_s = \left(\sum_{j=1}^{mT} |u_j|^s\right)^{\frac{1}{s}}, \tag{2.1}$$

for all $u \in E_{mT}$ and $s > 1$.

Since the norms $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants c_1, c_2 ($c_2 \geq c_1 > 0$), such that

$$c_1\|u\|_2 \leq \|u\|_s \leq c_2\|u\|_2, \quad u \in E_{mT}. \tag{2.2}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{mT}$, define the functional J on E_{mT} as follows:

$$J(u) = -\frac{1}{\delta+1}\sum_{n=1}^{mT} p_n (\Delta u_{n-1})^{\delta+1} + \sum_{n=1}^{mT} F(n, u_{n+M}, u_n),$$

$$:= -H(u) + \sum_{n=1}^{mT} F(n, u_{n+M}, u_n). \tag{2.3}$$

It is clear that $J \in C^1(E_{mT}, \mathbf{R})$, and using $u_0 = u_{mT}$, $u_1 = u_{mT+1}$, for any $u = \{u_n\}_{n\in\mathbf{Z}} \in E_{mT}$ we can compute the partial derivative $\frac{\partial J}{\partial u_n}$ to obtain

$$\frac{\partial J}{\partial u_n} = \Delta\left(p_n(\Delta u_{n-1})^\delta\right) + f(n, u_{n+M}, u_n, u_{n-M}).$$

Thus, u is a critical point of J on E_{mT} if and only if

$$\Delta\left(p_n(\Delta u_{n-1})^\delta\right) + f(n, u_{n+M}, u_n, u_{n-M}) = 0, \quad n \in \mathbf{Z}(1, mT).$$

Due to the periodicity of $u = \{u_n\}_{n\in\mathbf{Z}} \in E_{mT}$ and $f(n, v_1, v_2, v_3)$ in the first variable n, we reduce the existence of periodic solutions of equation (1.1) to the existence of critical points of J on E_{mT}. That is, the functional J is just the variational framework of equation (1.1). Let

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

be a $mT \times mT$ matrix. It is easy to check that the eigenvalues of P are given by

$$\lambda_k = 2\left(1 - \cos\frac{2k}{mT}\pi\right), \quad k = 0, 1, 2, \cdots, mT-1. \tag{2.4}$$

Thus, $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_{mT-1} > 0$, and we have

$$\lambda_{\min} = \min\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = 2\left(1 - \cos\tfrac{2}{mT}\pi\right),$$

$$\lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_{mT-1}\} = \begin{cases} 4, & \text{if } mT \text{ is even}, \\ 2\left(1 + \cos\frac{1}{mT}\pi\right), & \text{if } mT \text{ is odd}. \end{cases} \tag{2.5}$$

Denote $W = \ker P = \{u \in E_{mT} | Pu = 0 \in \mathbf{R}^{mT}\}$, and observe that $W = \{u \in E_{mT} | u = \{c\},\ c \in \mathbf{R}\}$.

Let V be the direct orthogonal complement of E_{mT} to W, that is, $E_{mT} = V \oplus W$. For convenience, we identify $u \in E_{mT}$ with $u = (u_1, u_2, \cdots, u_{mT})^*$.

Let E be a real Banach space and let $J \in C^1(E, \mathbf{R})$, that is, J is a continuously Fréchet-differentiable functional defined on E. We say that J satisfies the Palais-Smale condition (P.S. condition for short) if any sequence $\{u^{(k)}\} \subset E$ for which $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \to 0$ as $k \to \infty$, contains a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Saddle Point Theorem [14, 16]). *Let E be a real Banach space, and let $E = E_1 \oplus E_2$, where $E_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(E, R)$ satisfies the P.S. condition and*

(J_1) there exist constants σ, $\rho > 0$ such that $J|_{\partial B_\rho \cap E_1} \leq \sigma$;

(J_2) there exist $e \in B_\rho \cap E_1$ and a constant $\omega \geq \sigma$ such that $J_{e+E_2} \geq \omega$.

Then J possesses a critical value $c \geq \omega$, where

$$c = \inf_{h\in\Gamma} \max_{u\in B_\rho\cap E_1} J(h(u)), \quad \Gamma = \{h \in C(B_\rho \cap E_1, E) \mid h|_{\partial B_\rho \cap E_1} = id\}.$$

and id denotes the identity operator.

Lemma 2.2. *Assume that the conditions $(F_1)-(F_3)$ are satisfied. Then J satisfies the P.S. condition.*

Proof. Let $\{u^{(k)}\}\subset E_{mT}$ be such that $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)})\to 0$ as $k\to\infty$. Then there exists a positive constant M_2 such that $|J(u^{(k)})|\le M_2$.

Let $u^{(k)}=v^{(k)}+w^{(k)}\in V+W$. Taking into account that for large enough k,

$$-\|u\|_2\le\left\langle J'\left(u^{(k)}\right),u\right\rangle=-\left\langle H'\left(u^{(k)}\right),u\right\rangle+\sum_{n=1}^{mT}f\left(n,u_{n+M}^{(k)},u_n^{(k)},u_{n-M}^{(k)}\right)u_n,$$

in view of (F_2) and (F_3), we obtain

$$\left\langle H'\left(u^{(k)}\right),v^{(k)}\right\rangle\le\sum_{n=1}^{mT}f\left(n,u_{n+M}^{(k)},u_n^{(k)},u_{n-M}^{(k)}\right)v_n^{(k)}+\left\|v^{(k)}\right\|_2$$

$$\le 2M_0\sum_{n=1}^{mT}\left|v_n^{(k)}\right|+\left\|v^{(k)}\right\|_2\ \le\left(2M_0\sqrt{mT}+1\right)\left\|v^{(k)}\right\|_2.$$

On the other hand, we have

$$\left\langle H'\left(u^{(k)}\right),v^{(k)}\right\rangle=\sum_{n=1}^{mT}p_n\left(\left(\Delta v_{n-1}^{(k)}\right)^{\delta},\Delta v_{n-1}^{(k)}\right)$$

$$=\left[\left(\sum_{n=1}^{mT}p_{n+1}\left(\Delta v_n^{(k)}\right)^{\delta+1}\right)^{\frac{1}{\delta+1}}\right]^{\delta+1}=(\delta+1)H\left(v^{(k)}\right).$$

Since

$$\frac{\underline{p}}{\delta+1}c_1^{\delta+1}\left[\left(\sum_{n=1}^{mT}\left(\Delta v_n^{(k)}\right)^2\right)^{\frac{1}{2}}\right]^{\delta+1}\le H\left(v^{(k)}\right)\le\frac{\bar{p}}{\delta+1}c_2^{\delta+1}\left[\left(\sum_{n=1}^{mT}\left(\Delta v_n^{(k)}\right)^2\right)^{\frac{1}{2}}\right]^{\delta+1}$$

and

$$\lambda_{\min}\left\|v^{(k)}\right\|_2^2\le\sum_{n=1}^{mT}\left(\Delta v_n^{(k)}\right)^2=\left(v^{(k)}\right)^*P\left(v^{(k)}\right)\le\lambda_{\max}\left\|v^{(k)}\right\|_2^2,$$

we get

$$\frac{\underline{p}}{\delta+1}c_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\left\|v^{(k)}\right\|_2^{\delta+1}\le H\left(v^{(k)}\right)\le\frac{\bar{p}}{\delta+1}c_2^{\delta+1}\lambda_{\max}^{\frac{\delta+1}{2}}\left\|v^{(k)}\right\|_2^{\delta+1}. \tag{2.6}$$

Thus, we have

$$\underline{p}c_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\left\|v^{(k)}\right\|_2^{\delta+1}\le\left(2M_0\sqrt{mT}+1\right)\left\|v^{(k)}\right\|_2,$$

implying that $\{v^{(k)}\}$ is bounded. Next, we prove that $\{w^{(k)}\}$ is bounded. Since

$$M_2\ge J\left(u^{(k)}\right)=-H\left(u^{(k)}\right)+\sum_{n=1}^{mT}F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)= \tag{2.7}$$

$$= -H\left(v^{(k)}\right)+\sum_{n=1}^{mT}\left[F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)-F\left(n,w_{n+M}^{(k)},w_n^{(k)}\right)\right]+\sum_{n=1}^{mT}F\left(n,w_{n+M}^{(k)},w_n^{(k)}\right),$$

in view of (2.6) and (2.7), we can write

$$\sum_{n=1}^{mT}F\left(n,w_{n+M}^{(k)},w_n^{(k)}\right)\le M_2+\frac{\bar{p}}{\delta+1}c_2^{\delta+1}\lambda_{\max}^{\frac{\delta+1}{2}}\left\|v^{(k)}\right\|_2^{\delta+1}+$$

$$+\sum_{n=1}^{mT}\left|\frac{\partial F(n,\theta v_{n+M}^{(k)}+w_{n+M}^{(k)},u_n^{(k)})}{\partial v_1}v_{n+M}^{(k)}+\frac{\partial F(n,w_{n+M}^{(k)},\theta v_n^{(k)}+w_n^{(k)})}{\partial v_2}v_n^{(k)}\right|\le$$

$$\le M_2+\frac{\bar{p}}{\delta+1}c_2^{\delta+1}\lambda_{\max}^{\frac{\delta+1}{2}}\left\|v^{(k)}\right\|_2^{\delta+1}+2M_0\sqrt{mT}\left\|v^{(k)}\right\|_2,$$

where $\theta\in(0,1)$. It is not difficult to see that $\left\{\sum_{n=1}^{mT}F\left(n,w_{n+M},w_n^{(k)}\right)\right\}$ is bounded.

By (F_3), $\{w^{(k)}\}$ is bounded. Otherwise, assume that $\|w^{(k)}\|_2\to+\infty$ as $k\to\infty$. Since there exist $z^{(k)}\in\mathbf{R}$, $k\in\mathbf{N}$, such that $w^{(k)}=\left(z^{(k)},z^{(k)},\cdots,z^{(k)}\right)^*\in E_{mT}$, for $k\to\infty$ we have

$$\|w^{(k)}\|_2=\left(\sum_{n=1}^{mT}\left|w_n^{(k)}\right|^2\right)^{\frac12}=\left(\sum_{n=1}^{mT}\left|z^{(k)}\right|^2\right)^{\frac12}=\sqrt{mT}\left|z^{(k)}\right|\to+\infty.$$

Since $F\left(n,w_{n+M}^{(k)},w_n^{(k)}\right)=F\left(n,z^{(k)},z^{(k)}\right)$, then $F\left(n,w_{n+M}^{(k)},w_n^{(k)}\right)\to+\infty$ as $k\to\infty$. This contradicts the fact that $\left\{\sum_{n=1}^{mT}F\left(n,w_{n+M}^{(k)},w_n^{(k)}\right)\right\}$ is bounded, and shows that J satisfies the P.S. condition. Lemma 2.2 is proved. □

Lemma 2.3. *Assume that the conditions (F_1), (F_4) and (F_5) are satisfied. Then J satisfies the P.S. condition.*

Proof. Let $\{u^{(k)}\}\subset E_{mT}$ be such that $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)})\to0$ as $k\to\infty$. Then there exists a positive constant M_3 such that $\left|J\left(u^{(k)}\right)\right|\le M_3$. For k large enough, we have

$$\left|\left\langle J'\left(u^{(k)}\right),u^{(k)}\right\rangle\right|\le\left\|u^{(k)}\right\|_2.$$

Therefore

$$M_3+\frac{1}{\delta+1}\left\|u^{(k)}\right\|_2\ge J\left(u^{(k)}\right)-\frac{1}{\delta+1}\left\langle J'\left(u^{(k)}\right),u^{(k)}\right\rangle=$$

$$=\sum_{n=1}^{mT}\left[F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)-\frac{1}{\delta+1}\left(\frac{\partial F\left(n-M,u_n^{(k)},u_{n-M}^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)}+\right.\right.$$

$$\left.\left.+\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)}\right)\right]=\sum_{n=1}^{mT}\left[F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)-\right.$$

$$-\frac{1}{\delta+1}\left(\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_1}\cdot u_{n+M}^{(k)}+\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)}\right)\Bigg].$$

Next, taking

$$I_1=\left\{n\in\mathbf{Z}(1,mT)|\sqrt{\left(u_{n+M}^{(k)}\right)^2+\left(u_n^{(k)}\right)^2}\geq R_1\right\},$$

$$I_2=\left\{n\in\mathbf{Z}(1,mT)|\sqrt{\left(u_{n+M}^{(k)}\right)^2+\left(u_n^{(k)}\right)^2}< R_1\right\},$$

in view of (F_4), we can write

$$M_3+\frac{1}{\delta+1}\left\|u^{(k)}\right\|_2\geq\sum_{n=1}^{mT}F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)-$$

$$-\frac{1}{\delta+1}\sum_{n\in I_1}\left[\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_1}\cdot u_{n+M}^{(k)}+\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)}\right]$$

$$-\frac{1}{\delta+1}\sum_{n\in I_2}\left[\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_1}\cdot u_{n+M}^{(k)}+\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)}\right]$$

$$\geq\sum_{n=1}^{mT}F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)-\frac{\alpha}{2}\sum_{n\in I_1}F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)$$

$$-\frac{1}{\delta+1}\sum_{n\in I_2}\left[\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_1}\cdot u_{n+M}^{(k)}+\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)}\right]$$

$$=\left(1-\frac{\alpha}{2}\right)\sum_{n=1}^{mT}F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)+\frac{1}{\delta+1}\sum_{n\in I_2}\left[\frac{\alpha}{2}(\delta+1)F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)-\right.$$

$$\left.-\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_1}\cdot u_{n+M}^{(k)}-\frac{\partial F\left(n,u_{n+M}^{(k)},u_n^{(k)}\right)}{\partial v_2}\cdot u_n^{(k)}\right].$$

The continuity of $\frac{\alpha}{2}(\delta+1)F(n,v_1,v_2)-\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1-\frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2$ with respect to the second and third variables implies that there exists a constant $M_4>0$ such that

$$\frac{\alpha}{2}(\delta+1)F(n,v_1,v_2)-\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1-\frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2\geq -M_4,$$

for $n\in\mathbf{Z}(1,mT)$ and $\sqrt{v_1^2+v_2^2}\leq R_1$. Therefore, by (F_5), we get

$$M_3+\frac{1}{\delta+1}\left\|u^{(k)}\right\|_2\geq\left(1-\frac{\alpha}{2}\right)a_1\sum_{n=1}^{mT}\left|u_n^{(k)}\right|^{\frac{\gamma}{2}(\delta+1)}-M_5,$$

where $M_5=\left(1-\frac{\alpha}{2}\right)a_2mT+\frac{1}{\delta+1}mTM_4$.

Combining the last inequality with (2.4), we obtain

$$\left(1-\frac{\alpha}{2}\right)a_1c_1^{\frac{\alpha}{2}(\delta+1)}\left\|u^{(k)}\right\|_2^{\frac{\alpha}{2}(\delta+1)}-\frac{1}{\delta+1}\left\|u^{(k)}\right\|_2\leq M_3+M_5.$$

This implies that the sequence $\{\|u^{(k)}\|_2\}$ is bounded on the finite dimensional space E_{mT}, and as a consequence, it contains a convergent subsequence. Lemma 2.3 is proved. □

3. Proofs of the main results

In this Section, we prove our main results by using the critical point theory.

Proof of Theorem 1.1. Observe first that by Lemma 2.2, the functional J satisfies the P.S. condition. Hence, in order to prove Theorem 1.1 by using the Saddle Theorem, we need to verify the conditions (J_1) and (J_2). From (2.8) and (F_2'), for any $v\in V$, we have

$$J(v)=-H(v)+\sum_{n=1}^{mT}F(n,v_{n+M},v_n)$$

$$\leq-\frac{p}{\delta+1}c_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\|v\|_2^{\delta+1}+mTM_1+M_0\sum_{n=1}^{mT}(|v_{n+M}|+|v_n|)$$

$$\leq-\frac{p}{\delta+1}c_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\|v\|_2^{\delta+1}+mTM_1+2M_0\sqrt{mT}\|v\|_2\to-\infty \text{ as } \|v\|_2\to+\infty.$$

Therefore, it is easy to see that the condition (J_1) is satisfied.

To verify the condition (J_2), notice that for any $w\in W$, $w=(w_1,w_2,\cdots,w_{mT})^*$ there exists $z\in\mathbf{R}$ such that $w_n=z$ for all $n\in\mathbf{Z}(1,mT)$. Next, in view of (F_3), there exists a constant $R_0>0$ such that $F(n,z,z)>0$ for $n\in\mathbf{Z}$ and $|z|>\frac{R_0}{\sqrt{2}}$. Let $M_6=\min_{n\in\mathbf{Z},|z|\leq R_0/\sqrt{2}}F(n,z,z)$ and $M_7=\min\{0,M_6\}$. Then we have

$$F(n,z,z)\geq M_7,\ \ (n,z,z)\in\mathbf{Z}\times\mathbf{R}^2.$$

Therefore

$$J(w)=\sum_{n=1}^{mT}F(n,w_{n+M},w_n)=\sum_{n=1}^{mT}F(n,z,z)\geq mTM_7,\ \ w\in W,$$

implying that the condition (J_2) is satisfied. Thus, the conditions of (J_1) and (J_2) are satisfied. and the result follows. Theorem 1.1 is proved. □

Proof of Theorem 1.2. By Lemma 2.3, the functional J satisfies the P.S. condition. Hence to apply the Saddle Point Theorem, it is enough to show that J satisfies the conditions (J_1) and (J_2).

To this end, observe first that for any $w \in W$, since $H(w) = 0$, we have

$$J(w) = \sum_{n=1}^{mT} F(n, w_{n+M}, w_n),$$

and by (F_5)

$$J(w) \geq a_1 \sum_{n=1}^{mT} \left(\sqrt{w_{n+M}^2 + w_n^2}\right)^{\frac{\beta}{2}(\delta+1)} - a_2 mT \geq -a_2 mT.$$

Combining this with (F_4'), (2.4) and (2.8), for any $v \in V$, we can write

$$\begin{aligned} J(v) &\leq -\frac{p}{\delta+1} c_1^{\delta+1} \lambda_{\min}^{\frac{\delta+1}{2}} \|v\|_2^{\delta+1} + a_3 \sum_{n=1}^{mT} \left(\sqrt{v_{n+M}^2 + v_n^2}\right)^{\frac{\alpha}{2}(\delta+1)} + a_4 mT \\ &\leq -\frac{p}{\delta+1} c_1^{\delta+1} \lambda_{\min}^{\frac{\delta+1}{2}} \|v\|_2^{\delta+1} + a_3 c_2^{\frac{\alpha}{2}(\delta+1)} \left[\sum_{n=1}^{mT} \left(v_{n+M}^2 + v_n^2\right)\right]^{\frac{\alpha}{2}(\delta+1)} + a_4 mT \\ &\leq -\frac{p}{\delta+1} c_1^{\delta+1} \lambda_{\min}^{\frac{\delta+1}{2}} \|v\|_2^{\delta+1} + 2^{\frac{\alpha}{2}(\delta+1)} a_3 c_2^{\frac{\alpha}{2}(\delta+1)} \|v\|_2^{\frac{\alpha}{2}(\delta+1)} + a_4 mT. \end{aligned}$$

Let $\mu = -a_2 mT$. Since $1 < \alpha < 2$, there exists a constant $\rho > 0$ large enough such that

$$J(v) \leq \mu - 1 < \mu, \ \forall v \in V, \ \|v\|_2 = \rho.$$

Therefore, by Lemma 2.1, the equation (1.1) has at least one mT-periodic solution. Theorem 1.2 is proved. □

Proof of Theorem 1.3. Similar to the proof of Lemma 2.3, we can show that the functional J satisfies the P.S. condition. We prove the theorem by using the Saddle Point Theorem. We first verify the condition (J_1). To this end, observe that (F_4) clearly implies (F_4'). Hence for any $v \in V$, by (F_4') and (2.4), we have $J(v) \to -\infty$ as $\|v\|_2 \to +\infty$.

Next, we show that J satisfies the condition (J_2). For any given $v_0 \in V$ and $w \in W$, we set $u = v_0 + w$. Then we can write

$$\begin{aligned} J(u) &= -H(u) + \sum_{n=1}^{mT} F(n, u_{n+M}, u_n) = -H(v_0) + \sum_{n=1}^{mT} F(n, (v_0)_{n+M} + w_{n+M}, (v_0)_n + w_n) \\ &\geq -\frac{\bar{p}}{\delta+1} c_2^{\delta+1} \lambda_{\max}^{\frac{\delta+1}{2}} \|v_0\|_2^{\delta+1} + a_5 \sum_{n=1}^{mT} |(v_0)_n + w_n|^{\frac{\gamma}{2}(\delta+1)} \\ &\geq -\frac{\bar{p}}{\delta+1} c_2^{\delta+1} \lambda_{\max}^{\frac{\delta+1}{2}} \|v_0\|_2^{\delta+1} + a_5 c_1^{\frac{\gamma}{2}(\delta+1)} \left[\sum_{n=1}^{mT} |(v_0)_n + w_n|^2\right]^{\frac{\gamma}{2}(\delta+1)} \\ &= -\frac{\bar{p}}{\delta+1} c_2^{\delta+1} \lambda_{\max}^{\frac{\delta+1}{2}} \|v_0\|_2^{\delta+1} + a_5 c_1^{\frac{\gamma}{2}(\delta+1)} \left[\|v_0\|_2^2 + \|w\|_2^2\right]^{\frac{\gamma}{2}(\delta+1)} \end{aligned}$$

$$\geq -\frac{\bar{p}}{\delta+1} c_2^{\delta+1} \lambda_{\max}^{\frac{\delta+1}{2}} \|v_0\|_2^{\delta+1} + a_5 c_1^{\frac{\gamma}{2}(\delta+1)} \|v_0\|_2^{\frac{\gamma}{2}(\delta+1)} + a_5 c_1^{\frac{\gamma}{2}(\delta+1)} \|w\|_2^{\frac{\gamma}{2}(\delta+1)}.$$

Since $1 < \gamma < 2$, there exists a constant $\eta > 0$ small enough such that

$$J(v_0 + w) \geq \eta^{\frac{\gamma}{2}(\delta+1)} \left[a_5 c_1^{\frac{\gamma}{2}(\delta+1)} - \frac{\bar{p}}{\delta+1} c_2^{\delta+1} \lambda_{\max}^{\frac{\delta+1}{2}} \eta^{\delta+1-\frac{\gamma}{2}(\delta+1)} \right] =: \nu > 0,$$

for $v_0 \in V$ with $\|v_0\|_2 = \eta$ and for any $w \in W$. Then for $v_0 \in V$ and for any $w \in W$, we get $\|v_0\|_2 = \eta$ and $J(v_0 + w) \geq \nu > 0$.

Hence in view of Saddle Point Theorem there exists a critical point $\bar{u} \in E_{mT}$, which corresponds to a mT-periodic solution of equation (1.1).

Noting that $J(0) = 0$ and $J(\bar{u}) \geq \nu > 0$, we conclude that the critical point $\bar{u}$ of J is a nontrivial mT-periodic solution of equation (1.1). This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. The proof repeated the same arguments as those used in the proof of Theorem 1.3, and so we omit the details.

4. Examples

As an application of the main theorems, we give two examples to illustrate our results.

Example 4.1. For all $n \in \mathbf{Z}$ consider the equation:

$$\Delta\left(p_n (\Delta u_{n-1})^{\delta}\right) + \alpha(\delta+1) u_n \left[\psi(n) \left(u_{n+M}^2 + u_n^2\right)^{\frac{\alpha}{2}(\delta+1)-1} + \psi(n-M) \left(u_n^2 + u_{n-M}^2\right)^{\frac{\alpha}{2}(\delta+1)-1} \right] = 0, \tag{4.1}$$

where $\{p_n\}$ is a sequence of real numbers, $\delta > 0$ is the ratio of odd naturals M is a given nonnegative integer, ψ is continuously differentiable and $\psi(n) > 0$, T is a given positive integer, $p_{n+T} = p_n > 0$, $\psi(n+T) = \psi(n)$ and $1 < \alpha < 2$. We have

$$f(n, v_1, v_2, v_3) = \alpha(\delta+1) v_2 \left[\psi(n) \left(v_1^2 + v_2^2\right)^{\frac{\alpha}{2}(\delta+1)-1} + \psi(n-M) \left(v_2^2 + v_3^2\right)^{\frac{\alpha}{2}(\delta+1)-1} \right]$$

$$F(n, v_1, v_2) = \psi(n) \left(v_1^2 + v_2^2\right)^{\frac{\alpha}{2}(\delta+1)}.$$

Therefore

$$\frac{\partial F(n-M, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} =$$

$$= \alpha(\delta+1) v_2 \left[\psi(n) \left(v_1^2 + v_2^2\right)^{\frac{\alpha}{2}(\delta+1)-1} + \psi(n-M) \left(v_2^2 + v_3^2\right)^{\frac{\alpha}{2}(\delta+1)-1} \right].$$

It is easy to verify that all the assumptions of Theorem 1.3 are satisfied. Consequently, for any given natural m equation (4.1) has at least one nontrivial mT-periodic solution.

Example 4.2. For all $n \in \mathbf{Z}$ consider equation (4.1) for $\psi(n) = 6 + \cos^2(\frac{n\pi}{T})$, $\alpha \in (0, 2)$. It is easy to verify that all the assumptions of Theorem 1.4 are satisfied. Consequently, for any given natural m equation (4.1) has at least one nontrivial mT-periodic solution.

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