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FRAMES AND NON LINEAR APPROXIMATIONS IN HILBERT SPACES

K. T. POUMAI AND S. K. KAUSHIK

Motilal Nehru College, University of Delhi, India
Department of Mathematics, Kirori mal college. University of Delhi, India
E-mails: *kholetim@yahoo.co.in; shikk2003@yahoo.co.in*

Abstract. A simple proof of the proposition, stated in ([2], p. 346), asserting that in Hilbert spaces a Riesz basis is greedy, is given. Also, greedy approximant for frames in Hilbert spaces is defined and it is shown that frames satisfy the quasi greedy and almost greedy conditions. Finally, we give the characterizations of approximation spaces $\mathcal{A}^s(\Psi)$, $\mathcal{A}^s_q(\Psi)$ by means of weak-ℓ_p and Lorentz sequence spaces for frames.

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1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle .,. \rangle$.

Definition 1.1. (a) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ is called a *Riesz basis* for $\mathcal{H}$, if $\{x_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$ and there exist constants $A, B > 0$ such that

$$A\sum_{n=1}^{\infty} \|a_n\|^2 \leq \| \sum_{n=1}^{\infty} a_n x_n\|^2 \leq B\sum_{n=1}^{\infty} |a_n|^2 \text{ for all } \{a_n\}_{n=1}^{\infty} \in \ell^2.$$

(b) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2 \text{ for all } x \in \mathcal{H}. \tag{1.1}$$

The constants A and B in (1.1) are called the *lower* and *upper frame bounds* of the frame, respectively. They are not unique. If $A = B$, then $\{x_n\}$ is called an *A-tight frame* and if $A = B = 1$, then $\{x_n\}$ is called a *Parseval frame*. The inequality in (1.1) is called the *frame inequality*. The operator $T : \ell^2 \to \mathcal{H}$ defined as

$$T(\{c_k\}) = \sum_{k=1}^{\infty} c_k x_k, \ \{c_k\} \in \ell^2,$$

is called the *pre-frame operator* (or *synthesis operator*) and its adjoint operator T^* : $\mathcal{H} \to \ell^2$, is called the *analysis operator* and is given by

$$T^*(x) = \{\langle x, x_k \rangle\}, \; x \in \mathcal{H}.$$

Composing T and T^* we obtain the *frame operator* $S = TT^* : \mathcal{H} \to \mathcal{H}$ given by

$$S(x) = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k, \; x \in \mathcal{H}.$$

Observe that the frame operator S is a positive, self-adjoint and invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* for all $x \in \mathcal{H}$,

$$x = SS^{-1}x = \sum_{k=1}^{\infty} \langle S^{-1}x, x_k \rangle x_k = \sum_{k=1}^{\infty} \langle x, S^{-1}x_k \rangle x_k. \tag{1.2}$$

For more details related to frames and Riesz frames, see [4, 6].

The notion of *N-term error of approximation and thresholding greedy algorithm of order N* for Schauder basis in Banach spaces have been defined and studied in [9, 12, 13, 15].

Let $\mathcal{X}$ be a Banach space and (x_n, f_n) be Schauder basis for $\mathcal{X}$

$$\sum\nolimits_N = \{\sum_{n\in\sigma} a_n x_n : \sigma \subseteq \mathbb{N}, |\sigma| = N \in \mathbb{N}, a_n \texttt{ are scalars}\}$$

For $x \in \mathcal{X}$, $x = \sum_{n\in\mathbb{N}} f_n(x) x_n$ we define

$$\widetilde{\sum}_N = \{\sum_{n\in\sigma} f_n(x) x_n : \sigma \subseteq \mathbb{N}, |\sigma| = N \in \mathbb{N}\}.$$

For each $x \in \mathcal{X}$ the *N-term error of approximation* is defined by

$$\sigma_N(x) = \inf\{\|x - y\| : y \in \sum\nolimits_N\}.$$

$$\widetilde{\sigma}_N(x) = \inf\{\|x - y\| : y \in \widetilde{\sum}_N\}.$$

Let $\gamma = \{n_k\}_{k=1}^{\infty}$ be a permutation of natural numbers such that

$$|f_{n_1}(x)| \geq |f_{n_2}(x)| \geq |f_{n_3}(x)| \geq .....$$

We now define the N-greedy approximant as

$$G_N(x) = \sum_{k=1}^{N} f_{n_k}(x) x_{n_k}.$$

Definition 1.2. A Schauder basis (x_n, f_n) is said to be quasi greedy if there exists a constant C such that $\|G_N(x)\| \leq C\|x\|$ for all $x \in \mathcal{X}$.

Definition 1.3. A Schauder basis (x_n, f_n) is said to be almost greedy if there exists a constant C such that $\|x - G_N(x)\| \leq C\widetilde{\sigma}_N(x)$ for all $x \in \mathcal{X}$.

Definition 1.4. A Schauder basis (x_n, f_n) is said to be greedy if there exists a constant C such that $\|x - G_N(x)\| \leq C\sigma_N(x)$ for all $x \in \mathcal{X}$.

In [9], the following relation between greedy basis, almost greedy basis and quasi greedy basis has been given:

Greedy basis ⇒ almost greedy basis ⇒ quasi greedy basis.

Remark 1.1. Let $\{x_n\}$ be an orthonormal basis for Hilbert space $\mathcal{H}$, then $\{x_n\}$ is greedy.

2. Riesz basis and greedy approximation

We begin this section with a simple new proof of the proposition, stated in ([2], p. 346), asserting that in Hilbert spaces Riesz bases are greedy. This theorem follows from the fact that a Riesz basis is L_2-equivalent to the Haar basis and using Theorem 2.1 of [13]. It can also be deduced from the fact that a Riesz basis is democratic and using Theorem 1 of [14]. Here we give a formal proof of this result with improved constant in the Greedy estimate.

Theorem 2.1. *Let $\{x_n\}$ be a Riesz basis for Hilbert space $\mathcal{H}$ with bounds A and B. Then, for any $N \in \mathbb{N}$*

$$\|x - G_N(x)\| \leq \sqrt{\frac{B}{A}}\sigma_N(x) \text{ for all } x \in \mathcal{H}.$$

We first provide some terminology which is required in the proof of Theorem 2.1. Let $\{x_n\}$ be a frame for Hilbert space $\mathcal{H}$ and $\sigma \subseteq \mathbb{N}$ with $|\sigma| = N \in \mathbb{N}$. We denote $\mathcal{H}_\sigma = \text{span}_{n\in\sigma}\{x_n\}$. As in [5], $\{x_n\}_{n\in\sigma}$ is a frame for $\mathcal{H}_\sigma$. Let V_σ be a frame operator for the frame $\{x_n\}_{n\in\sigma}$ of $\mathcal{H}_\sigma$. Since $\mathcal{H}_\sigma$ is a closed subspace of $\mathcal{H}$, there is an orthogonal projection P_σ from $\mathcal{H}$ onto $\mathcal{H}_\sigma$. Thus, for $x \in \mathcal{H}$ we have

$$\begin{aligned} P_\sigma(x) &= \sum_{n\in\sigma}\langle P_\sigma(x), V_\sigma^{-1}x_n\rangle x_n = \sum_{n\in\sigma}\langle x, P_\sigma V_\sigma^{-1}x_n\rangle x_n \\ &= \sum_{n\in\sigma}\langle x, V_\sigma^{-1}x_n\rangle x_n. \end{aligned}$$

Lemma 2.1. *Let $\{x_n\}$ be a frame for Hilbert space $\mathcal{H}$, σ be a finite subset of $\mathbb{N}$ and P_σ be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_\sigma$. Then, for $x \in \mathcal{H}$ we have*

$$\sigma_N(x) = \inf_\sigma\{\|x - P_\sigma(x)\| : \sigma \subseteq \mathbb{N},\ |\sigma| = N\}.$$

Proof. For $x \in \mathcal{H}$ we have

$$\|x - P_\sigma(x)\| = \text{dist}(x, \mathcal{H}_\sigma) = \inf\{\|x - y\| : y \in \mathcal{H}_\sigma \}.$$

So, for any $y \in \mathcal{H}_\sigma$ we have $\|x - P_\sigma(x)\| \leq \|x - y\|$. Hence

$$\sigma_N(x) = \inf_\sigma\{\|x - P_\sigma(x)\| : \sigma \subseteq \mathbb{N}, |\sigma| = N\}.$$

Let $\{x_n\}$ be a Riesz basis for $\mathcal{H}$ and S be a frame operator for $\{x_n\}$. For $\sigma \subseteq \mathbb{N}$, define the following operators:

(i) $S_\sigma : \mathcal{H} \to \mathcal{H}$ as

$$S_\sigma(x) = \sum_{n\in\sigma} \langle x, x_n\rangle x_n, \text{ for } \sigma \subseteq \mathbb{N}, \ |\sigma| = N.$$

(ii) $Q_\sigma : \mathcal{H} \to \mathcal{H}$ as

$$Q_\sigma(x) = S_\sigma S^{-1}(x) = \sum_{n\in\sigma} \langle x, S^{-1}x_n\rangle x_n.$$

Let $\rho = \{n_k\}_{k=1}^{\infty}$ be a permutation of natural numbers such that

$$|\langle x, S^{-1}x_{n_1}\rangle| \geq |\langle x, S^{-1}x_{n_2}\rangle| \geq |\langle x, S^{-1}x_{n_3}\rangle| \geq \ldots.$$

The N-greedy approximant is given by $G_N(x) = \sum\limits_{k=1}^{N} \langle x, S^{-1}x_{n_k}\rangle x_{n_k}$.

Remark 2.1. ([4]) A Riesz basis $\{x_n\}$ for $\mathcal{H}$ is a frame for $\mathcal{H}$, and the Riesz basis bounds coincide with the frame bounds.

Proof. of Theorem 2.1 by Remark 2.1, a Riesz basis is a frame for $\mathcal{H}$ with the same bounds. Since A and B are the bounds of the Riesz basis $\{x_n\}$, by the frame inequality we have

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \ x \in \mathcal{H}.$$

Note that

$$\begin{aligned}\|x - G_N(x)\|^2 &= \|\sum_{k>N} \langle x, S^{-1}x_{n_k}\rangle x_{n_k}\|^2 = \sup_{y\in H, \|y\|=1} |\sum_{k>N} \langle x, S^{-1}x_{n_k}\rangle \langle x_{n_k}, y\rangle|^2 \\ &\leq \sum_{k>N} |\langle x, S^{-1}x_{n_k}\rangle|^2 \sup_{y\in\mathcal{H}, \|y\|=1} \sum_{k>N} |\langle x_{n_k}, y\rangle|^2 \leq B \sum_{k>N} |\langle x, S^{-1}x_{n_k}\rangle|^2\end{aligned}$$

Also, by the definition of greedy approximant, for $x \in \mathcal{H}$, we have

$$\sum_{k>N} |\langle x, S^{-1}x_{n_k}\rangle|^2 \leq \sum_{n\in\mathbb{N}\setminus\sigma} |\langle x, S^{-1}x_n\rangle|^2, \text{for any } \sigma \subseteq \mathbb{N}, |\sigma| = N.$$

Thus, for any $\sigma \subseteq \mathbb{N}$ with $|\sigma| = N$, we have

$$\|x - G_N(x)\|^2 \leq B \sum_{n\in\mathbb{N}\setminus\sigma} |\langle x, S^{-1}x_n\rangle|^2, \text{ for } x \in \mathcal{H}.$$

Let V_σ be the frame operator for the frame $\{x_n\}_{n\in\sigma}$ of $\operatorname{span}\{x_n\}_{n\in\sigma}$, and let P_σ be the orthogonal projection from $\mathcal{H}$ onto $\operatorname{span}\{x_n\}_{n\in\sigma}$, given by $P_\sigma(x) = \sum_{n\in\sigma} \langle x, V_\sigma^{-1}x_n\rangle x_n$ (see Theorem 1.1.8 in [4]). Also, by inequalities of canonical dual frame of $\{x_n\}$, we have

$$\frac{1}{A}\|x - P_\sigma(x)\|^2 \geq \sum_{n\in\mathbb{N}} |\langle x - P_\sigma(x), S^{-1}x_n\rangle|^2 \geq \sum_{n\in\mathbb{N}\setminus\sigma} |\langle x, S^{-1}x_n\rangle - \langle P_\sigma(x), S^{-1}x_n\rangle|^2$$

$$= \sum_{n\in\mathbb{N}\setminus\sigma} |\langle x, S^{-1}x_n\rangle - \sum_{j\in\sigma}\langle x, V_\sigma^{-1}x_j\rangle\langle x_j, S^{-1}x_n\rangle|^2 = \sum_{n\in\mathbb{N}\setminus\sigma} |\langle x, S^{-1}x_n\rangle|^2.$$

Thus, for any $\sigma \subseteq \mathbb{N}$ with $|\sigma| = N$, we obtain

$$\|x - G_N(x)\|^2 \leq \frac{B}{A}\|x - P_\sigma(x)\|^2 \text{ for } x \in \mathcal{H}.$$

Hence, by Lemma 2.2, we get

$$\|x - G_N(x)\| \leq \sqrt{\frac{B}{A}}\, \sigma_N(x) \text{ for } x \in \mathcal{H}.$$

Theorem 2.1 is proved.

3. Frames and Greedy approximation

Let $\Psi = \{x_n\}$ be a frame for Hilbert space $\mathcal{H}$ with canonical dual frame $\{S^{-1}x_n\}$, and let $x = \sum_{n=1}^{\infty} \langle x, S^{-1}x_n\rangle x_n$ for all $x \in \mathcal{H}$. Define the nonlinear *N-term approximation manifolds* for frames, in the similar manner as we have defined for Schauder basis, as follows:

$$\sum\nolimits_N(\Psi) = \left\{\sum_{n\in\sigma} a_n x_n : \sigma \subseteq \mathbb{N}, |\sigma| = N,\ a_n \text{ are scalars}\right\},$$

$$\overline{\sum}_N(\Psi) = \left\{\sum_{n\in\sigma} \langle x, S^{-1}x_n\rangle x_n : \sigma \subseteq \mathbb{N},\ x \in \mathcal{H},\ |\sigma| = N\right\}.$$

We define the *N-term approximation errors* as

$$\sigma_N(x) = \inf\{\|x - y\| : y \in \sum\nolimits_N(\Psi)\},$$

$$\bar{\sigma}_N(x) = \inf\{\|x - y\| : y \in \overline{\sum}_N(\Psi)\}.$$

Also, define S_σ, $Q_\sigma : \mathcal{H} \to \mathcal{H}$ as

$$S_\sigma(x) = \sum_{n\in\sigma}\langle x, x_n\rangle x_n, \ \sigma \subseteq \mathbb{N}, \ |\sigma| = N,$$

$$Q_\sigma(x) = S_\sigma(S^{-1}(x)) = \sum_{n\in\sigma}\langle x, S^{-1}x_n\rangle x_n.$$

Let $\rho = \{n_k\}_{k=1}^{\infty}$ be a permutation of natural numbers such that

$$|\langle x, S^{-1}x_{n_1}\rangle|^2 \geq |\langle x, S^{-1}x_{n_2}\rangle|^2 \geq |\langle x, S^{-1}x_{n_3}\rangle|^2 \geq \cdots$$

Now, define the *N-greedy approximant* for a frame $\{x_n\}$ as

$$G_N(x) = \sum_{k=1}^{N}\langle x, S^{-1}x_{n_k}\rangle x_{n_k} \text{ for } x \in \mathcal{H}.$$

We have $G_N(x) = Q_{\sigma_0}(x)$ for some $\sigma_0 \subseteq \mathbb{N}$, $|\sigma_0| = N$.

Lemma 3.1. ([6]) *Let $\{x_n\}$ be a frame with bounds A, B, and let T be its synthesis operator. Then $\ell^2 = \ker T \oplus \operatorname{Ran} T^*$. Moreover, we have*

$$A\sum_n |\alpha_n|^2 \leq \|\sum_n \alpha_n x_n\|^2 \leq B\sum_n |\alpha_n|^2 \ \textit{for all}\ \alpha = \{\alpha_n\} \in \operatorname{Ran} T^*.$$

Let $\{x_n\}$ be a frame for Hilbert space $\mathcal{H}$ and $\sigma \subset \mathbb{N}$. Define the operators T_σ, T_σ^* as follows:

$$T_\sigma(\{\alpha_j\}_{j\in\sigma}) = \sum_{j\in\sigma}\alpha_j x_j, \ \{\alpha_j\}_{j\in\sigma} \in \ell^2; \quad T_\sigma^*(x) = \{\langle x, x_j\rangle\}_{j\in\sigma}, \ x \in \mathcal{H}.$$

Observe that $S_\sigma(x) = T_\sigma T_\sigma^*(x) = \sum_{j\in\sigma}\langle x, x_j\rangle x_j$ for all $j \in \sigma$, and T_σ^* is the adjoint of operator T_σ.

Lemma 3.2. *Let $\Psi = \{x_n\}$ be a frame for Hilbert space $\mathcal{H}$ with bounds A and B, and let $\sigma \subset \mathbb{N}$. Then $\|S_\sigma(x)\| \leq B\|x\|$ for all $x \in \mathcal{H}$.*

Proof. Using the frame inequality, for $x \in \mathcal{H}$ we obtain

$$\|T_\sigma^*(x)\|^2 = \sum_{j\in\sigma}|\langle x, x_j\rangle|^2 \leq \sum_{j=1}^{\infty}|\langle x, x_j\rangle|^2 \leq B\|x\|^2.$$

So, we have $\|T_\sigma^*(x)\| \leq \sqrt{B}\|x\|$ for all $x \in \mathcal{H}$. Thus, we get

$$\|S_\sigma(x)\| = \|T_\sigma T_\sigma^*(x)\| \leq \|T_\sigma\|\|T_\sigma^*\|\|x\| \leq B\|x\|, \ x \in \mathcal{H}.$$

The next result shows that frames satisfy the quasi greedy condition.

Theorem 3.1. *Let $\Psi = \{x_n\}$ be a frame with bounds A and B. Then*

(1) $\|G_N(x)\| \le \frac{B}{A}\|x\|$ *for all* $x \in \mathcal{H}$.

(2) $\|x - G_N(x)\| \to 0$ *as* $N \to \infty$.

Proof. (1). Let $G_N(x) = Q_{\sigma_0}(x)$ for some $\sigma_0 \subseteq \mathbb{N}$ with $|\sigma_0| = N$. Then we have

$$\|G_N(x)\| = \|Q_{\sigma_0}(x)\| = \|S_{\sigma_0}S^{-1}(x)\| \le \|S_{\sigma_0}\|\|S^{-1}\| \cdot \|x\|.$$

Therefore, by Lemma 3.2, $\|S_{\sigma_0}\| \le B$ and $\|S^{-1}\| \le A^{-1}$. Hence $\|G_N(x)\| \le \frac{B}{A}\|x\|$.
To prove the assertion (2) observe that

$$\|x - G_N(x)\|^2 = \| \sum_{k>N} \langle x, S^{-1}x_{n_k}\rangle x_{n_k}\|^2 = \sup_{y\in\mathcal{H}, \|y\|=1} | \sum_{k>N} \langle x, S^{-1}x_{n_k}\rangle\langle x_{n_k}, y\rangle|^2$$

$$\le \sum_{k>N} |\langle x, S^{-1}x_{n_k}\rangle|^2 \sup_{y\in\mathcal{H}, \|x\|=1} \sum_{k>N} |\langle x_{n_k}, y\rangle|^2 \le B \sum_{k>N} |\langle x, S^{-1}x_{n_k}\rangle|^2.$$

By the definition of greedy algorithm, for any $\sigma \subseteq \mathbb{N}$ with $|\sigma| = N$, we have

$$\sum_{k>N} |\langle x, S^{-1}x_{n_k}\rangle|^2 \le \sum_{n\in\mathbb{N}\setminus\sigma} |\langle x, S^{-1}x_n\rangle|^2.$$

Thus, we obtain

$$\|x - G_N(x)\|^2 \le B \sum_{n=N+1}^{\infty} |\langle x, S^{-1}x_n\rangle|^2 \to 0 \, as \;\; N \to \infty$$

Theorem 3.1 is proved. Next, we show that frames also satisfy the almost greedy condition.

Theorem 3.2. *Let* $\Psi = \{x_n\}$ *be a frame for* H *with bounds* A *and* B. *Then*

$$\|x - G_N(x)\| \le \sqrt{\frac{B}{A}}\tilde{\sigma}_N(x).$$

Proof. As in the proof of Theorem 3.3, we have $\|x-G_N(x)\|^2 \le B \sum_{n\in\mathbb{N}\setminus\sigma} |\langle x, S^{-1}x_n\rangle|^2$ for any $\sigma \subset \mathbb{N}$ with $|\sigma| = N$ and $x \in H$. Also, by Lemma 3.1 we have

$$A\|a\|^2 \le \| \sum_{n=1}^{\infty} a_n x_n\|^2 \le B\|a\|^2 \;\; \text{for all } a = \{a_n\} \in \ker T^{\perp}.$$

Moreover, $\{\langle x, S^{-1}x_n\rangle\} = \{\langle S^{-1}x, x_n\rangle\} = T^*S^{-1}(x) \in \mathrm{Ran}T^* = \ker T^{\perp}$. So, for any $\sigma \in \mathbb{N}$ with $|\sigma| \subset N$ we obtain

$$\|x - G_N(x)\|^2 \le \frac{B}{A}\| \sum_{n\in\mathbb{N}\setminus\sigma} \langle x, S^{-1}x_n\rangle x_n\|^2 \text{ for all } x \in \mathcal{H}.$$

Now, let $y = \sum_{n\in\sigma} \langle x, S^{-1}x_n\rangle x_n \in \tilde{\sum}_N(\Psi)$. Then

$$\|x - y\|^2 = \| \sum_{n\in\mathbb{N}\setminus\sigma} \langle x, S^{-1}x_n\rangle x_n\|^2 \text{ for any } \sigma \in \mathbb{N} \text{ with } |\sigma| = N.$$

Thus,

$$\|x - G_N(x)\|^2 \leq \frac{B}{A}\|x - y\|^2 \text{ for any } y \in \widetilde{\sum}_N(\Psi).$$

Hence, $\|x - G_N(x)\| \leq \sqrt{\frac{B}{A}}\, \tilde{\sigma}_N(x)$ for $x \in \mathcal{H}$. Theorem 3.4 is proved.
Next, we define the *weak*-l_p and Lorentz sequence spaces. We also define the approximation spaces $\mathcal{A}^s(\Psi)$ and $\mathcal{A}^s_q(\Psi)$ for frames $\Psi = \{x_n\}$ in Hilbert spaces. In [8, 10, 12], greedy algorithms such as *Pure Greedy Algorithm*, *Relaxed Greedy Algorithm*, *Orthogonal Greedy Algorithm* for general *dictionaries* in Hilbert spaces have been defined and proved various Lebesgue-type inequalities for greedy approximations. In the following, we give the characterizations of approximation spaces by means of sequence spaces. We first define the *weak*-ℓ_p and Lorentz sequence spaces.

Definition 3.1. ([7, 11]) For $0 < p < \infty$, the *weak*-ℓ_p sequence space, denoted by $w\ell_p$, is defined to be the space of all sequences $\{a_n\}_{n=1}^{\infty}$ satisfying

$$\|\{a_n\}\|_{w\ell_p} = \sup_{n\geq 1} n^{1/p} a_n^* < \infty,$$

where $\{a_n^*\}_{n=1}^{\infty}$ is a nonincreasing rearrangement of $\{|a_n|\}_{n=1}^{\infty}$.

Definition 3.2. ([1, 7]) For $0 < p < \infty$ and $1 \leq q \leq \infty$, the Lorentz sequence space, denoted by $\ell_{p,q}$, is defined to be the space of all sequences $\{a_n\}_{n=1}^{\infty}$ satisfying

$$\|\{a_n\}\|_{\ell_{p,q}} = \left(\sum_{n=1}^{\infty} a_n^{*q} n^{q/p-1}\right)^{1/q} < \infty,$$

where $\{a_n^*\}_{n=1}^{\infty}$ is a nonincreasing rearrangement of $\{|a_n|\}_{n=1}^{\infty}$.

The notation $A \asymp B$ stands for $C_1 A \leq B \leq C_2 A$ with some constants $C_1, C_2 > 0$.

Remark 3.1. Note that

$$\|\{a_n\}\|_{\ell_{p,q}} \asymp \left(\sum_{j=0}^{\infty} 2^{jq/p} a_{2^j}^{*q}\right)^{1/q}. \tag{3.1}$$

In the following, we define the approximation spaces for frames.

Definition 3.3. ([7]) Let $\Psi = \{x_n\}$ be a frame. For $0 < s < \infty$, we define the approximation space $\mathcal{A}^s(\Psi)$ as the set of $x \in \mathcal{H}$ satisfying

$$\|x\|_{\mathcal{A}^s(\Psi)} = \sup_{n\geq 1} n^s \sigma_n(x) < \infty.$$

Definition 3.4. ([7]) Let $\Psi = \{x_n\}$ be a frame. For $0 < s < \infty$ and $1 \leq q \leq \infty$, we define the approximation space $\mathcal{A}_q^s(\Psi)$ as the set of $x \in \mathcal{H}$ satisfying

$$\|x\|_{\mathcal{A}_q^s(\Psi)} = \left(\sum_{n=1}^{\infty} \sigma_n^q(x) n^{qs-1}\right)^{1/q} < \infty.$$

We have (see [7])

$$\|x\|_{\mathcal{A}_q^s(\Psi)} = \left(\sum_{j=0}^{\infty} \sigma_{2^j}^q(x) 2^{jqs}\right)^{1/q}. \tag{3.2}$$

For the sake of completeness of our discussion related to the characterizations of approximation spaces, in the form of a remark, we state a result from [7].

Remark 3.2. Let $\Psi = \{x_n\}$ be an orthonormal basis for $\mathcal{H}$. Then

(a) $x \in \mathcal{A}^s(\Psi)$ if and only if $\{\langle x, x_n\rangle\} \in w\ell_p$, $\dfrac{1}{p} = s + \dfrac{1}{2}$, $p < 2$.

(b) $x \in \mathcal{A}_q^s(\Psi)$ if and only if $\{\langle x, x_n\rangle\} \in \ell_{p,q}$, $\dfrac{1}{p} = s + \dfrac{1}{2}$, $q < 2$.

Now the question of interest is: given a real number $s > 0$ such that for $x \in \mathcal{H}$ the error of N-term approximation for frames satisfies $\sigma_N(x) \leq M \cdot N^{-s}$; $N = 1, 2, 3, \ldots$, for some constant $M > 0$. The next result concerns this question.

Theorem 3.3. *Let $\Psi = \{x_n\}$ be a frame for $\mathcal{H}$ with bounds A and B. If $\{\langle x, S^{-1}x_n\rangle\} \in w\ell_p$, then $x \in \mathcal{A}^s(\mathcal{D})$, where $\dfrac{1}{p} = s + \dfrac{1}{2}$ and $p < 2$.*

Proof. Let $M = \|\{\langle x, S^{-1}x_n\rangle\}\|_{w\ell_p}$ and $\{n_k\}$ be a permutation of $\mathbb{N}$ such that

$$|\langle x, S^{-1}x_{n_1}\rangle| \geq |\langle x, S^{-1}x_{n_2}\rangle| \geq |\langle x, S^{-1}x_{n_3}\rangle| \ldots.$$

Now, take $c_k^* = |\langle x, S^{-1}x_{n_k}\rangle|$ for $k = 1, 2, 3, 4, \ldots$, and observe that by the assumption we have $k^{1/p} c_k^* \leq M$ for all $k \in \mathbb{N}$. Also, by Theorem 3.1, for $N \in \mathbb{N}$ we have

$$\|x - G_N(x)\|^2 \leq B \sum_{k=N+1}^{\infty} |\langle x, S^{-1}(x_{n_k})\rangle|^2 = B \sum_{k=N+1}^{\infty} c_k^{*2}, \; x \in \mathcal{H}.$$

Consider the dyadic sums $F_m^2 = \sum_{k=2^m}^{2^{m+1}-1} c_k^{*2}$, $m = 1, 2, 3, \ldots$ Then we have

$$F_m^2 \leq \sum_{k=2^m}^{2^{m+1}-1} M^2 k^{-2/p} \leq M^2 2^{-2ms} \text{ for all } m = 1, 2, 3, \ldots$$

Also, let $2^l \leq N \leq 2^{l+1}$ for $l = 0, 1, 2, 3, \dots$ Then we can write

$$N^{2s}\sigma_N^2(x) \leq N^{2s}\|x - G_N(x)\|^2 \leq N^{2s}B \sum_{k=N+1}^{\infty} c_k^{*2} \leq N^{2s}B \sum_{k=2^l}^{\infty} c_k^{*2}$$

$$\leq B2^{2s(l+1)} \sum_{k=2^l}^{\infty} c_k^{*2} = B2^{2s(l+1)} \sum_{m=l}^{\infty} F_m^2 \leq BM^2 2^{2s(l+1)} \sum_{m=l}^{\infty} 2^{-2ms} = BM^2 \frac{2^{4s}}{2^{2s}-1}.$$

Thus, $\sup_{1 \leq N < \infty} N^s \sigma_N(x) < \infty$ and the result follows. Theorem 3.3 is proved.

Next, we show that the converse of Theorem 3.3 is true for Riesz bases.

Theorem 3.4. *Let $\Psi = \{x_n\}$ be a Riesz basis for $\mathcal{H}$ with bounds A and B. Then*

$$x \in \mathcal{A}^s(\Psi) \text{ if and only if } \{\langle x, S^{-1}x_n\rangle\} \in w\ell_p,\ \frac{1}{p} = s + \frac{1}{2},\ p < 2.$$

Proof. If $\{\langle x, S^{-1}x_n\rangle\} \in w\ell_p$, then the result follows from Theorem 3.3.

Conversely, let $x \in \mathcal{A}^s$. Then for any finite subset $\sigma \subset \mathbb{N}$ with $|\sigma| = N$, $x \in \mathcal{H}$ and from the proof of Theorem 2.1 we have

$$\sum_{k=N+1}^{\infty} |\langle x, S^{-1}x_{n_k}\rangle|^2 \leq \frac{1}{A}\sigma_N^2(x) \text{ for all } x \in \mathcal{H}.$$

As in the proof of Theorem 3.3, take $c_k^* = |\langle x, S^{-1}x_{n_k}\rangle|$ for k=1, 2, 3,..., to obtain

$$c_{2N}^{*\,2} \leq N^{-1} \sum_{k=N+1}^{2N} c_k^{*2} \leq N^{-1} \sum_{k=N+1}^{\infty} c_k^{*2} \leq N^{-1}\frac{1}{A}\sigma_N^2(x).$$

Since a similar inequality holds for c_{2N+1}^*, we have the other implication of the asserted equivalence. Moreover, by assumption, $\sigma_N^2(x) \leq N^{-2s}\|x\|_{\mathcal{A}^s}^2$ for $N \geq 1$. Therefore

$$N^{2/p}c_N^* = N^{2s+1}c_N^* \leq \frac{1}{A}\|x\|_{\mathcal{A}^s}^2 < \infty \text{ for any } N \geq 1,$$

implying that $\{\langle x, S^{-1}x_n\rangle\} \in w\ell_p$. Theorem 3.4 is proved.

Now, we give a characterization of approximation space $\mathcal{A}_q^s(\Psi)$ by means of Lorentz sequence spaces.

Theorem 3.5. *Let $\Psi = \{x_n\}$ be a frame for $\mathcal{H}$ with bounds A and B. If $\{\langle x, S^{-1}x_n\rangle\} \in \ell_{p,q}$, then $x \in \mathcal{A}_q^s(\Psi)$, where $\frac{1}{p} = s + \frac{1}{2}$ and $0 \leq q < 2$.*

Proof. As in the proof of Theorem 3.3, take $c_k^* = |\langle x, S^{-1}x_{n_k}\rangle|$ for $k = 1, 2, 3, \dots$. Then, by Theorem 3.1 for $m \in \mathbb{N}$ we have

$$\sigma_m^2(x) \leq \|x - G_m(x)\|^2 \leq B \sum_{k=m+1}^{\infty} c_k^{*2},\ x \in \mathcal{H}.$$

Also, since an ℓ_2-norm does not exceed an ℓ_q-norm, we can write

$$\begin{aligned}\sigma_{2^m}(x) &\leq \sqrt{B}(\sum_{k=2^m+1}^{\infty} c_k^{*2})^{1/2} \leq \sqrt{B}(\sum_{k=2^m}^{\infty} c_k^{*2})^{1/2} \leq \sqrt{B}(\sum_{k=m}^{\infty} 2^k c_{2^k}^{*}{}^2)^{1/2} \\ &\leq \sqrt{B}(\sum_{k=m}^{\infty} 2^{kq/2} c_{2^k}^{*}{}^q)^{1/q}.\end{aligned}$$

So, it follows that with some constant C

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{msq}\sigma_{2^m}^q(x) &\leq B^{q/2}\sum_{m=1}^{\infty} 2^{msq}\sum_{k=m}^{\infty} 2^{kq/2}c_{2^k}^{*}{}^q \leq B^{q/2}C\sum_{k=1}^{\infty} 2^{ksq}2^{kq/2}c_{2^k}^{*}{}^q \\ &\leq B^{q/2}C\sum_{k=1}^{\infty} 2^{kq/p}c_{2^k}^{*}{}^q < \infty,\end{aligned}$$

implying that $x \in \mathcal{A}_q^s(\mathcal{D})$. Theorem 3.5 is proved.

Finally, we show that the converse of Theorem 3.5 is true for Riesz bases in Hilbert spaces. In [8], a similar characterization of $\mathcal{A}_q^s(\Psi)$ by orthonormal basis is given.

Theorem 3.6. *Let $\Psi = \{x_n\}$ be a Riesz basis for $\mathcal{H}$ with bounds A and B. Then*

$$x \in \mathcal{A}_q^s(\Psi) \text{ if and only if } \{\langle x, S^{-1}x_n\rangle\} \in \ell_{p,q},\ \frac{1}{p} = s + \frac{1}{2},\ q < 2.$$

Proof. If $\{\langle x, S^{-1}x_n\rangle\} \in \ell_{p,q}$, then the result immediately follows from Theorem 3..

Conversely, let $x \in \mathcal{A}_q^s(\mathcal{D})$. Then, by Theorem 3.12 we have

$$c_n^* \leq n^{-1/2}\frac{1}{\sqrt{A}}\sigma_n(x), \text{ for } n \in \mathbb{N},\ x \in \mathcal{H}.$$

Therefore

$$\sum_{n=1}^{\infty} n^{q/p-1}c_n^{*q} \leq \frac{1}{A^{q/2}}\sum_{n=1}^{\infty} n^{q/p-1}n^{-q/2}\sigma_n^q(x) = \frac{1}{A^{q/2}}\sum_{n=1}^{\infty} n^{qs-1}\sigma_n^q(x) < \infty.$$

Hence, $\{\langle x, S^{-1}x_n\rangle\} \in l_{p,q}$. Theorem 3.6 is proved.

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