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TWO RESULTS ON THE NORMALITY CRITERION CONCERNING HOLOMORPHIC FUNCTIONS

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Abstract. In this paper we generalize two known results concerning normal families of meromorphic functions. We first improve and extend a theorem of Liu and Nevo [10], using a completely different approach. Then we obtain a generalization of Gu's normality criterion stated in [5].

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1. Introduction and the main results

Let $\mathcal{F}$ be a family of meromorphic functions defined in D. The family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$ there exists a subsequence f_{n_j} that converges spherically locally uniformly on D to a meromorphic function or to ∞ (see [13]).

Let f and g be two meromorphic functions defined in a domain D, and let a be a complex number. If $g(z) = a$ whenever $f(z) = a$, then we write $f(z) = a \Rightarrow g(z) = a$. If $f(z) = a \Rightarrow g(z) = a$ and $g(z) = a \Rightarrow f(z) = a$, then we write $f(z) = a \Leftrightarrow g(z) = a$, and say that f and g share the value a. It is assumed that the reader is familiar with the standard notions and fundamental results of Nevanlinna theory, as found in [7, 15, 16].

In 2004, Chang, Fang and Zalcamn [2], have obtained a normal family of holomorphic functions related to a non-vanishing function, which improved the results of Chen and Hua ([3], Theorem 1), Pang ([11], Theorem 1), and Fang and Xu ([4], Theorem 3).

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Theorem A. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain $D \subset \mathbb{C}$. Let $k \geq 2$ be an integer, and let $h(z) \neq 0$ be a holomorphic function in D. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Rightarrow f'(z) = h(z)$, and

(b) $f'(z) = h(z) \Rightarrow |f^{(k)}(z)| \leq c$, where c is a constant.

Then the family $\mathcal{F}$ is normal on D.

Recently, Liu and Nevo [10] have improved the above theorem by allowing $h(z)$ to have zeros. Specifically, Liu and Nevo have proved the following theorem.

Theorem B. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain D. Let $k \geq 2$ be an integer, and let $h(z) \not\equiv 0$ be a holomorphic function on $D \subset \mathbb{C}$ that has no common zeros with any $f \in \mathcal{F}$. Assume also that the following two conditions hold for every $f \in \mathcal{F}$:*

(a) $f(z) = 0 \Rightarrow f'(z) = h(z)$, and

(b) $f'(z) = h(z) \Rightarrow |f^{(k)}(z)| \leq c$, where c is a constant.

Then the family $\mathcal{F}$ is normal on D.

Remark 1.1. In fact, Liu and Nevo firstly proved Theorem B in [9] under the additional condition that all zeros of function $h(z)$ have multiplicity at most $k - 1$. Then they removed this additional condition in [10].

From the above theorems, we see that for each $f \in \mathcal{F}$, the function h is a fixed function. So, we now pose the following question: can h be different for different functions f in the above theorems? In this paper give an affirmative answer to this question.

To state our result, we first recall a notation. Let $\mathcal{F}$ be a family of meromorphic functions. Denote by $\mathcal{F}'$ the family of limit functions, and set $\overline{\mathcal{F}} = \mathcal{F} \cup \mathcal{F}'$. Motivated by an idea of Grahl and C. Meng [6], we prove the following generalization of the above theorems using a completely different approach.

Theorem 1.1. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain D. Let $k \geq 2$ be an integer, and let $\mathcal{H}$ be a normal family of holomorphic functions on D such that $0, \infty \notin \overline{\mathcal{H}}$ on D. Assume also that for each $f \in \mathcal{F}$ there exists $h_f \in \mathcal{H}$ such that the following conditions hold:*

(a) f and h_f has no common zeros,

(b) $f(z) = 0 \Rightarrow f'(z) = h_f(z)$,

(c) $f'(z) = h_f(z) \Rightarrow |f^{(k)}(z)| \leq c$, where c is a constant.

Then the family $\mathcal{F}$ is normal on D.

Remark 1.2. The condition (a) is necessary, even in the case where all h_f are the same and the multiplicities of zeros of f are very large. The next example illustrates this point.

Example 1.1. Let $f_n = nz^p$, where p is an integer, and let Δ be the unit disc. Then $f_n'(z) = pnz^{p-1}$ and $f_n^{(k)}(z) = 0$ for $k > p$. Let $h(z) = z$. Then, it is easy to see that, the family $\{f_n\}$ satisfies the conditions (b) and (c) of Theorem 1.1, but $\{f_n\}$ is not normal at 0, no matter how large the integer p is.

Now we consider another problem concerning a normal family.

In 1959, Hayman [7] proved the following seminal result: if f is a meromorphic function on $\mathbb{C}$ and if $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for some fixed positive integer k for all $z \in \mathbb{C}$, then f is constant.

The corresponding normality criterion is due to Gu [5]. It states that a family $\mathcal{F}$ of functions meromorphic on D is normal if $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ on D for each $f \in \mathcal{F}$. Yang [15] extended the above criterion from a value to a holomorphic function. In 2007, Nevo, Pang and Zalcman studied the normality problem from different viewpoint, and proved the following result (see [14, Lemma 3]).

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, $k \in \mathbb{N}$, and let $\mathcal{H}$ be a normal family of holomorphic functions on D such that $h \neq 0, \infty$ on D for each function $h \in \overline{\mathcal{H}}$. If for each $f \in \mathcal{F}$, $f(z) \neq 0$ on D, and there exists a $h_f \in \mathcal{H}$ such that $f^{(k)}(z) \neq h_f(z)$ on D, then $\mathcal{F}$ is normal on D.*

Recently, Liu and Chang [8] generalized Theorem C by allowing $\mathcal{H}$ to consist of meromorphic functions.

Theorem D. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, $k \in \mathbb{N}$, and let $\mathcal{H}$ be a normal family of meromorphic functions on D such that $h \neq 0, \infty$ on D for each function $h \in \overline{\mathcal{H}}$. If for each $f \in \mathcal{F}$, $f(z) \neq 0$ on D, and there exists an $h_f \in \mathcal{H}$ such that $f^{(k)}(z) \neq h_f(z)$ on D, then $\mathcal{F}$ is normal on D.*

In [8], the authors also gave an example showing that the condition $f^{(k)}(z) \neq h_f(z)$ cannot be replaced by $f^{(k)}(z) - h_f(z) \neq 0$, even when all h_f are the same.

Example 1.2. Let $f_n(z) = \frac{1}{nz}$ for every $n \in \mathbb{N}$, and $h(z) = \frac{(-1)^k k!}{z^{k+1}}$. Then we have $f_n(z) \neq 0$ and $f_n^{(k)}(z) - h_f(z) \neq 0$ on $\mathbb{C}$ for $n > 1$. However, the family $\{f_n\}$ is not normal at 0.

In fact, the condition $f^{(k)}(z) \neq h_f(z)$ implies that f and h_f have no common poles. So, if f is holomorphic, then the condition $f^{(k)}(z) \neq h_f(z)$ coincides with $f^{(k)}(z) - h_f(z) \neq 0$.

From Example 1.2 we see that the functions $f_n^{(k)}$ and h have the same pole at 0 with the same multiplicity. So, it is natural to ask what if all the common poles of $f_n^{(k)}$ and h have different multiplicities?

In view of paper [8], we obtain a generalization of Theorems C and D.

In what follows, we use the following notation:

$$L[f] = a_0 f^{(k)} + a_1 f^{(k-1)} + \cdots + a_{k-1} f' + a_k f \tag{1.1}$$

denotes a linear differential polynomial of f, where $a_0, \cdots, a_k$ are holomorphic functions with $a_0(z) \neq 0$.

Theorem 1.2. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, and let $\mathcal{H}$ be a normal family of meromorphic functions on D such that $h \neq 0, \infty$ on D for each function $h \in \overline{\mathcal{H}}$. Assume also that for each $f \in \mathcal{F}$ there exists a function $h_f \in \mathcal{H}$ such that the following conditions hold:*

1. *$f(z) \neq 0$,*
2. *all the zeros of $L(f) - h_f$ come from the zeros of h_f,*
3. *the multiplicity of zeros of $L(f) - h_f$ is not larger than that of zeros of h_f at the common zeros of $L(f) - h_f$ and h_f,*
4. *the multiplicity of poles of $L(f)$ is larger than that of poles of h_f at the common poles of $L(f)$ and h_f.*

Then the family $\mathcal{F}$ is normal in D.

The next example shows that the condition (4) in Theorem 2 is necessary.

Example 1.3. Let $f_n(z) = \frac{1}{nz}$ for every $n \in \mathbb{N}$, $h(z) = \frac{(-1)^k k!}{z^{k+1}}$ and $D = \{z : |z| < 1\}$. Then for $n > 1$ we have $f_n(z) \neq 0$ and $f_n^{(k)}(z) - h(z) \neq 0$ on D. However, the family $\{f_n\}$ is not normal at 0.

Finally, we give an example to show that there exists a normal family $\mathcal{F}$ satisfying all the conditions of Theorem 1.2.

Example 1.4. Let $f_n(z) = \frac{e^{z^2}}{n}$ for every $n \in \mathbb{N}$, $h(z) = ze^{z^2}$ and $D = \{z : |z| < 1\}$. Then for $n > 1$ we have $f_n(z) \neq 0$ and $f_n'(z) - h(z) = ze^{z^2}(\frac{1}{n} - 1)$ on $\mathbb{C}$. Hence $f_n'(z) - h(z)$ and $h(z)$ have the same zeros with the same multiplicities. It is easy to see that the family $\{f_n\}$ satisfies all the conditions of Theorem 1.2, and hence $\{f_n\}$ is normal on D.

Remark 1.3. The condition (3) plays an important role in the proof of Theorem 1.2. However, we don't know whether it is necessary or not.

Throughout the paper we use the following notation: D denotes a domain in $\mathbb{C}$; $\Delta(z_0, r) = \{z : |z - z_0| < r\}$ and $\Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\}$ for $z_0 \in \mathbb{C}$ and $r > 0$.

2. Proof of Theorem 1.1

To prove our main results, we need some lemmas (see [2], [7], [12]).

Lemma 2.1 ([7]). *Let f be a meromorphic function on $\mathbb{C}$ such that $f(z) \neq 0$ and $f^{(k)}(z) \neq c$ for some constant $c \neq 0$ and all $z \in \mathbb{C}$. Then f is a constant.*

Lemma 2.2 ([2]). *Let g be a nonconstant entire function with $\rho(g) \leq 1$. Let $k \geq 2$ be an integer and let a be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$, and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then*

$$g(z) = a(z - z_0),$$

where z_0 is a constant.

Lemma 2.3 ([12]). *Let $\mathcal{F}$ be a family of functions meromorphic (resp. holomorphic) in the unit disc Δ, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If $\mathcal{F}$ is not normal at z_0 in the unit disc, then for each $0 \leq \alpha \leq k$ there exist:*

(a) points $z_n \in \Delta$, $z_n \to z_0$

(b) functions $f_n \in \mathcal{F}$ and

(c) a sequence of positive numbers $\rho_n \to 0$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a non-constant meromorphic (resp. entire) function in $\mathbb{C}$ with order at most 2 (resp. 1) such that $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$.

Here, as usual, $g^\sharp(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative of $g(\xi)$.

We are required to prove that a given sequence $\{f_n\}$ contains a subsequence that converges spherically locally uniformly on D or $\{f_n\}$ is normal on D. By the assumptions, there exists a corresponding sequence $\{h_n\} \in \mathcal{H}$ such that the functions f_n, h_n satisfy the conditions (a)-(c) of the theorem.

Since $\mathcal{H}$ is normal, the sequence $\{h_n\}$ contains a subsequence, which we again denote by $\{h_n\}$, such that $\{h_n\}$ converges spherically locally uniformly on D to a holomorphic function h_0, which may be ∞ identically. Note that since $0, \infty \notin \overline{\mathcal{H}}$, we have $h_0 \neq 0,\ \infty$.

Taking into account that normality is a local property, it is enough to show that $\{f_n\}$ is normal at each $z_0 \in D$. We set $E = h_0^{-1}(0)$, and continue the proof by distinguishing two cases.

Case 1. Let $z_0 \notin E$.

In this case we have $h_0(z_0) \neq 0$. Hence, noting that h_n converges to h_0 spherically locally uniformly on D, we conclude that there exists a positive constant δ such that $h_n(z) \neq 0$ and $|h_n(z)| \leq |h(z)| + 1$ on $\Delta(z_0, \delta)$ for large enough n.

Next, it follows from condition (b) that

$$f_n(z) = 0 \Rightarrow |f_n'(z)| = |h_n(z)| \leq A,$$

on $\Delta(z_0, \delta)$, where $A = \max\{|h(z)| :\ z \in \Delta(z_0, \delta)\} + 1$.

Suppose, to the contrary, that $\{f_n\}$ is not normal at z_0. Then by lemma 2.3, there exist a subsequence of the sequence $\{f_n\}$ (which we again denote by $\{f_n\}$), a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\xi) = \rho_n^{-1} f_n(z_n + \rho_n \xi) \to g(\xi) \tag{2.1}$$

where the convergence is spherically locally uniformly in $\mathbb{C}$, and g is a nonconstant holomorphic function of order at most 1 and $g^\sharp(\xi) \leq g^\sharp(0) = A + 1$.

By differentiating (2.1) one time and k-times, we obtain

$$g_n'(\xi) = f_n'(z_n + \rho_n \xi) \to g'(\xi) \tag{2.2}$$

and

$$g_n^{(k)}(\xi) = \rho_n^{k-1} f_n^{(k)}(z_n + \rho_n \xi) \to g^{(k)}(\xi), \tag{2.3}$$

where the convergence is spherically locally uniformly in $\mathbb{C}$.

Then, it follows from (2.2) that

$$H_n(\xi) = f_n'(z_n + \rho_n \xi) - h_n(z_n + \rho_n \xi) \to g'(\xi) - h_0(z_0), \tag{2.4}$$

where the convergence is spherically locally uniformly in $\mathbb{C}$.

We claim that

(I) $g(\xi) = 0 \Rightarrow g'(\xi) = h_0(z_0)$

(II)$g'(\xi) = h_0(z) \Rightarrow g^{(k)}(\xi) = 0$.

Suppose that $g(\xi_0) = 0$. Then by Hurwitz's theorem and (2.1), there exists a sequence $\{\xi_n\}$ such that $\xi_n \to \xi_0$, and for large enough n we have

$$f_n(z_n + \rho_n\xi_n) = 0.$$

Then, by assumption (b) we get

$$f'_n(z_n + \rho_n\xi_n) = h_n(z_n + \rho_n\xi_n). \tag{2.5}$$

A combination of (2.2) and (2.5) yields

$$g'(\xi_0) = \lim_{n\to\infty} f'_n(z_n + \rho_n\xi_n) = \lim_{n\to\infty} h_n(z_n + \rho_n\xi_n) = h_0(z_0),$$

implying that the claim (I) holds.

Similarly we can show the validity of the claim (II).

Then, by Lemma 2 we have $g(\xi) = h_0(z_0)(\xi - p_0)$, where p_0 is a constant. Therefore

$$A + 1 = g^\sharp(0) \le |h_0(z_0)| \le A,$$

which is a contradiction.

Case 2. Let $z_0 \in E$.

Without loss of generality, we can assume that $z_0 = 0$. Then there exists a positive constant δ_1 such that $h_0(z) \neq 0$ on $\Delta'(0,\delta_1)$. It follows from Case 1 that $\{f_n\}$ is normal on $\Delta'(0,\delta_1)$.

By taking a subsequence and renumbering, we can assume that

$$f_n \to f \quad on \quad \Delta'(0,\delta_1). \tag{2.6}$$

Now if f is a holomorphic in $\Delta'(0,\delta_1)$, then by the maximum modulus principle we conclude that $f_n \to f$ on $\Delta(0,\delta_1)$, and the result follows. So, let us assume that $f_n \to \infty$ in $\Delta'(0,\delta_1)$. Let $r < \delta_1$, then $f_n \to \infty$ uniformly on $|z| = r$. Without loss of generality, we can assume that $f_n \neq 0$ on $|z| = r$ for large enough n. Note that since f_n and h_n have no common zeros, f_n can have only simple zeros. Then from condition (b), we deduce that $\frac{f'_n - h_n}{f_n}$ is holomorphic on $\Delta(0,r)$.

Then by the argument principle and the Caushy theorem, for large enough n, we have

$$\begin{aligned} n(r,\frac{1}{f_n}) &= |\frac{1}{2\pi i}\int_{|z|=r}\frac{f'_n}{f_n}dz| \\ &= |\frac{1}{2\pi i}\int_{|z|=r}\frac{f'_n - h_n}{f_n}dz + \frac{1}{2\pi i}\int_{|z|=r}\frac{h_n}{f_n}dz| \\ &= |\frac{1}{2\pi i}\int_{|z|=r}\frac{h_n}{f_n}dz| \le \frac{1}{2\pi}\int_{|z|=r}|\frac{h_n}{f_n}|ds \to 0, \end{aligned} \tag{2.7}$$

which implies that f_n has no zeros in $|z| < r$ and $\frac{1}{f_n}$ is holomorphic in $|z| < r$. In view of $\frac{1}{f_n} \to 0$ in $0 < |z| < r$, we have $\frac{1}{f_n} \to 0$ in $|z| < r$. Thus, $f_n \to \infty$ in $|z| < r$. So, $\{f_n\}$ is normal at $z = 0$. Theorem 1.1 is proved.

3. Proof of the Theorem 1.2

We are required to prove that a given sequence $\{f_n\}$ contains a subsequence that converges spherically locally uniformly on D or $\{f_n\}$ is normal on D. By the assumption, there exists a corresponding sequence $\{h_n\} \in \mathcal{H}$ such that the functions f_n, h_n and $L(f_n)$ satisfy the conditions (1)-(4) of the theorem.

Since $\mathcal{H}$ is normal, the sequence $\{h_n\}$ contains a subsequence, which we again denote by $\{h_n\}$, such that $\{h_n\}$ converges spherically locally uniformly on D to a meromorphic function h_0, which may be ∞ identically. Note that since $0, \infty \notin \overline{\mathcal{H}}$, we have $h_0 \not\equiv 0, \infty$.

Taking into account that normality is a local property, it is enough to show that $\{f_n\}$ is normal at each $z_0 \in D$. We set $E = h_0^{-1}(0) \cup h_0^{-1}(\infty)$, and continue the proof by distinguishing two cases.

Case 1. Let $z_0 \notin E$.

In this case we have $h_0(z_0) \neq 0$. Hence, noting that h_n converges to h_0 spherically locally uniformly on D, we conclude that there exists a positive constant δ such that $h_n(z) \neq 0$ on $\Delta(z_0, \delta)$ for large enough n. Then it follows from the condition (2) of the theorem that $L(f_n)(z) - h_n(z) \neq 0$ on $\Delta(z_0, \delta)$.

Suppose, to the contrary, that $\{f_n\}$ is not normal at z_0. Then by lemma 2.3, there exist a subsequence of the sequence $\{f_n\}$, (which we again denote by $\{f_n\}$), a sequence of complex numbers $z_n \to z_0$ and a sequence of positive numbers $\rho_n \to 0$, such that

$$g_n(\xi) = \rho_n^{-k} f_n(z_n + \rho_n \xi) \to g(\xi), \tag{3.1}$$

where the convergence is spherically locally uniformly in $\mathbb{C}$, and g is a non-constant meromorphic function.

By differentiating (3.1) we obtain

$$g_n^{(l)}(\xi) = \rho_n^{l-k} f_n^{(l)}(z_n + \rho_n \xi) \to g^{(l)}(\xi), \tag{3.2}$$

where the convergence is spherically locally uniformly in $\mathbb{C}$, and $1 \leq l \leq k$.

Furthermore, we have

$$
\begin{aligned}
&L(f_n)(z_n+\rho_n\xi)-h_n(z_n+\rho_n\xi)\\
=&a_0(z_n+\rho_n\xi)f_n^{(k)}(z_n+\rho_n\xi)+a_1(z_n+\rho_n\xi)f_n^{(k-1)}(z_n+\rho_n\xi)\\
&+\cdots+a_kf_n(z_n+\rho_n\xi)-h_n(z_n+\rho_n\xi)\\
=&a_0(z_n+\rho_n\xi)g_n^{(k)}(\xi)+\rho_na_1(z_n+\rho_n\xi)g_n^{(k-1)}(\xi)\\
&+\cdots+\rho_n^ka_k(z_n+\rho_n\xi)g_n(\xi)-h_n(z_n+\rho_n\xi)\to a_0(z_0)g^{(k)}(\xi)-h_0(z_0),
\end{aligned}
\tag{3.3}
$$

where the convergence is spherically locally uniformly in $\mathbb{C}$.

Noting that $f_n(z)\neq 0$ in view of Hurwitz's theorem we have $g(z)\neq 0$. Moreover, it follows from Hurwitz's theorem and $L(f_n)(z)-h_n(z)\neq 0$ on $\triangle(z_0,\delta)$ that $g^{(k)}(z)\neq \frac{h_0(z_0)}{a_0(z_0)}$. Hence, in view of Lemma 2.1, we get a contradiction. Thus, we have proved that $\{f_n\}$ is normal at z_0.

Case 2. Let $z_0\in E$.

Without loss of generality, we can assume that $z_0=0$. Then there exists a positive constant δ_1 such that $h_0(z)\neq 0,\infty$ on $\triangle'(0,\delta_1)$. It follows from Case 1 that $\{f_n\}$ is normal on $\triangle'(0,\delta_1)$. Moreover, we have $f_n\to f_0$ on $\triangle'(0,\delta_1)$, where f_0 is meromorphic on $\triangle'(0,\delta_1)$ or $f_0=\infty$.

Suppose first that $f_0\neq 0$. Then we have $\frac{1}{f_n}\to\frac{1}{f_0}$ on $\triangle'(0,\delta_1)$. Since $f_n(z)\neq 0$ on D, $\frac{1}{f_n}$ is holomorphic on D. Hence, by the maximum modulus principle, we have $\frac{1}{f_n}\to\frac{1}{f_0}$ on $\triangle'(0,\delta_1)$, implying that $\{f_n\}$ is normal at 0.

Now, we assume that $f_0=0$. In this case, for any positive constant $r<\delta_1$, we have $f_n\to 0$ and $f_n^{(l)}\to 0$ as $n\to\infty$ on $|z|=r$ for all positive integers l. Hence $L(f_n)\to 0$ and $L(f_n)'\to 0$ on $|z|=r$.

Next, from the argument principle, for sufficiently large n we have

$$
\begin{aligned}
&n(r,\ \frac{1}{L(f_n)-h_n})-n(r,\ L(f_n)-h_n)\\
=&\frac{1}{2\pi i}\int_{|z|=r}\frac{L(f_n)'-h_n'}{L(f_n)-h_n}dz\to\frac{1}{2\pi i}\int_{|z|=r}\frac{h_0'}{h_0}dz\\
=&n(r,\ \frac{1}{h_0})-n(r,\ h_0),
\end{aligned}
$$

where $n(r,\ \frac{1}{g})$ and $n(r,\ g)$ are the numbers of zeros and poles of g in $|z|<r$, respectively, counting multiplicities. Then, it follows that

$$
n(r,\ \frac{1}{L(f_n)-h_n})-n(r,\ L(f_n)-h_n)=n(r,\ \frac{1}{h_0})-n(r,\ h_0).
\tag{3.4}
$$

We consider two subcases.

Subcase 2.1. Let $h_0(0) = 0$.

Noting that the convergence $h_n \to h_0$ is spherically locally uniformly on D, we conclude that there exists a positive $r_1 < \delta_1$ such that $n(r_1, \frac{1}{h_n}) = n(r_1, \frac{1}{h_0})$ and $n(r_1, h_0) = 0$ for sufficiently large n. Then it follows from conditions (2) and (3) that

$$n(r_1, \frac{1}{L(f_n) - h_n}) \le n(r_1, \frac{1}{h_n}) = n(r_1, \frac{1}{h_0}),$$

and in view of (3.4) we have

$$n(r_1,\ L(f_n) - h_n) \le n(r_1,\ h_0) = 0.$$

Then, it follows from condition (4) that

$$n(r_1,\ L(f_n)) = n(r_1,\ L(f_n) - h_n) \le n(r_1,\ h_0) = 0,$$

implying that $n(r_1,\ L(f_n)) = 0$. Hence, f_n has no pole on Δ_{r_1}. Then, taking into account that $f_n \to 0$ on $\Delta'(0, r_1)$ and using the maximum modulus principle, we conclude that $f_n \to 0$ on $\Delta(0, r_1)$, showing that $\{f_n\}$ is normal at 0.

Subcase 2.2. Let $h_0(0) = \infty$.

Noting that the convergence $h_n \to h_0$ is spherically locally uniformly on D, we conclude that there exists a positive $r_2 < \delta_1$ such that for sufficiently large n

$$n(r_2, h_n) = n(r_2, h_0)$$

and

$$n(r_2, \frac{1}{h_n}) = n(r_2, \frac{1}{h_0}) = 0.$$

Then it follows from condition (2) that $n(r_2, \frac{1}{L(f_n)-h_n}) = 0$, and in view of (3.4), we obtain

$$n(r,\ L(f_n) - h_n) = n(r,\ h_0) = n(r_2, h_n). \tag{3.5}$$

Thus, by condition (4), we have

$$\overline{n}(r_2,\ L(f_n)) + n(r_2,\ h_n) \le n(r_1,\ L(f_n) - h_n), \tag{3.6}$$

where $\overline{n}(r_2,\ g)$ is the number of poles of g in $\Delta(0, r_2)$, ignoring multiplicities. Combing (3.5) and (3.6), we get $\overline{n}(r_2,\ L(f_n)) \le 0$, implying that $\overline{n}(r_2,\ L(f_n)) = 0$. So $n(r_2,\ L(f_n)) = 0$, and hence f_n has no pole on $\Delta(0, r_2)$. The arguments, similar to that of used in Subcase 2.1, show that $\{f_n\}$ is normal at 0. Combing the Cases 1 and 2 we conclude that the family $\mathcal{F}$ is normal at each $z_0 \in D$.

This completes the proof of Theorem 1.2.

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