Известия НАН Армении. Математика, том 50, н. 6, 2015, стр. 3-15.

ON ABSOLUTE CONVERGENCE OF THE SERIES OF FOURIER COEFFICIENTS WITH RESPECT TO HAAR-LIKE SYSTEMS IN THE CLASS $BV(p(n) \uparrow p, \phi)$

A. APLAKOV

Georgian National University - SEU, Tbilisi, Georgia
E-mail: *aplakovi@gmail.com*

Abstract. In this paper we study a question of absolute convergence of the series of Fourier coefficients with respect to Haar-like systems in the class of functions of generalized bounded variation $BV(p(n) \uparrow p, \phi)$.

MSC2010 numbers: 42C10.

Keywords: Haar-like system; Vilenkin system; Fourier series; absolute convergence.

1. Introduction

It is well-known that the notion of variation of a function was introduced by Jordan [18] in 1881. In 1924, Wiener [25] generalized this notion and introduced the notion of p-variation. In 1937, Young [26] introduced the notion of Φ-variation. Waterman [24] has studied the class of functions of bounded Λ-variation, and Chanturia [5] has defined the notion of modulus of variation of a function. Later the notion of modulus of variation for a continuous function was generalized by Karchava [19]. In 1990, Kita and Yoneda [20] have introduced the notion of generalized Wiener's class $BV(p(n) \uparrow p)$.

Let f be a function defined on $(-\infty, +\infty)$ with period 1. Δ is said to be a partition with period 1, if

$$\Delta: \quad ...t_{-1} < t_0 < t_1 < t_2 < ... < t_m < t_{m+1} < ...$$

satisfies $t_{k+m} = t_k + 1$ for $k = 0, \pm1, \pm2, ...,$ where m is a positive integer. Let $\rho(\Delta) = \inf |t_k - t_{k-1}|$.

Definition 1.1 ([20]). *Let $p(n)$ be an increasing sequence such that $1 \leq p(n) \uparrow p$ as $n \to \infty$, where $1 \leq p \leq \infty$. We say that a function f belongs to the class $BV(p(n) \uparrow p)$*

if

$$V\left(f,p\left(n\right)\uparrow p\right):=\underset{n\geq1}{\sup}\underset{\Delta}{\sup}\left\{\left(\sum_{k=1}^{m}\left|f\left(t_{k}\right)-f\left(t_{k-1}\right)\right|^{p(n)}\right)^{\frac{1}{p(n)}}:\rho\left(\Delta\right)\geq\frac{1}{2^{n}}\right\}<\infty.$$

If $p(n)=p$ for all n, then the class $BV(p(n)\uparrow p)$ coincides with the Wiener class V_p of functions of bounded p-variation.

Properties of functions of class $BV(p(n)\uparrow p)$ as well as uniform convergence and divergence of their Fourier series by trigonometric and Walsh systems have been studied by Kita [21], Goginava [7] – [9] and Goginava, Nagy [12].

In 2000, Akhobadze [1] has generalized the class $BV(p(n)\uparrow p)$ by introducing the class $BV(p(n)\uparrow p,\phi)$, defined below.

Let ϕ be an increasing function defined on the set of natural numbers $\mathbb{N}$, such that $\phi(1)\geq 2$ and

$$\lim_{n\to\infty}\phi\left(n\right)=+\infty.$$

Let f be a finite 1-periodic function defined on $(-\infty,+\infty)$, and let Δ be a partition with period 1, defined above.

Definition 1.2 (Akhobadze [1]). *Let $p(n)$ be an increasing sequence such that $1\leq p(n)\uparrow p$ as $n\to\infty$, where $1\leq p\leq\infty$, and let the function ϕ be as above. We say that a function f belongs to the class $BV(p(n)\uparrow p,\phi)$ if*

$$V\left(f,p\left(n\right)\uparrow p,\phi\right)=\underset{n\geq1}{\sup}\underset{\Delta}{\sup}\left\{\left(\sum_{k=1}^{m}\left|f\left(t_{k}\right)-f\left(t_{k-1}\right)\right|^{p(n)}\right)^{\frac{1}{p(n)}}:\rho\left(\Delta\right)\geq\frac{1}{\phi\left(n\right)}\right\}<\infty.$$

If $p(n)=p$ for each natural n, where $1<p<\infty$, then the class $BV(p(n)\uparrow p,\phi)$ coincides with the Wiener class V_p, and if $\phi(n)=2^n$, $n=1,2,...$, then the class $BV(p(n)\uparrow p,\phi)$ coincides with the class $BV(p(n)\uparrow p)$.

Some generalizations of the notion of variation of functions has been considered by Akhobadze [2], Goginava [8, 9], and Goginava and Sahakian [13, 14].

Let $C([0,1])$ denote the space of continuous functions with period 1.

If $f\in C([0,1])$, then the function

$$\omega\left(\delta,f\right)=\max\left\{\left|f\left(x\right)-f\left(y\right)\right|:\left|x-y\right|\leq\delta,\ \ x,y\in\left[0.1\right]\right\}$$

is called the modulus of continuity of f.

Given a modulus of continuity $\omega(\delta)$, by H^{ω} we denote the class of functions $f\in C([0,1])$ satisfying $\omega(\delta,f)=O(\omega(\delta))$ as $\delta\to 0+$. Given an increasing function

φ defined on the set of naturals $\mathbb{N}$, such that $\varphi(1) \geq 2$ and $\lim_{n\to\infty} \varphi(n) = +\infty$. Denote

$$\tau(r) = \min\{k : k \in \mathbb{N},\ \varphi(k) \geq r\}, \quad r \geq 2.$$

Definition 1.3. *We say that the sequence $\{p(n) : n \geq 1\}$ and the function φ satisfy condition (B, φ), if there exists a constant $c > 0$ such that for every $n \geq 1$*

$$\sum_{k=n+1}^{\infty} \frac{1}{m_k^{1/p(\ \tau(k))}} \leq \frac{c}{m_n^{1/p(\ \tau(n))}}.$$

2. Haar-like systems

Let $\{p_n\}$ be a sequence of prime numbers, such that $2 \leq p_n \leq N$ $(n = 1, 2, ...)$, where $N \geq 2$ is a natural number. Let $\{m_n,\ n \geq 0\}$ be a sequence of integers defined by

$$m_0 = 1;\ m_n = p_1 p_2 ... p_n,\ n \geq 1,$$

and let

$$Q = \left\{\frac{l}{m_n}\right\},\ l = 0, 1, ..., m_n,\ n \geq 0.$$

Then for each $t \in [0, 1] \setminus Q$ there is only one expression of the form

$$t = \sum_{k=1}^{\infty} \frac{j_k(t)}{m_k},\quad 0 \leq j_k(t) \leq p_k - 1.$$

Each integer $m \geq 2$ can be uniquely written in the form

$$m = m_n + r(p_{n+1} - 1) + s,$$

where $n = 0, 1, ..., r = 0, ..., m_n - 1$ and $s = 1, ..., p_{n+1} - 1$.

We set $\chi_1(t) = 1$ on $[0, 1]$, and for $m \geq 2$ define

$$\chi_m(t) = \chi_{nr}^{(s)}(t) = \begin{cases} \sqrt{m_n} \exp\{2\pi i \frac{j_{n+1}(t)}{p_{n+1}}\} & \text{for } t \in \left(\frac{r}{m_n}, \frac{r+1}{m_n}\right) \setminus Q, \\ 0 & \text{for } t \notin \left[\frac{r}{m_n}, \frac{r+1}{m_n}\right]. \end{cases}$$

At the interior discontinuity points $t \in Q$, we define the function $\chi_m(t)$ to be the average of the limits on either side, while at the endpoints of the interval $[0, 1]$, we define $\chi_m(t)$ to be the limits from the interior.

Thus we have defined the system $\chi\{p_n\}$ of the class χ. The class χ, which is the set of all systems $\chi\{p_n\}$ with bounded sequences $\{p_n \geq 2\}$, was introduced by Vilenkin [23] in 1947. If $p_n = 2, n = 1, 2, ...,$ then we have the classical Haar system, introduced by Haar [17] in 1909.

Let $L([0,1])$ be the class of all measurable 1−periodic functions defined on $[0,1]$ with the following norm

$$\|f\|_1 = \int_0^1 |f(x)|\, dx < \infty.$$

Denote by $a_m(f)$ the Fourier coefficients of a function $f \in L([0,1])$ with respect to Haar-like system $\{\chi_m(t)\}$:

$$a_m(f) = \int_0^1 f(t)\, \overline{\chi_m(t)}dt, \quad m = 1, 2, \ldots.$$

The problem of estimation of Fourier-Haar coefficients and absolute convergence of series of Fourier-Haar coefficients has been studied in a number of papers. We mention, for instance, the papers by Ul'ianov [22], Golubov [15], Chanturia [5], Goginava [10], Gát and Toledo [6], Aplakov [4].

The absolute convergence of series of Fourier-Haar-like coefficients has been studied by Golubov and Rubinshtein [16], where, in particular, the following theorems were proved.

Theorem GR1 ([16]). *Let $\chi\{p_n\} \in \chi$. The following assertions hold.*

1) *If $f \in V_p$ for $1 \le p < \infty$ and $\beta > \frac{2p}{2+p}$, then*

$$\sum_{m=1}^{\infty} |a_m(f)|^{\beta} < \infty.$$

2) *For any $1 \le p < \infty$ there exists a function $f_0 \in V_p$ (moreover $f_0 \in Lip(1/p)$), for which*

$$\sum_{m=1}^{\infty} |a_m(f_0)|^{\beta} = \infty$$

when $\beta = \frac{2p}{2+p}$.

Theorem GR2 ([16]). *Let $\chi\{p_n\} \in \chi$. The following assertions hold.*

1) *If $f \in V_p$ for $1 \le p < \infty$ and $\alpha < \frac{1}{p} - \frac{1}{2}$, then*

$$\sum_{m=1}^{\infty} m^{\alpha} |a_m(f)| < \infty.$$

2) *For any $1 \le p < \infty$ there exists a function $f_0 \in V_p$ (moreover $f_0 \in Lip(1/p)$), for which*

$$\sum_{m=1}^{\infty} m^{\alpha} |a_m(f_0)| = \infty$$

when $\alpha = \frac{1}{p} - \frac{1}{2}$.

In this paper we study absolute convergence of the series of Fourier coefficients with respect to Haar-like systems in the class of functions of generalized bounded variation. Specifically, we study convergence of the series

$$\sum_{m=1}^{\infty} m^{\alpha} \left|a_m(f)\right|^{\beta}$$

for functions f from the class $BV(p(n) \uparrow p, \phi)$, for various values of the parameters α and β.

3. The main results

The main results of this paper are the following theorems.

Theorem 3.1. *The following assertions hold.*

a) *Let $f \in BV\left(p(n) \uparrow \frac{2\beta}{2-\beta}, \varphi\right)$ for some $\beta \in (2/3, 2)$ and*

$$\sum_{n=1}^{\infty} \frac{1}{m_n^{\beta(1/2+1/p(\tau(m_n)))-1}} < \infty.$$

Then

$$\sum_{n=1}^{\infty} \left|a_n(f)\right|^{\beta} < \infty.$$

b) *Let the sequence $\{p(n) : n \geq 1\}$ and the function φ be such that the condition (B, φ) is satisfied, and let for some $\beta \in (2/3, 2)$*

$$\sum_{n=1}^{\infty} \frac{1}{m_n^{\beta(1/2+1/p(\tau(m_n)))-1}} = \infty.$$

Then there exists a function $f_0 \in BV\left(p(n) \uparrow \frac{2\beta}{2-\beta}, \phi\right)$ such that

$$\sum_{n=1}^{\infty} \left|a_n(f_0)\right|^{\beta} = \infty.$$

Theorem 3.2. *The following assertions hold.*

a) *Let $f \in BV\left(p(n) \uparrow \frac{2}{1+2\alpha}, \phi\right)$ for some $\alpha \in (-1/2, 1/2)$ and*

$$\sum_{n=1}^{\infty} \frac{1}{m_n^{1/p(\tau(m_n))-1/2-\alpha}} < \infty.$$

Then

$$\sum_{n=1}^{\infty} n^{\alpha} \left|a_n(f)\right| < \infty.$$

b) *Let the sequence* $\{p(n) : n \geq 1\}$ *and the function* φ *be such that the condition* (B, φ) *is satisfied, and let for some* $\alpha \in (-1/2, 1/2)$

$$\sum_{n=1}^{\infty} \frac{1}{m_n^{1/p(\tau(m_n))-1/2-\alpha}} = \infty.$$

Then there exists a function $f_0 \in BV\left(p(n) \uparrow \frac{2}{1+2\alpha}, \phi\right)$ *such that*

$$\sum_{n=1}^{\infty} n^{\alpha} |a_n(f_0)| = \infty.$$

Combining Theorems 3.1 and 3.2 we obtain the following result.

Theorem 3.3. *The following assertions hold.*

a) *Let* $f \in BV\left(p(n) \uparrow \frac{2\beta}{2+2\alpha-\beta}, \phi\right)$ *for some* $\beta > 0$ *with* $\frac{2+2\alpha}{3} < \beta < 2 + 2\alpha$, *and let*

$$\sum_{n=1}^{\infty} \frac{1}{m_n^{\beta(1/p(\tau(m_n))+1/2)-1-\alpha}} < \infty.$$

Then

$$\sum_{n=1}^{\infty} n^{\alpha} |a_n(f)|^{\beta} < \infty.$$

b) *Let the sequence* $\{p(n) : n \geq 1\}$ *and the function* φ *be such that the condition* (B, φ) *is satisfied, and let for some* $\beta > 0$ *with* $\frac{2+2\alpha}{3} < \beta < 2 + 2\alpha$

$$\sum_{n=1}^{\infty} \frac{1}{m_n^{\beta(1/p(\tau(m_n))+1/2)-1-\alpha}} = \infty.$$

Then there exists a function $f_0 \in BV\left(p(n) \uparrow \frac{2\beta}{2+2\alpha-\beta}, \phi\right)$ *such that*

$$\sum_{n=1}^{\infty} n^{\alpha} |a_n(f_0)|^{\beta} = \infty.$$

4. Auxiliary results

To prove the theorems stated in Section 3, we need the following lemmas.

Lemma 4.1 ([3, 11]). *The embedding* $H^{\omega} \subset BV(p(n) \uparrow p, \phi)$ *holds if and only if*

$$\omega(t) = O\left(t^{1/p(\tau(1/t))}\right) \quad as \quad t \to 0+.$$

Lemma 4.2 ([16]). *Let* $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n < \infty$ *and let* f_0 *be a function defined by*

$$f_0(x) = \sum_{k=1}^{\infty} (-1)^{k-1} c_k e^{2\pi i m_k x},$$

then $\sum_{m=m_n+1}^{m_{n+1}} |a_m(f_0)| \geq c_0 m_n^{1/2} c_n$, *where* c_0 *is any positive constant.*

Lemma 4.3. *Let* $p(n) \uparrow p$ *with* $p \in (1,\infty)$ *and let the sequence* $\{p(n) : n \geq 1\}$ *and the function* ϕ *satisfy condition* (B,ϕ). *If*

$$f_0(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{e^{2\pi i m_k x}}{m_k^{1/p(\tau(m_k))}}.$$

then $f_0 \in BV(p(n) \uparrow p, \phi)$.

Proof. For every $0 < \delta < 1$, there exists a natural n such that

$$\frac{1}{m_{n+1}} \leq \delta < \frac{1}{m_n}.$$

Let $x, y \in [0,1]$ be such that $|x-y| < \delta < 1$. Then we can write

$$\begin{aligned}
|f_0(x) - f_0(y)| &\leq \sum_{k=1}^{\infty} (|\cos(2\pi m_k x) - \cos(2\pi m_k y)| + \\
&+ |\sin(2\pi m_k x) - \sin(2\pi m_k y)|) m_k^{-1/p(\tau(m_k))} \\
&= \sum_{k=1}^{\infty} (2|\sin(m_k\pi(x-y))\sin(m_k\pi(x+y))| \\
&+2|\sin(m_k\pi(x-y))\cos(m_k\pi(x+y))|) m_k^{-1/p(\tau(m_k))} \\
&\leq 4\sum_{k=1}^{\infty} \frac{|\sin(m_k\pi(x-y))|}{m_k^{1/p(\tau(m_k))}} \\
&\leq 4\sum_{k=1}^{n} \frac{|\sin(m_k\pi(x-y))|}{m_k^{1/p(\tau(m_k))}} + 4\sum_{k=n+1}^{\infty} \frac{1}{m_k^{1/p(\tau(m_k))}} = I + II. \qquad (4.1)
\end{aligned}$$

We first estimate I to obtain

$$\begin{aligned}
I &\leq 4\sum_{k=1}^{n} \frac{m_k\pi\delta}{m_k^{1/p(\tau(m_k))}} = 4\pi\delta \sum_{k=1}^{n} m_k^{1-1/p(\tau(m_k))} \qquad (4.2) \\
&\leq 4\pi\delta \sum_{k=1}^{n} m_k^{1-1/p(\tau(m_n))} \leq c\delta m_n^{1-1/p(\tau(m_n))} \\
&\leq c\delta \left(\frac{1}{\delta}\right)^{1-1/p(\tau(m_n))} = c\delta^{1/p(\tau(m_n))} \leq c\delta^{1/p(\tau(1/\delta))}.
\end{aligned}$$

From the condition of the lemma for II we have

$$\begin{aligned}
II &\leq \frac{c}{m_n^{1/p(\tau(m_n))}} \qquad (4.3) \\
&= \frac{c p_{n+1}^{1/p(\tau(m_n))}}{m_{n+1}^{1/p(\tau(m_n))}} \leq c\delta^{1/p(\tau(m_n))} \leq c\delta^{1/p(\tau(1/\delta))}.
\end{aligned}$$

Combining (4.1)-(4.3) we get

$$\omega(\delta, f_0) = O\left(\delta^{1/p(\tau(1/\delta))}\right) \text{ as } \delta \to 0+.$$

Applying Lemma 4.1 we obtain

$$f_0 \in BV(p(n) \uparrow p, \phi),$$

and the result follows. □

5. Proofs

Proof of Theorem 3.1. In [16] it is proved that the following estimation holds

$$\tag{5.1} \sum_{m=m_n+1}^{m_{n+1}} |a_m(f)|^p$$

$$\leq \; c m_n^{1-\frac{p}{2}} (p_{n+1}-1) \sum_{r=0}^{m_n-1} \sum_{k=1}^{p_{n+1}-1} \int_{\delta_{n,r,k}} \left| f(t) - f\left(t + \frac{1}{m_{n+1}}\right)\right|^p dt,$$

where

$$\delta_{n,r,k} = \left(\frac{r}{m_n} + \frac{k-1}{m_{n+1}}, \frac{r}{m_n} + \frac{k}{m_{n+1}}\right).$$

Then we can write

$$\sum_{m=m_n+1}^{m_{n+1}} |a_m(f)|^{p(\tau(m_n))} \leq c m_n^{1-\frac{p(\tau(m_n))}{2}} (p_{n+1}-1)$$

$$\times \sum_{r=0}^{m_n-1} \sum_{k=1}^{p_{n+1}-1} \int_{\frac{r}{m_n}+\frac{k-1}{m_{n+1}}}^{\frac{r}{m_n}+\frac{k}{m_{n+1}}} \left| f(t) - f\left(t + \frac{1}{m_{n+1}}\right)\right|^{p(\tau(m_n))} dt$$

$$= c m_n^{1-\frac{p(\tau(m_n))}{2}} (p_{n+1}-1)$$

$$\times \sum_{r=0}^{m_n-1} \int_0^{\frac{1}{m_n}-\frac{1}{m_{n+1}}} \left| f\left(t+\frac{r}{m_n}\right) - f\left(t + \frac{r}{m_n} + \frac{1}{m_{n+1}}\right)\right|^{p(\tau(m_n))} dt$$

$$\leq c m_n^{1-\frac{p(\tau(m_n))}{2}} (p_{n+1}-1) \left(\frac{1}{m_n} - \frac{1}{m_{n+1}}\right)$$

$$\times \sup_{t \in \left[0, \frac{1}{m_n} - \frac{1}{m_{n+1}}\right]} \sum_{r=0}^{m_n-1} \left| f\left(t+\frac{r}{m_n}\right) - f\left(t + \frac{r}{m_n} + \frac{1}{m_{n+1}}\right)\right|^{p(\tau(m_n))}$$

$$\leq c \sup_n p_n m_n^{-\frac{p(\tau(m_n))}{2}} \left(V\left(f, p(n) \uparrow \frac{2\beta}{2-\beta}, \phi\right)\right)^{p(\tau(m_n))}.$$

Let $\beta < p(\tau(m_n)) < \frac{2\beta}{2-\beta}$ and $n \geq n_0$. Then applying Hölder's inequality we get

$$\sum_{m=m_n+1}^{m_{n+1}} |a_m(f)|^\beta \leq \left(\sum_{m=m_n+1}^{m_{n+1}} |a_m(f)|^{p(\tau(m_n))}\right)^{\beta/p(\tau(m_n))} (m_{n+1}-m_n)^{1-\frac{\beta}{p(\tau(m_n))}}$$

$$\leq \ c m_n^{\frac{\beta}{2}} \left(V\left(f, p(n) \uparrow \frac{2\beta}{2-\beta}, \phi\right)\right)^\beta m_n^{1-\frac{\beta}{p(\tau(m_n))}} \leq \frac{c}{m_n^{\beta(1/2+1/p(\tau(m_n)))-1}} .$$

Hence

$$\sum_{m=2}^{\infty} |a_m(f)|^\beta = \sum_{n=0}^{\infty} \sum_{m=m_n+1}^{m_{n+1}} |a_m(f)|^\beta$$

$$= \sum_{n=0}^{n_0} \sum_{m=m_n+1}^{m_{n+1}} |a_m(f)|^\beta + \sum_{n=n_0+1}^{\infty} \sum_{m=m_n+1}^{m_{n+1}} |a_m(f)|^\beta$$

$$\leq c + \sum_{n=n_0+1}^{\infty} \frac{c}{m_n^{\beta(1/2+1/p(\tau(m_n)))-1}} < \infty .$$

Part a) of the theorem is proved.

b) Let $p(n) \uparrow \frac{2\beta}{2-\beta}$ and $\beta \in (2/3, 2)$. Define

$$f_0(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{e^{2\pi i m_k x}}{m_k^{1/p(\tau(m_k))}},$$

and show that f_0 is the required function. To this end, observe first that by Lemma 4.3 we have

$$f_0 \in BV\left(p(n) \uparrow \frac{2\beta}{2-\beta}, \phi\right).$$

Denote by $E_n^{'}$ the set of integers r from $[0, m_n - 1]$ satisfying $\left|a_{nr}^{(s)}(f_0)\right| \geq \frac{c_0}{2} m_n^{-\left(\frac{1}{2}+1/p(\tau(m_n))\right)}$, and let $E_n^{''} = [0, m_n - 1] \setminus E_n^{'}$.

In [16] it is proved that

$$|a_m(f)| = \left|a_{nr}^{(s)}(f)\right| \leq c\sqrt{m_n} \sum_{k=1}^{p_{n+1}-1} \int_{\delta_{n,r,k}} \left|f(t) - f\left(t + \frac{1}{m_{n+1}}\right)\right| dt.$$

Consequently, we can write

$$\sum_{r \in E_n^{'}} \left|a_{nr}^{(s)}(f_0)\right| \leq c m_n^{1/2} \sum_{r \in E_n^{'}} \sum_{k=1}^{p_{n+1}-1} \int_{\frac{r}{m_n}+\frac{k-1}{m_{n+1}}}^{\frac{r}{m_n}+\frac{k}{m_{n+1}}} \left|f(t) - f\left(t + \frac{1}{m_{n+1}}\right)\right| dt$$

$$\leq c m_n^{1/2} \sum_{r \in E_n^{'}} \int_{\frac{r}{m_n}}^{\frac{r}{m_n}+\frac{1}{m_n}-\frac{1}{m_{n+1}}} \left|f(t) - f\left(t + \frac{1}{m_{n+1}}\right)\right| dt$$

$$\le c m_n^{1/2} \sum_{r\in E_n'} \int_0^{\frac{1}{m_n}-\frac{1}{m_{n+1}}} \left| f\left(t+\frac{r}{m_n}\right) - f\left(t+\frac{r}{m_n}+\frac{1}{m_{n+1}}\right)\right| dt$$

$$\le c m_n^{1/2}\left(\frac{1}{m_n}-\frac{1}{m_{n+1}}\right)$$

$$\times \sup_{t\in\left[0,\frac{1}{m_n}-\frac{1}{m_{n+1}}\right]} \sum_{r\in E_n'} \left| f\left(t+\frac{r}{m_n}\right) - f\left(t+\frac{r}{m_n}+\frac{1}{m_{n+1}}\right)\right| \le c m_n^{1/2}\frac{p_{n+1}-1}{m_{n+1}}$$

$$\times \sup_{t\in\left[0,\frac{1}{m_n}-\frac{1}{m_{n+1}}\right]} \left(\sum_{r\in E_n'} \left| f\left(t+\frac{r}{m_n}\right) - f\left(t+\frac{r}{m_n}+\frac{1}{m_{n+1}}\right)\right|^{p(\tau(m_n))}\right)^{1/p(\tau(m_n))}$$

$$\times \left|E_n'\right|^{1-1/p(\tau(m_n))} \le c\frac{\left|E_n'\right|^{1-1/p(\tau(m_n))}}{m_n^{1/2}}.$$

Next, from Lemma 4.2 we get

$$c_0\frac{m_n^{1/2}}{m_n^{1/p(\tau(m_n))}} \le \sum_{r=0}^{m_n-1}\left|a_{nr}^{(s)}(f_0)\right|$$

$$\le \sum_{r\in E_n'}\left|a_{nr}^{(s)}(f_0)\right| + \sum_{r\in E_n''}\left|a_{nr}^{(s)}(f_0)\right|$$

$$\le c\frac{\left|E_n'\right|^{1-1/p(\tau(m_n))}}{m_n^{1/2}} + \frac{c_0}{2}m_n^{-(1/2+1/p(\tau(m_n)))}\left|E_n''\right|$$

$$\le c\frac{\left|E_n'\right|^{1-1/p(\tau(m_n))}}{m_n^{1/2}} + \frac{c_0}{2}m_n^{1/2-1/p(\tau(m_n))}.$$

Therefore

$$c\frac{\left|E_n'\right|^{1-1/p(\tau(m_n))}}{m_n^{1/2}} \ge \frac{c_0}{2}m_n^{1/2-1/p(\tau(m_n))};\quad \left|E_n'\right| \ge c m_n. \tag{5.2}$$

In view of (5.2) we get

$$\sum_{r=0}^{m_n-1}\left|a_{nr}^{(s)}(f_0)\right|^{\beta} \ge \sum_{r\in E_n'}\left|a_{nr}^{(s)}(f_0)\right|^{\beta}$$

$$\ge \left(\frac{c_0}{2}\frac{1}{m_n^{1/2+1/p(\tau(m_n))}}\right)^{\beta}\left|E_n'\right| \ge \frac{c}{m_n^{\beta(1/2+1/p(\tau(m_n)))-1}}. \tag{5.3}$$

From condition of the theorem and (5.3) we obtain

$$\sum_{m=2}^{\infty}\left|a_m(f_0)\right|^{\beta} = \sum_{n=0}^{\infty}\sum_{m=m_n+1}^{m_{n+1}}\left|a_m(f_0)\right|^{\beta} \ge c\sum_{n=1}^{\infty}\frac{1}{m_n^{\beta(1/2+1/p(\tau(m_n)))-1}} = \infty.$$

This completes the proof of part b) of the theorem. □

Proof of Theorem 3.2. To prove part a) of the theorem, we apply (5.1) for $p = 1$ and use the fact $\sup_n p_n \leq N$, to obtain

$$\sum_{m=1}^{\infty} m^{\alpha} |a_m(f)| = \sum_{n=0}^{\infty} \sum_{m=m_n+1}^{m_{n+1}} m^{\alpha} |a_m(f)| \leq \sum_{n=0}^{\infty} m_{n+1}^{\alpha} \sum_{m=m_n+1}^{m_{n+1}} |a_m(f)|$$

$$\leq \sum_{n=0}^{\infty} m_{n+1}^{\alpha} C m_n^{1/2} (p_{n+1}-1) \sum_{r=0}^{m_n-1} \sum_{k=1}^{p_{n+1}-1} \int_{\frac{r}{m_n}+\frac{k-1}{m_{n+1}}}^{\frac{r}{m_n}+\frac{k}{m_{n+1}}} \left| f(t) - f\left(t + \frac{1}{m_{n+1}}\right)\right| dt$$

$$\leq c \sum_{n=0}^{\infty} m_{n+1}^{\alpha} m_n^{1/2} (p_{n+1}-1) \sum_{r=0}^{m_n-1} \int_{\frac{r}{m_n}}^{\frac{r}{m_n}+\frac{1}{m_n}-\frac{1}{m_{n+1}}} \left| f(t) - f\left(t + \frac{1}{m_{n+1}}\right)\right| dt$$

$$= c \sum_{n=0}^{\infty} m_{n+1}^{\alpha} m_n^{1/2} (p_{n+1}-1)$$

$$\times \sum_{r=0}^{m_n-1} \int_{0}^{\frac{1}{m_n}-\frac{1}{m_{n+1}}} \left| f\left(t + \frac{r}{m_n}\right) - f\left(t + \frac{r}{m_n} + \frac{1}{m_{n+1}}\right)\right| dt$$

$$\leq c \sum_{n=0}^{\infty} m_{n+1}^{\alpha} m_n^{1/2} (p_{n+1}-1) \left(\frac{1}{m_n} - \frac{1}{m_{n+1}}\right)$$

$$\times \sup_{t \in \left[0, \frac{1}{m_n}-\frac{1}{m_{n+1}}\right]} \sum_{r=0}^{m_n-1} \left| f\left(t + \frac{r}{m_n}\right) - f\left(t + \frac{r}{m_n} + \frac{1}{m_{n+1}}\right)\right|$$

$$\leq c \sum_{n=0}^{\infty} m_{n+1}^{\alpha} m_n^{1/2} (p_{n+1}-1) \frac{p_{n+1}-1}{m_{n+1}} \times \sup_{t \in \left[0, \frac{1}{m_n}-\frac{1}{m_{n+1}}\right]} \left(\sum_{r=0}^{m_n-1} \left| f\left(t + \frac{r}{m_n}\right) \right.\right.$$

$$\left.\left. - f\left(t + \frac{r}{m_n} + \frac{1}{m_{n+1}}\right)\right|^{p(\tau(m_{n+1}))} \right)^{1/p(\tau(m_{n+1}))} \times m_n^{1-1/p(\tau(m_{n+1}))}$$

$$\leq c \sum_{n=0}^{\infty} m_{n+1}^{\alpha+1/2-1+1-1/p(\tau(m_{n+1}))} V\left(f, p(n) \uparrow \frac{2}{1+2\alpha}, \phi\right)$$

$$\leq c \sum_{n=1}^{\infty} \frac{1}{m_n^{1/p(\tau(m_n))-1/2-\alpha}} < \infty \, .$$

Thus, part a) of the theorem is proved.

To prove part b) of the theorem, we define

$$f_0(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{e^{2\pi i m_k x}}{m_k^{1/p(\tau(m_k))}},$$

and apply Lemma 4.3 to conclude that $f_0 \in BV\left(p(n) \uparrow \frac{2}{1+2\alpha}, \phi\right)$.

Next, from condition of the theorem and Lemma 4.2 we obtain

$$\sum_{m=1}^{\infty} m^{\alpha} |a_m(f_0)| = \sum_{n=0}^{\infty} \sum_{m=m_n+1}^{m_{n+1}} m^{\alpha} |a_m(f_0)|$$

$$\geq \sum_{n=0}^{\infty} m_n^{\alpha} \sum_{m=m_n+1}^{m_{n+1}} |a_m(f_0)| \geq c_0 \sum_{n=0}^{\infty} m_n^{\alpha} \frac{m_n^{1/2}}{m_n^{1/p(\tau(m_n))}}$$

$$\geq c_0 \sum_{n=1}^{\infty} \frac{1}{m_n^{1/p(\tau(m_n))-1/2-\alpha}} = \infty,$$

and the result follows. □

Список литературы

[1] T. Akhobadze, "Functions of generalized Wiener classes $BV(p(n) \uparrow p, \phi)$ and their Fourier coefficients", Georgian Math. J. **7**, no.3, 401 416 (2000).

[2] T. Akhobadze, "$BA(p(n) \uparrow p, \phi)$ classes of functions of bounded variation. Bull. Georgian Acad. Sci. **164**, no. 1, 18 – 20 (2001).

[3] T. Akhobadze, "Generalized $BV(p(n) \uparrow p, \phi)$ class of bounded variation", Bull. Georgian Acad. Sci., **163**, no. 3, 426 – 428 (2001).

[4] A. Aplakov, "On the absolute convergence of the series of Fourier-Haar coefficients", Bull. Georgian Acad. Sci., **164**, no. 2, 238 – 241 (2001).

[5] Z. Chanturia, "On the absolute convergence of the series of Fourier-Haar coefficients", Commentationes Math. **11**, 25 – 35 (1979).

[6] G. Gát, R. Toledo, "Fourier coefficients and absolute convergence on compact totally disconnected groups", Math. Pannon., **10** (2), 223 – 233 (1999).

[7] U. Goginava, "On the divergence of trigonometric Fourier series of the class $H^{\omega} \cap BV(p(n) \uparrow \infty)$", Proc. A. Razmadze Math. Inst., **121**, 63 – 70 (1999).

[8] U. Goginava, "Uniform convergence of Cesáro means of negative order of double Walsh-Fourier series", J. Approx. Theory. **124**, no. 1, 96 – 108 (2003).

[9] U. Goginava, "On the uniform convergence Walsh-Fourier series", Acta Math. Hung. **93**, no. 1-2, 59 – 70 (2001).

[10] U. Goginava. "On the absolute convergence of the series of Fourier-Haar coefficients". Bull. Georgian Acad. Sci., **164**, no. 1, 21 – 23 (2001).

[11] U. Goginava, "Relations between some classes of functions", Sci. Math. Jpn. **53**, no. 2, 223 232 (2001).

[12] U. Goginava, K. Nagy, "On the maximal operator of Walsh-Kaczmarz-Fejér means", Czechoslovak Math. J., **61** (136), no. 3, 673 – 686 (2011).

[13] U. Goginava, A. A. Sahakian, "Convergence of double Fourier series and generalized Λ-variation", Georgian Math. J. **19**, no. 3, 497 – 509 (2012).

[14] U. Goginava, A. A. Sahakian, "On the convergence of multiple Walsh-Fourier series of functions of bounded generalized variation", Journal of Contemporary Mathematical Analysis **5**, 47, 221 – 233 (2012).

[15] B. Golubov, "On the Fourier series of continuous functions with respect to Haar system", Izv. AN SSSR, seria matem. **6**, no. 28, 1271 – 1296 (1964).

[16] B. Golubov, A. Rubinshtein, "About one class of systems of convergence", Matem. sb. **71** (113), 96 – 115 (1966).

[17] Haar A. Zur teorie der ortogonalen Funktionensysteme. Inaugural dissertation. Gottingen (1909).

[18] G. Jordan, G. C.R. Acad. Sci. Paris. **92**, 228 – 230 (1881).

[19] T. Karchava, "On the converges of Fourier series", Reports of Enlarged Sessions of the Seminar of I.Vekua institute of Applied Mathematics, **3**, no. 2, 93 – 95 (1988).

[20] H. Kita, K. Yoneda, "A generalization of bounded variation", Acta Math. Hung. **56**, 229 – 238 (1990).

[21] H. Kita, "Convergence of Fourier series of functions on generalized Wiener's class $BV(p(n)\uparrow\infty)$", Acta Math. Hung. **57**, 233 – 243 (1991).

[22] P. L. Ulianov, "On the Haar series", Matem. sb. **63**, no. 3, 356 – 391 (1964).

[23] N. Vilenkin, "About one class complete ortogonal system", Izv. AN SSSR, seria matem. **11**, 363 – 400 (1947).

[24] D. Waterman, "On Λ-bounded variation", Studia Math. **57**, 33 – 45 (1976).

[25] N. Wiener, "The quadratic variation of a function and its Fourier coefficients", Massachusetts J. Math. **3**, 72 – 92 (1924).

[26] L. Young, "Sur un generalization de la notion de variation de Poissance p-ieme bornee an sonce de Wiener et sur la convergence de series de Fourier", C.R. Acad. Sci. Paris. **204**, 470 – 472 (1937).

Поступила 10 августа 2014