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HARDY-TYPE INEQUALITIES FOR FUNCTIONS WHOSE FOURIER TRANSFORMS HAVE GAPS

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Abstract. The original proof of the Littlewood conjecture was a special case of a more general inequality of functions whose Fourier coefficients have gaps. In this article, we prove similar inequalities, but treating the Fourier transform of a function integrable on the real line, rather than on the unit circle.

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1. Introduction

The problem of finding a lower bound of the L^1−norm of exponential sums, known as a Littlewood conjecture, was posed by Littlewood [2], stating that a constant K exists such that for any set $\{n_1 < n_2 < ... < n_N\} \subset \mathbb{Z}$,

$$\left\|\sum_{k=1}^{N} e^{in_k t}\right\|_1 \geq K \log N, \tag{1.1}$$

where the L^1−norm is taken on $\mathbb{T}$, that is, over unit circle. In [5], the Littlewood conjecture was proved as a special case of the following general result.

Theorem 1.1. *(McGehee, Pigno and Smith) There is an absolute constant $c > 0$ such that for any function $f \in L^1(\mathbb{T})$ whose spectrum is contained in the set $\{n_1 < n_2 < ...\} \subset \mathbb{Z}$ we have*

$$\sum_{k=1}^{\infty} \frac{|\hat{f}(n_k)|}{k} \leq c\|f\|_1, \tag{1.2}$$

where $\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}dt$ is the $n-th$ Fourier coefficient of f.

In this context, the spectrum of f is the support of $\hat{f}$. Since then, many attempts have been done to generalize the original Hardy's inequality, which is given by (1.2) with $n_k = k$.

In this paper we prove the corresponding inequality of (1.2) for functions $f \in L^1(\mathbb{R})$ and their Fourier transforms $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$. To this end, we will prove first a continuous version of the inequality

$$\sum_{j=1}^{\infty}\left(4^{-j}\sum_{4^{j-1}\leq k<4^j}|\hat{f}(n_k)|^2\right)^{1/2} \leq C\|f\|_1, \tag{1.3}$$

for all $f \in L^1(\mathbb{T})$ whose spectrum lies in $\{n_1, n_2, \cdots\} \subset \mathbb{Z}$. This inequality was proved in [9] as a gapped version of the following inequality of [3]:

$$\sum_{j=1}^{\infty}\left(4^{-j}\sum_{n=4^{j-1}}^{4^j-1}|\hat{f}(n)|^2\right)^{1/2} \leq C\|f\|_1 + C\sum_{j=1}^{\infty}\left(4^{-j}\sum_{n=4^{j-1}}^{4^j-1}|\hat{f}(-n)|^2\right)^{1/2} \tag{1.4}$$

for all functions $f \in L^1(\mathbb{T})$.

We emphasize here that although inequalities (1.3) and (1.4) look very similar, the authors in [3] and [9] used completely different constructions to prove these inequalities. In [9] the author used the construction applied in [5] to prove the Littlewood conjecture, while [3] used what we call the algebraic construction. In this paper we use the algebraic construction to prove some gapped versions of such inequalities. Hence, the importance of this paper is two folded: the results themselves and the treatment of the algebraic construction with gaps.

We refer the reader to [1] where these two, and two other constructions were reported as alternatives to prove the Littlewood conjecture.

It is worth to note that the recent proofs of inequalities of type (1.3) and (1.4) used a duality idea, where a bounded function with certain decay properties must be constructed, a powerful idea that has been used extensively on the circle.

In our recent works we have focused on how to deal with such inequalities on the real-line, that is, when having a function $f \in L^1(\mathbb{R})$ rather than $L^1(\mathbb{T})$. We refer the reader to [6] – [8] for the study of the real-line versions of Hardy's inequality.

In particular, in [8] we have proved the continuous version of (1.4), stating that a constant $C > 0$ exists such that for all $f \in L^1(\mathbb{R})$,

$$\sum_{j=1}^{\infty}\left(4^{-j}\int_{4^{j-1}}^{4^j}|\hat{f}(\xi)|^2d\xi\right)^{1/2} \leq C\|f\|_1 + C\sum_{j=1}^{\infty}\left(4^{-j}\int_{4^{j-1}}^{4^j}|\hat{f}(-\xi)|^2d\xi\right)^{1/2}. \tag{1.5}$$

Now we propose the following question: What are the real-line (continuous) versions of the inequalities (1.1) – (1.3)?

Observe that an analogue of the Littlewood original conjecture (1.1) would be

$$\left\|\int_E e^{i\xi t}d\xi\right\|_1 \geq K\log(A),$$

where E is a subset of $\mathbb{R}$ with a finite positive Lebesgue measure A, and K is an absolute constant. However, this inequality is trivially true by virtue of the following proposition.

Proposition 1.1. *Let $E \subset \mathbb{R}$ be a set of finite positive Lebesgue measure $m(E)$. Then,*

$$\left\|\int_E e^{i\xi t}d\xi\right\|_1 = \infty.$$

Proof. Let f be the characteristic function of the set E. Then we have $\int_E e^{i\xi t}d\xi = \hat{f}(-t)$. We claim that $\|\hat{f}\|_1 = \infty$. Otherwise we would have $\hat{f} \in L^1(\mathbb{R})$ and f is continuous. But f, being a characteristic function, is continuous if and only if $E = \mathbb{R}$, contradicting the fact that $m(E) < \infty$. □

This concludes the study of the continuous version of (1.1). The purpose of this paper is to present the continuous versions of inequalities (1.2) and (1.3). The corresponding results are stated in Section 3 (see (3.1) and (3.2) below).

To state the aimed results, we first introduce some notation.

For $f \in L^1(\mathbb{R})$, we set $\text{supp}(\hat{f}) = \{\xi \in \mathbb{R} : \hat{f}(\xi) \neq 0\}$ and suppose that $\hat{f}(\xi) = 0$ for all $\xi \leq \xi_0$ with some ξ_0.

We then define a new sequence $\{b_j,\ j \geq 1\}$ by setting

$$b_j = \inf\left\{b : m\left((b_{j-1}, b) \cap \text{supp}(\hat{f})\right) \geq 3 \times 4^{j-1}\right\}, \quad j \geq 1,$$

where $b_0 = \xi_0$ and $m(\cdot)$ is the Lebesgue measure.

Next, we define a new sequence $\{I_j\}$ of disjoint sets by

$$I_j = (b_{j-1}, b_j) \cap \text{supp}(\hat{f}), \quad j = 1, 2, 3, \cdots.$$

We remark that at each step j of the construction, if

$$m\left((b_{j-1}, b) \cap \text{supp}(\hat{f})\right) < 3 \times 4^{j-1}$$

for all $b > b_{j-1}$, then we set $b_j = \sup\text{supp}(\hat{f})$ and we stop the process to get finitely many I_j's.

Thus, for $f \in L^1(\mathbb{R})$ satisfying $\hat{f}(\xi) = 0$ for all $\xi \leq \xi_0$ with some ξ_0, we have constructed the sets $\{I_j\}$, possibly finitely many, with the following properties:

i) $\text{supp}(\hat{f}) = \bigcup_j I_j$.

ii) If $x \in I_j$ and $y \in I_{j+1}$, then $x < y$.

iii) $m(I_j) = 3 \times 4^{j-1}$, except possibly for the last one I_n, if there are finitely many of them, where we would have $m(I_n) \leq 3 \times 4^{n-1}$.

iv) If there are finitely many of the I_j, say $\{I_1, \cdots, I_n\}$, then by convention let $I_{n+1} = I_{n+2} = \cdots = \phi$.

If $f \in L^1(\mathbb{R})$ satisfies the properties i)-iv), then we refer to it as a gapped function with partition $\{I_j\}$.

Keeping these notation in mind, we prove the continuous version of (1.3). That is, for functions f possessing the above properties, we prove existence of a constant $C > 0$ such that

$$\sum_{j=1}^{\infty} \left(4^{-j} \int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \leq C\|f\|_1 \tag{1.6}$$

In this context, C is an absolute constant that does not depend on f nor on the partition $\{I_j\}$. It is worth to mention the advantage of (1.6) over (1.5). If, for example, $\hat{f}(\xi) = 0$ for all $\xi < 4^{100}$, then in (1.5) the first nonzero integral will be multiplied by 4^{-99}, while in the new form, this integral will be multiplied by 4^{-1}. Although we can shift the first block to be multiplied by 4^{-1} in (1.5), there is no way then we shift all other blocks. However, inequality (1.6) shifts and merges the support of $\hat{f}$ to behave like a function whose support is continuously extended over the real line. We remark that the ideas of the forthcoming proofs are similar to those of [8].

2. Preliminaries

Let $f \in L^1$ be a gapped function with partition $\{I_j\}$, and assume that $\hat{f}$ is of compact support. For $j \geq 1$ we define the following sequence of functions:

$$f_j(x) = \frac{1}{\sqrt{2\pi}} 4^{-j/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{-1/2} \int_{I_j} \hat{f}(\xi) e^{i\xi x} d\xi. \tag{2.1}$$

Since $\text{supp}(\hat{f})$ is assumed to be compact, for large j we have $f_j = 0$ and $I_j = \phi$. The following lemma gives the basic properties of the sequence $\{f_j\}$. The proof of this lemma is similar to the proof given in [8], and so is omitted.

Lemma 2.1. *Let f_j be as above. Then $\|f_j\|_2 = 4^{-j/2}$, unless $f_j = 0$, $\|f_j\|_\infty \leq 1$, and*

$$\hat{f}_j(\xi) = \sqrt{2\pi} 4^{-j/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{-1/2} g_j(\xi), \tag{2.2}$$

where

(2.3)
$$g_j(\xi) = \begin{cases} \hat{f}(\xi), & \xi \in I_j \\ 0, & \textit{otherwise.} \end{cases}$$

Now we construct a new sequence of functions $\{F_j,\ j \geq 0\}$ as follows. Put $F_0 = 0$ and for $j \geq 0$ define

(2.4)
$$F_{j+1} = \frac{\epsilon}{2} f_{j+1} + (1 - \epsilon^2 |f_{j+1}|^2) F_j - \frac{\epsilon}{2} \overline{f}_{j+1} F_j^2,$$

where $0 < \epsilon < 1$ is a number to be specified later.

Since $\hat{f}$ is of compact support, we have $f_j \equiv 0$ for large j. Therefore, there exists an index k such that $F_k = F_{k+1} = F_{k+2} = \cdots$. Let $F = \frac{2}{\epsilon} F_k$.

Some properties of the sequence $\{F_j,\ j \geq 0\}$ are given in the following sequence of lemmas. Again, the proofs of these results are similar, and some times identical to those in [8]. Hence, we omit the proofs, unless it is necessary.

Lemma 2.2. *For each* $j \geq 0$, *we have* $\|F_j\|_\infty \leq 1$.

Lemma 2.3. *For each* $j \geq 0$, *we have* $\mathrm{spec}(F_j) \subset (A_j, b_j)$ *for some* $A_j \in \mathbb{R}$. *Here* $\mathrm{spec}(F) = \mathrm{supp}(\hat{F})$.

Proof. We proceed by induction on j. The result is true for F_0 because $\mathrm{spec}(F_0) = \phi$. Suppose that $\mathrm{spec}(F_j) \subset (A_j, b_j)$ for some $A_j \in \mathbb{R}$. Observe that f_j is a scalar multiple of the Fourier transform of g_j, where g_j is given in (2.3). Therefore, $\mathrm{spec}(f_j) = I_j \subset (b_{j-1}, b_j)$, and using the fact $b_j < b_{j+1}$, we can write

$$\mathrm{spec}(f_{j+1}) \subset (b_j, b_{j+1}); \qquad \mathrm{spec}(F_j) \subset (A_j, b_j)$$

$$\mathrm{spec}(|f_{j+1}|^2 F_j) \subset \mathrm{spec}(f_{j+1}) + \mathrm{spec}(\overline{f}_{j+1}) + \mathrm{spec}(F_j)$$

$$\subset (b_j, b_{j+1}) + (-b_{j+1}, -b_j) + (A_j, b_j) \subset (b_j - b_{j+1} + A_j, b_{j+1})$$

$$\mathrm{spec}(\overline{f}_{j+1} F_j^2) \subset -\mathrm{spec}(f_{j+1}) + 2\mathrm{spec}(F_j) \subset (-b_{j+1}, -b_j) + (2A_j, 2b_j)$$

$$\subset (-b_{j+1} + 2A_j, b_j) \subset (-b_{j+1} + 2A_j, b_{j+1}).$$

Therefore $\mathrm{spec}(F_{j+1}) \subset (A_{j+1}, b_{j+1})$,
where $A_{j+1} = \min\{b_j, A_j, b_j - b_{j+1} + A_j, -b_{j+1} + 2A_j\}$. This completes the proof. □

Lemma 2.4. *For any* $k > j \geq 1$ *we have*

(2.5)
$$\left(\int_{\xi \geq b_{j-1}} |\hat{F}_k(\xi)|^2 d\xi \right)^{1/2} \leq 16\epsilon\sqrt{2\pi} 4^{-j/2}.$$

Lemma 2.5. *Let $j \geq 1$ and $k > j$. Then*

$$\left(\int_{I_j} |\hat{F}_k(\xi) - \frac{\epsilon}{2}\hat{f}_j(\xi)|^2 d\xi\right)^{1/2} \leq 18\epsilon^2 4^{-j/2}. \tag{2.6}$$

Now, recall that $F = \frac{2}{\epsilon}F_k$ for some $k \geq 1$.

Lemma 2.6. *Let $F = \frac{2}{\epsilon}F_k$, $k \geq 1$, and let $\epsilon := \frac{1}{72\sqrt{2\pi}}$. The following inequalities hold:*

$$\|F\|_\infty \leq c, \text{ where } c = 144\sqrt{2\pi}, \tag{2.7}$$

$$|\hat{F}(\xi) - \hat{f}_j(\xi)| \leq \frac{1}{2}4^{-j} \text{ for } \xi \in I_j. \tag{2.8}$$

The following basic result will be needed in our proofs in Section 3.

Lemma 2.7. *If $f, g \in L^2$, then*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Observe that if $f \in L^1$ is such that $\hat{f}$ is compactly supported, then $\hat{f} \in L^2$. Then Plancherel theorem guarantees that $f \in L^2$.

Now, if F is as above, then $F \in L^2$, and hence by lemma 2.7

$$\int_{\mathbb{R}} f(x)\overline{F(x)}dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi. \tag{2.9}$$

3. The main results

Now we are ready to prove our first main result, which is the continuous version of (1.3). Notice that although the proof is identical to that of (1.5), we present it here for completeness.

Theorem 3.1. *There exists an absolute constant $C > 0$, such that for all gapped $f \in L^1(\mathbb{R})$ with partition $\{I_j\}$,*

$$\sum_{j=1}^{\infty}\left(4^{-j}\int_{I_j} |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \leq C\|f\|_1. \tag{3.1}$$

Proof. We first prove the result for $f \in L^1$ whose Fourier transform $\hat{f}$ is of compact support. Let f_j and F be as above. Recall (2.7) and observe that (2.9) holds because

$\hat{f}$ is of compact support. Therefore,

$$c\|f\|_1 \geq \left|\int_{\mathbb{R}} f(x)\overline{F(x)}dx\right| = \frac{1}{2\pi}\left|\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi\right|$$

$$= \frac{1}{2\pi}\left|\int_{\cup I_j} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi\right| \geq \frac{1}{2\pi}\Re\left(\int_{\cup I_j} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi\right) = \frac{1}{2\pi}\sum_{j=1}^{\infty}\Re\left(\int_{I_j} \hat{f}(\xi)\overline{\hat{F}(\xi)}d\xi\right).$$

But in view of (2.8), for $\xi \in I_j$ we have

$$\left|\overline{\hat{F}(\xi)} - \overline{\hat{f}_j(\xi)}\right| \leq \frac{1}{2}4^{-j},$$

and hence

$$\left|\overline{\hat{F}(\xi)}\hat{f}(\xi) - \overline{\hat{f}_j(\xi)}\hat{f}(\xi)\right| \leq \frac{1}{2}4^{-j}|\hat{f}(\xi)|,$$

implying that

$$\Re\left(\overline{\hat{f}_j(\xi)}\hat{f}(\xi) - \overline{\hat{F}(\xi)}\hat{f}(\xi)\right) \leq \frac{1}{2}4^{-j}|\hat{f}(\xi)|.$$

Consequently, for $\xi \in I_j$, we have

$$\Re\left(\overline{\hat{F}(\xi)}\hat{f}(\xi)\right) \geq \Re\left(\hat{f}(\xi)\overline{\hat{f}_j(\xi)}\right) - \frac{1}{2}4^{-j}|\hat{f}(\xi)| =$$

$$= \sqrt{2\pi}4^{-j/2}\left(\int_{I_j}|\hat{f}(\tau)|^2 d\tau\right)^{-1/2}|\hat{f}(\xi)|^2 - \frac{1}{2}4^{-j}|\hat{f}(\xi)|,$$

where we have used (2.2) to obtain the last line.

Integrate both sides and then use Cauchy-Schwartz inequality to get

$$\int_{I_j}\Re\left(\overline{\hat{F}(\xi)}\hat{f}(\xi)d\xi\right) \geq \sqrt{2\pi}4^{-j/2}\left(\int_{I_j}|\hat{f}(\xi)|^2 d\xi\right)^{1/2} - \frac{4^{-j}}{2}\int_{I_j}|\hat{f}(\xi)|d\xi$$

$$\geq \sqrt{2\pi}4^{-j/2}\left(\int_{I_j}|\hat{f}(\xi)|^2 d\xi\right)^{1/2} - \frac{4^{-j}}{2}\left(\int_{I_j}|\hat{f}(\xi)|^2 d\xi\right)^{1/2}\left(\int_{I_j}d\xi\right)^{1/2}$$

$$= \left(\sqrt{2\pi} - \frac{\sqrt{3}}{4}\right)\left(4^{-j}\int_{I_j}|\hat{f}(\xi)|^2 d\xi\right)^{1/2} \geq \left(4^{-j}\int_{I_j}|\hat{f}(\xi)|^2 d\xi\right)^{1/2}.$$

This proves (3.1) with $C = 2\pi c$. This completes the proof of the theorem in the case where $\hat{f}$ is compactly supported.

For the general case, let $f \in L^1$, then apply the above arguments to $f * K_\lambda$, where K_λ is the Fejer kernel of order λ. □

To proceed to the next result, we recall Hardy's inequality and its gapped version (1.2), and (1.5) and its gapped version (3.1).

It is natural to propose a gapped version of the real-line Hardy inequality that states

$$\int_0^\infty \frac{|\hat{f}(\xi)|}{\xi} d\xi \le c\|f\|_1, \tag{3.2}$$

for $f \in L^1(\mathbb{R})$ satisfying $\hat{f}(\xi) = 0$ for all $\xi < 0$. See [8] for a proof, deduced from (1.5). To state and prove a gapped version of this inequality, we look at the gapped generalization of the discrete inequality in the following way: the inequality (1.2) can be thought of as

$$\sum_{k=1}^{\infty} \frac{|\hat{f}(n_k)|}{g(k)} \le C\|f\|_1,$$

where g is the mapping $g(n_k) = k$, which maps the supporting integers into the original set of integers.

To realize this idea in the real-line case, we need the following setup.

Observe first that I_j is open, being the intersection of two open sets. Hence, we can write

$$I_j = \bigcup_{k=1}^{n_j} (\alpha_{j,k}, \beta_{j,k}),$$

for some $\alpha_{j,k}, \beta_{j,k} \in \mathbb{R}$ such that $\sum_{k=1}^{n_j} (\beta_{j,k} - \alpha_{i,k}) \le 3 \times 4^{j-1}$. Consequently, we can find $\{\gamma_{j,k}\}$ and $\{\eta_{j,k}\}$ to satisfy

$$4^{j-1} \le \gamma_{j,k} < \eta_{j,k} \le 4^j \text{ and } \eta_{j,k} - \gamma_{j,k} = \beta_{j,k} - \alpha_{j,k}.$$

For each j and $k = 1, ..., n_j$, let $g_{j,k} : (\alpha_{j,k}, \beta_{j,k}) \longrightarrow (\gamma_{j,k}, \eta_{j,k})$ be defined by

$$g_{j,k}(\xi) = \xi - \alpha_{j,k} + \gamma_{j,k}.$$

Then we set

$$g_j = \sum_{k=1}^{n_j} g_{j,k}\chi_{j,k} \text{ and } g = \sum_j g_j,$$

where $\chi_{j,k}$ is the characteristic function of $(\alpha_{j,k}, \beta_{j,k})$.

Now we are in position to state our result that gives a gapped version of the continuous Hardy inequality (3.2).

Theorem 3.2. *Let f be a gapped function with partition $\{I_j\}$, and let g be as above. Then for some absolute constant C', we have*

$$\int_{\mathbb{R}} \frac{|\hat{f}(\xi)|}{g(\xi)} d\xi \le C'\|f\|_1 \tag{3.3}$$

Proof. Since $\operatorname{supp}(\hat{f}) \subset \bigcup_j I_j$, we have

$$\int_{\mathbb{R}} \frac{|\hat{f}(\xi)|}{g(\xi)} d\xi = \sum_j \int_{I_j} \frac{|\hat{f}(\xi)|}{g(\xi)} d\xi \le \sum_j \left(\int_{I_j} \frac{1}{g_j^2(\xi)} d\xi \right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$$

$$= \sum_j \left(\sum_{k=1}^{n_j} \int_{\alpha_{j,k}}^{\beta_{j,k}} \frac{1}{g_{j,k}^2(\xi)} d\xi \right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

Making the substitution $g_{j,k}(\xi) = \tau$ in the first integral, we can write

$$\int_{\mathbb{R}} \frac{|\hat{f}(\xi)|}{g(\xi)} d\xi \le \sum_j \left(\sum_{k=1}^{n_j} \int_{\gamma_{j,k}}^{\eta_{j,k}} \frac{1}{\tau^2} d\tau \right)^{1/2} \cdot \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$$

$$= \sum_j \left(\sum_{k=1}^{n_j} \frac{\eta_{j,k} - \gamma_{j,k}}{\eta_{j,k}\gamma_{j,k}} \right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$$

$$\le \sum_j \left(\frac{1}{4^{2j-2}} \sum_{k=1}^{n_j} (\eta_{j,k} - \gamma_{j,k}) \right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$$

$$\le \sum_j \left(\frac{3 \times 4^{j-1}}{4^{2j-2}} \right)^{1/2} \left(\int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \le \sqrt{12} \sum_j \left(4^{-j} \int_{I_j} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \le C' \|f\|_1,$$

where $C' = \sqrt{12}C$, and (3.3) follows. □

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