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SPECIFIC PROPERTIES OF SOLUTIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS WITH SEVERAL DELAY ARGUMENTS

R. KOPLATADZE

Tbilisi State University, Georgia
E-mail: *roman.koplatadze@tsu.ge*

Abstract. The paper considers the following differential equation $x'(t)+\sum_{i=1}^{m} p_i(t)\, x(\tau_i(t))=0$, $t\geq 0$, where $p_i\in L_{\text{loc}}(R_+;R_+)$, $\tau_i\in C(R_+;R)$, $\tau_i(t)\leq t$ and $\lim\limits_{t\to+\infty}\tau_i(t)=+\infty$, $i=1,\dots,m$. New oscillation criteria of all solutions for this equation are established.

MSC2010 numbers:* 34K11
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1. Introduction

It is a trivial consequence of the uniqueness of solutions of initial value problems that first order linear ordinary differential equations cannot have oscillatory solutions. As for the equation

$$x'(t)+p(t)\, x(\tau(t))=0, \tag{1.1}$$

the presence of a delay leads to the fact that oscillatory solutions do appear. Moreover, if p is nonnegative and the delay is sufficiently large, all proper solutions (see Definition 2.1 below) turn out to be oscillatory. Specific criteria for the oscillation of proper solutions of linear delay equations were for the first time proposed by A. D. Myshkis [1]. It follows from the results of [2, 3] that if the functions $p: R_+\to R_+$ ($R_+=[0,+\infty)$) and $\tau: R_+\to R$ are continuous, $\tau(t)\leq t$ for $t\in R_+$, $\lim\limits_{t\to+\infty}\tau(t)=+\infty$,

$$p^*=\limsup_{t\to+\infty}\int_{\tau(t)}^{t} p(s)ds, \quad p_*=\liminf_{t\to+\infty}\int_{\tau(t)}^{t} p(s)ds$$

and

$$\text{either}\quad p^*>1 \quad\text{or}\quad p_*>\frac{1}{e}, \tag{1.2}$$

then the equation (1.1) is oscillatory. Note that if $p_*\leq\frac{1}{e}$, the condition $p^*>1$ can be improved.

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For $\tau(t) = t - \tau$ ($\tau = \text{const} > 0$) such an improvement was carried out successively in [4–6], where the condition $p^* > 1$ was replaced, respectively, by $p^* > 1 - \frac{p_*^2}{4}$, $p^* > 1 - \frac{p_*^2}{2(1-p_*)}$ and

$$p^* > 1 - \frac{1 - p_* - \sqrt{1 - 2p_* - p_*^2}}{2}.$$

On the other hand, in [7, 8] sufficient conditions for oscillation of all proper solutions of equation (1.1) were given, which involve classes of inequalities not satisfying condition (1.2). In the present paper, using the ideas of [9], we establish some new criteria for the equation

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x(\tau_i(t)) = 0, \tag{1.3}$$

where

$$\begin{gathered} p_i \in L_{\text{loc}}(R_+; R_+), \quad \tau_i \in C(R_+; R), \quad \tau_i(t) \le t \quad \text{for} \quad t \in R_+ \\ \text{and} \quad \lim_{t\to+\infty} \tau_i(t) = +\infty \quad (i = 1, \dots, m), \end{gathered} \tag{1.4}$$

to be oscillatory.

2. The main results

Throughout the paper we assume that $\tau_*(t) = \min\{\tau_i(t) : i = 1, \dots, m\}$. Put

$$\begin{gathered} \eta^{\tau_*}(t) = \max\{s : \tau_*(s) \le t\} \quad \text{for} \quad t \in R_+, \\ \eta_1^{\tau_*}(t) = \eta^{\tau_*}(t), \quad \eta_i^{\tau_*} = \eta^{\tau_*} \circ \eta_{i-1}^{\tau_*} \quad (i = 2, 3, \dots). \end{gathered} \tag{2.1}$$

Definition 2.1. *Let $t_0 \in R_+$. A continuous function $x : [t_0, +\infty) \to R$ is said to be a proper solution of equation* (1.3) *if it is locally absolutely continuous on $[\eta^{\tau_*}(t_0), +\infty)$, satisfies* (1.3) *almost everywhere on $[\eta^{\tau_*}(t_0), +\infty)$, and*

$$\sup\{|x(s)| : t \le s < +\infty\} > 0 \quad \textit{for} \quad t \ge t_0.$$

Definition 2.2. *A proper solution of equation* (1.3) *is said to be oscillatory if the set of its zeros is unbounded from above. Otherwise it is said to be nonoscillatory.*

Definition 2.3. *The equation* (1.3) *is said to be oscillatory, if any of its proper solutions is oscillatory. Define*

$$\psi_1(t) = 0, \quad \psi_i(t) = \exp\left\{\sum_{j=1}^{m} \int_{\tau_j(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s)\Big)^{\frac{1}{m}} \psi_{i-1}(s)\, ds\right\}, \quad i = 2, 3, \dots. \tag{2.2}$$

Theorem 2.1. *Let there exist $k \in N$ and non-decreasing functions $\sigma_i \in C(R_+; R)$ such that*

$$\tau_i(t) \le \sigma_i(t) \le t, \quad i = 1, \dots, m \tag{2.3}$$

and

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds \right]^{\frac{1}{m}} >$$
$$> \frac{1}{m^m} - \prod_{\ell=1}^{m} c_\ell(p_{*\ell}). \tag{2.4}$$

Then the equation (1.3) is oscillatory, where

$$p_{*\ell} = \liminf_{t\to+\infty} \int_{\sigma_\ell(t)}^{t} p_\ell(s)\, ds,$$
$$c_\ell(p_{*\ell}) = \frac{1 - p_{*\ell} - \sqrt{1 - 2p_{*\ell} - p_{*\ell}^2}}{2}, \quad \ell = 1, \dots, m. \tag{2.5}$$

Theorem 2.1$'$. *Let $\overline{p}_* \le \frac{1}{e}$ and there exist nondecreasing functions $\sigma_i \in C(R_+; R)$ such that the conditions (2.3) are fulfilled and for some $\varepsilon \in (0, \lambda^*(\overline{p}_*))$*

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left\{ (\lambda^*(\overline{p}_*) - \varepsilon) \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} d\xi \right\} ds \right]^{\frac{1}{m}} >$$
$$> \frac{1}{m^m} - \prod_{\ell=1}^{m} c_\ell(p_{*\ell}). \tag{2.6}$$

Then equation (1.3) is oscillatory, where $p_{*\ell}$ and $c_\ell(p_{*\ell})$ are given by (2.5) and $\lambda^*(\overline{p}_*)$ is the smallest root of the equation

$$e^{\overline{p}_* \lambda} = \lambda, \tag{2.7}$$

$$\overline{p}_* = \liminf_{t\to+\infty} \sum_{i=1}^{m} \int_{\tau_i(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s) \Big)^{\frac{1}{m}} ds > 0. \tag{2.8}$$

Corollary 2.1. *Let τ_i be nondecreasing functions, $p_i(t) \ge p(t)$ $(i = 1, \dots, m)$, $p \in L_{\mathrm{loc}}(R_+ : R_+)$ and for some $\varepsilon \in (0, \lambda^*(\overline{p}_*))$*

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\tau_j(t)}^{t} p(s) \exp\left\{ m(\lambda^*(\overline{p}_*) - \varepsilon) \int_{\tau_i(s)}^{\tau_i(t)} p(\xi)\, d\xi \right\} ds \right]^{\frac{1}{m}} >$$
$$> \frac{1}{m^m} - \prod_{i=1}^{m} c_\ell(p_{*\ell}). \tag{2.9}$$

Then equation (1.3) is oscillatory, where $p_{*\ell}$ and c_ℓ are given by (2.5) and $\lambda^*(\overline{p}_*)$ is the smallest root of equation (2.7).

Theorem 2.2. *Let there exist nondecreasing functions $\sigma_i \in C(R_+;R)$ such that conditions* (2.3) *are fulfilled,*

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} d\xi\, ds \right]^{\frac{1}{m}} > 0, \tag{2.10}$$

and let

$$\liminf_{t\to+\infty} \sum_{i=1}^{m} \int_{\tau_i(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s) \Big)^{\frac{1}{m}} ds > \frac{1}{\ell}. \tag{2.11}$$

Then equation (1.3) *is oscillatory.*

Corollary 2.2. *Let τ_i be nondecreasing functions,*

$$\liminf_{t\to+\infty} \int_{\tau_j(t)}^{t} p_i(s) \int_{\tau_i(s)}^{\tau_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} d\xi\, ds > 0, \quad i,j = 1,\dots,m, \tag{2.12}$$

and let condition (2.11) *be fulfilled. Then equation* (1.3) *is oscillatory.*

Theorem 2.3. *Let τ_i be nondecreasing functions,*

$$\liminf_{t\to+\infty} \int_{\tau_i(t)}^{t} p_i(s)\, ds > 0, \quad i = 1,\dots,m, \tag{2.13}$$

and let condition (2.11) *be fulfilled. Then the equation* (1.3) *is oscillatory.*

3. Some auxiliary statements

In this section we establish estimates of the quotient

$$\frac{\Big(\prod\limits_{i=1}^{m} x(\tau_i(t)) \Big)^{\frac{1}{m}}}{x(t)},$$

where x is a nonoscillatory solution of equation (1.3).

Lemma 3.1. *Let $t_0 \in R_+$ and $x : [t_0,+\infty) \to (0,+\infty)$ be a solution of equation* (1.3)*. Then for any $i \in \{1,2,\dots\}$*

$$\Big(\prod_{\ell=1}^{m} x(\tau_\ell(t)) \Big)^{\frac{1}{m}} \ge \psi_i(t)\, x(t) \quad \textit{for} \quad t \ge \eta_i^{\tau_*}(t_0), \tag{3.1}$$

where the functions $\eta_i^{\tau_*}$ and ψ_i are defined by (2.1) *and* (2.2)*, respectively.*

Proof. From (1.3) for $t \ge \eta_1^{\tau_*}(t_0)$ we have

$$\frac{x(\tau_j(t))}{x(t)} \ge \exp\left\{ \int_{\tau_j(t)}^{t} \sum_{i=1}^{m} p_i(s) \frac{x(\tau_i(s))}{x(s)}\, ds \right\} \quad (j = 1,\dots,m).$$

Using the arithmetic mean-geometric mean inequality, from the last inequality we get

$$\frac{x(\tau_j(t))}{x(t)} \geq \exp\left\{m\int_{\tau_j(t)}^{t}\Big(\prod_{\ell=1}^{m}p_\ell(s)\Big)^{\frac{1}{m}}\frac{\Big(\prod\limits_{\ell=1}^{m}x(\tau_\ell(s))\Big)^{\frac{1}{m}}}{x(s)}\,ds\right\} \quad (j=1,\dots,m).$$

Therefore for $t \geq \eta_1^{\tau_*}(t_0)$,

$$\frac{\Big(\prod\limits_{j=1}^{m}x(\tau_j(t))\Big)^{\frac{1}{m}}}{x(t)} \geq \exp\left\{\sum_{j=1}^{m}\int_{\tau_j(t)}^{t}\Big(\prod_{\ell=1}^{m}p_\ell(s)\Big)^{\frac{1}{m}}\frac{\Big(\prod\limits_{\ell=1}^{m}x(\tau_\ell(s))\Big)^{\frac{1}{m}}}{x(s)}\,ds\right\}. \tag{3.2}$$

Inequality (3.1) is obviously fulfilled for $i=1$. Assuming its validity for some $i \in \{1,2,\dots\}$, by (3.2) for $t \geq \eta_{i+1}^{\tau_*}(t_0)$, we obtain

$$\Big(\prod_{\ell=1}^{m}x(\tau_\ell(t))\Big)^{\frac{1}{m}} \geq \exp\left\{\sum_{j=1}^{m}\int_{\tau_j(t)}^{t}\Big(\prod_{\ell=1}^{m}p_\ell(s)\Big)^{\frac{1}{m}}\psi_i(s)\,ds\right\}x(t) = \psi_{i+1}(t)\,x(t),$$

and the result follows. □

Lemma 3.2. *Let $\overline{p}_* \leq \frac{1}{\ell}$, where $\overline{p}_*$ is defined by (2.8). Then*

$$\lim_{i\to+\infty}\Big(\liminf_{t\to+\infty}\psi_i(t)\Big) \geq \lambda^*(\overline{p}_*), \tag{3.3}$$

where functions ψ_i are given by (2.2) and $\lambda^*(\overline{p}_*)$ is the smallest root of equation (2.7).

Proof. Suppose on the contrary, that (3.3) is not true. Let

$$\lim_{i\to+\infty}\Big(\liminf_{t\to+\infty}\psi_i(t)\Big) = \gamma < \lambda^*(\overline{p}_*). \tag{3.4}$$

Then there exists $\varepsilon_0 > 0$ such that

$$\frac{e^{\gamma\overline{p}_*}}{\gamma} \geq 1+\varepsilon_0. \tag{3.5}$$

On the other hand, for any $\varepsilon > 0$, by (2.8) and (3.4) there exist $k_\varepsilon \in N$ and $t_\varepsilon \in R_+$ such that

$$\sum_{j=1}^{m}\int_{\tau_j(t)}^{t}\Big(\prod_{\ell=1}^{m}p_\ell(s)\Big)^{\frac{1}{m}}ds \geq \overline{p}_*-\varepsilon, \quad \psi_k(t) \geq \gamma-\varepsilon \quad \text{for} \quad t \geq t_\varepsilon \quad \text{and} \quad k \geq k_\varepsilon. \tag{3.6}$$

According to (3.6) from (2.2) we get

$$\psi_{k+1}(t) \geq e^{(\gamma-\varepsilon)(\overline{p}_*-\varepsilon)} \quad \text{for} \quad t \geq t_\varepsilon \quad \text{and} \quad k \geq k_\varepsilon\,.$$

Therefore

$$\lim_{k\to+\infty}\Big(\liminf_{t\to+\infty}\psi_k(t)\Big) \geq e^{(\gamma-\varepsilon)(\overline{p}_*-\varepsilon)},$$

which implies $\gamma \geq e^{\gamma\cdot\overline{p}_*}$. In view (3.5), this is a contradiction, and the proof of the lemma 3.2 is complete.

Quite similarly one can prove the next result.

Lemma 3.3. *Let*

$$\bar{p}_* = \liminf_{t\to+\infty} \sum_{j=1}^{m} \int_{\tau_j(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s)\Big)^{\frac{1}{m}} ds > \frac{1}{e}, \tag{3.7}$$

then

$$\lim_{k\to+\infty} \big(\liminf_{t\to+\infty} \psi_k(t)\big) = +\infty, \tag{3.8}$$

where ψ_k are given by (2.2).

The proof of the next lemma can be found in [9].

Lemma 3.4. *Let there exist nondecreasing functions $\sigma_i \in C(R_+; R)$ such that condition* (2.3) *is fulfilled and equation* (1.3) *has an eventually positive solution $x : [t_0, +\infty) \to (0, +\infty)$. Then*

$$\liminf_{t\to+\infty} \frac{x(t)}{x(\sigma_i(t))} \ge c_i(p_{*i}) \quad (i = 1, \dots, m), \tag{3.9}$$

where p_{*i} and $c_i(p_{*i})$ are defined by (2.5)

4. Proofs of the theorems

Proof of Theorem 2.1. Suppose on the contrary that the equation (1.3) has a nonoscillatory proper solution $x : [t_0, +\infty) \to R$. Since $-x(t)$ is also a solution for (1.3), we confine ourselves only to the case where $x(t)$ is an eventually positive solution of equation (1.3). Then there exists $t_1 > t_0$ such that $x(\tau_i(t)) > 0$ for $t \ge t_1$ $(i = 1, \dots, m)$. As we have seen, while proving Lemma 3.1

$$\Big(\prod_{\ell=1}^{m} x(\tau_\ell(t))\Big)^{\frac{1}{m}} \ge \psi_i(t)\, x(t) \quad \text{for} \quad t \ge \eta_i^{\tau_*}(t_1) \quad (i = 1, 2, \dots), \tag{4.1}$$

where the functions $\eta_i^{\tau_*}$ and ψ_i are defined by (2.1) and (2.2), respectively. From (1.3) we have

$$x(s) = x(t) \exp\Big\{ \int_s^t \sum_{i=1}^{m} p_i(\xi) \frac{x(\tau_i(\xi))}{x(\xi)}\, d\xi \Big\} \quad \text{for} \quad t \ge s \ge t_1. \tag{4.2}$$

Integrating (1.3) from $\sigma_j(t)$ to t, for sufficiently large t, we get

$$x(\sigma_j(t)) = \sum_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s)\, x(\tau_i(s))\, ds + x(t). \tag{4.3}$$

On the other hand, taking into account (2.3), from (4.2) for sufficiently large t, we obtain

$$\frac{x(\tau_j(t))}{x(t)} = \exp\Big\{ \int_{\tau_j(t)}^{t} \sum_{i=1}^{m} p_i(s) \frac{x(\tau_i(s))}{x(s)}\, ds \Big\} \quad (j = 1, \dots, m) \tag{4.4}$$

and

$$x(\tau_j(s)) = x(\sigma_j(t)) \exp\left\{ \int_{\tau_j(s)}^{\sigma_j(t)} \sum_{i=1}^{m} p_i(\xi) \frac{x(\tau_i(\xi))}{x(\xi)}\, d\xi \right\} \tag{4.5}$$
$$\text{for} \quad t \geq s > \eta_1^{\tau_*}(t_1) \quad (j = 1, \dots, m).$$

Combining (4.1), (4.3) – (4.5) and using the arithmetic mean-geometric mean inequality for $j = 1, 2, \dots, m$, we can write

$$x(\sigma_j(t)) \geq \sum_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s)\, x(\sigma_i(t)) \times$$
$$\times \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \frac{\Big(\prod_{\ell=1}^{m} x(\tau_\ell(\xi)) \Big)^{\frac{1}{m}}}{x(\xi)}\, d\xi \right\} + x(t) \geq$$
$$\geq \sum_{i=1}^{m} x(\sigma_i(t)) \int_{\sigma_j(t)}^{t} p_i(s) \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds + x(t) \geq$$
$$\geq m \Big[\prod_{i=1}^{m} x(\sigma_i(t)) \Big]^{\frac{1}{m}} \Big[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \cdot$$
$$\cdot \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds \Big]^{\frac{1}{m}} + x(t).$$

Therefore

$$w(t) \geq m^m w(t) \prod_{j=1}^{m} \Big[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \times$$
$$\times \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds \Big]^{\frac{1}{m}} + x^m(t),$$

where $w(t) = \prod\limits_{j=1}^{m} x(\sigma_j(t))$. Hence

$$\limsup_{t \to +\infty} \left(\prod_{j=1}^{m} \Big[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left\{ m \int_{\tau_i(s)}^{\sigma_i(t)} \Big(\prod_{\ell=1}^{m} p_\ell(\xi) \Big)^{\frac{1}{m}} \psi_k(\xi)\, d\xi \right\} ds \Big]^{\frac{1}{m}} + \right.$$
$$\left. + \frac{x^m(t)}{\prod\limits_{\ell=1}^{m} x(\sigma_\ell(t))} \right) \leq \frac{1}{m^m}. \tag{4.6}$$

On the other hand, by Lemma 3.4 we have

$$\liminf_{t \to +\infty} \frac{x(t)}{x(\sigma_\ell(t))} \geq c_\ell(p_{*\ell}), \quad \ell = 1, \dots, m, \tag{4.7}$$

where $p_{\ell *}$ and c_ℓ are given by (2.5). Therefore, according to (4.6) and (4.7), we get

$$\limsup_{t\to+\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t}p_i(s)\cdot\exp\left\{m\int_{\tau_i(s)}^{\sigma_i(t)}\Big(\prod_{\ell=1}^{m}p_\ell(\xi)\Big)^{\frac{1}{m}}\psi_k(\xi)\,d\xi\right\}ds\right]^{\frac{1}{m}}+$$
$$+\prod_{\ell=1}^{m}c_\ell(p_{*\ell})\le$$
$$\le\limsup_{t\to+\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t}p_i(s)\exp\left\{m\int_{\tau_i(s)}^{\sigma_i(t)}\Big(\prod_{\ell=1}^{m}p_\ell(\xi)\Big)^{\frac{1}{m}}\psi_k(\xi)\,d\xi\right\}ds\right]^{\frac{1}{m}}+$$
$$+\liminf_{t\to+\infty}\frac{x^m(t)}{\prod\limits_{\ell=1}^{m}x(\sigma_\ell(t))}\le$$
$$\le\limsup_{t\to+\infty}\left(\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t}p_i(s)\exp\left\{m\int_{\tau_i(s)}^{\sigma_i(t)}\Big(\prod_{\ell=1}^{m}p_\ell(\xi)\Big)^{\frac{1}{m}}\psi_k(\xi)\,d\xi\right\}ds\right]^{\frac{1}{m}}+\right.$$
$$\left.+\frac{x^m(t)}{\prod\limits_{\ell=1}^{m}x(\sigma_\ell(t))}\right)\le\frac{1}{m^m},$$

which contradicts (2.4). □

Proof of Theorem 2.1′. Suppose on the contrary that equation (1.3) has a nonoscillatory solution $x:[t_0,+\infty)\to(0,+\infty)$. Then by Lemma 3.2, the inequality (3.3) is fulfilled. Therefore, for any $\varepsilon>0$, there exist $t_\varepsilon\in R_+$ and $k_0\in N$ such that

$$\psi_{k_0}(t)\ge\lambda^*(\overline{p}_*)-\varepsilon\quad\text{for}\quad t\ge t_\varepsilon,\tag{4.8}$$

where $\overline{p}_*$ is defined by (2.8) and $\lambda^*(\overline{p}_*)$ is the smallest root of equation (2.7). Taking into account that (2.6) and (2.8) imply (2.4), it is easy to see that Theorem 2.1′ is a straightforward consequence of Theorem 2.1. □

Proof of Corollary 2.1 immediately follows from Theorem 2.1′, if we take $\tau_i(t)\equiv\sigma_i(t)$. □

Proof of Theorem 2.2. Suppose on the contrary that equation (1.3) has a nonoscillatory solution $x:[t_0,+\infty)\to(0,+\infty)$. Then by Lemma 3.3, condition (3.8) holds. In view of (2.10), we can choose $M>0$ to satisfy

$$(e\,M)^{\frac{1}{m}}\cdot\limsup_{t\to+\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t}p_i(s)\int_{\tau_i(s)}^{\sigma_i(t)}\Big(\prod_{\ell=1}^{m}p_\ell(\xi)\Big)^{\frac{1}{m}}d\xi\,ds\right]^{\frac{1}{m}}>$$
$$>\frac{1}{m^m}-\prod_{\ell=1}^{m}c_\ell(p_{*\ell}),\tag{4.9}$$

where $p_{*\ell}$ and c_ℓ are given by (2.5).

On the other hand, according to (3.8), there exist $k_0 \in N$ and $t_0 \in R_+$ such that $\psi_{k_0}(t) \geq M$ for $t \geq t_0$. Since $e^x \geq ex$, by (4.9) the condition (2.4) is fulfilled for $k = k_0$, and the result follows. □

Proof of Corollary 2.2 is similar to that of Theorem 2.2, and so is omitted.

Proof of Theorem 2.3. Suppose on the contrary that equation (1.3) has an eventually positive solution $x(t)$. Then by (2.13) and Lemma 3.4 we have

$$\limsup_{t\to+\infty} \frac{x(\tau_i(t))}{x(t)} < +\infty, \quad i = 1, \dots, m. \tag{4.10}$$

On the other hand, by Lemmas 3.3 and 3.4 conditions (3.1) and (3.8) hold. But this contradicts condition (4.10), and the result follows. □

Remark 4.1. Condition (2.11) for any $\varepsilon > 0$, cannot be replaced by the condition

$$\liminf_{t\to+\infty} \sum_{i=1}^{m} \int_{\tau_i(t)}^{t} \Big(\prod_{\ell=1}^{m} p_\ell(s)\Big)^{\frac{1}{m}} ds \geq \frac{1-\varepsilon}{e}. \tag{4.11}$$

Consider the differential equation

$$x'(t) + \sum_{i=1}^{m} c_i x(t - \Delta_i) = 0, \tag{4.12}$$

where $c_i, \Delta_i \in (0, +\infty)$, $i = 1, \dots, m$. Choose $\delta > 0$ such that, if

$$\sum_{j,i=1}^{m} (|c_i - c_j| + |\Delta_i - \Delta_j|) < \delta,$$

then

$$\sum_{i=1}^{m} c_i e^{\lambda\Delta_i} \leq \Big(1 + \frac{\varepsilon}{2}\Big) m \Big(\prod_{i=1}^{m} c_i\Big)^{\frac{1}{m}} e^{\frac{\lambda}{m}\sum\limits_{i=1}^{m}\Delta_i} \tag{4.13}$$

and

$$\frac{1-\varepsilon}{e} \leq \Big(\prod_{i=1}^{m} c_i\Big)^{\frac{1}{m}} \sum_{i=1}^{m} \Delta_i \leq \frac{1-\frac{\varepsilon}{2}}{e}. \tag{4.14}$$

By (4.13) and (4.14) we have

$$\max\left\{ \frac{\lambda}{\sum\limits_{i=1}^{m} c_i e^{\lambda\Delta_i}} : \lambda \in [0, +\infty) \right\} \geq$$

$$\geq \frac{1}{m(1 + \frac{\varepsilon}{2})\big(\prod\limits_{i=1}^{m} c_i\big)^{\frac{1}{m}}} \max\left\{ \frac{\lambda}{e^{\frac{\lambda}{m}\sum\limits_{i=1}^{m}\Delta_i}} : \lambda \in [0, +\infty) \right\} =$$

$$= \frac{1}{m(1+\frac{\varepsilon}{2})\left(\prod_{i=1}^{m} c_i\right)^{\frac{1}{m}}} \cdot \frac{m}{e\sum_{i=1}^{m} \Delta_i} \geq \frac{1}{(1+\frac{\varepsilon}{2})(1-\frac{\varepsilon}{2})} = \frac{1}{1-\frac{\varepsilon^2}{4}} > 1.$$

According to the last inequality, it is obvious that $e^{-\lambda_0 \cdot t}$ is a solution of equation (4.12), where λ_0 is the root of equation $\lambda = \sum_{i=1}^{m} c_i e^{\lambda \Delta_i}$. On the other hand, by (4.14), condition (4.11) holds, where $c_i = p_i(t)$ and $t - \Delta_i = \tau_i(t)$.

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