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EXPONENTIAL CONVERGENCE OF NONLINEAR TIME-VARYING DIFFERENTIAL EQUATIONS

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Abstract. In this paper we present a converse Lyapunov theorem for global practical uniform exponential stability of nonlinear time-varying systems. The main result shows that the system is practically globally uniformly exponentially stable if and only if it admits a Lyapunov function which satisfies some conditions. An example is also discussed to illustrate the advantage of the proposed result.

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1. Introduction

The investigation of practical stability of nonlinear systems using the second Lyapunov function method has produced a number of important results and has been widely studied in the literature (see, e.g., [1]-[5], and references therein). It is known that requiring the existence of a Lyapunov function that satisfies certain conditions implies practical exponential stability (see, e.g., [4], [6]-[9]). Requiring the existence of an auxiliary function $V(t, x)$ that satisfies certain conditions is typical in many results obtained by Lyapunov's method. The question of the validity of the converse results, which arises naturally, originates the problem of the existence of such a Lyapunov function, that is, the problem of the existence of converse Lyapunov theorems.

It is known that practical stability is neither weaker nor stronger than the usual stability; an equilibrium can be stable in the usual sense, but not practically stable, and vice versa. Practical stability is, in a sense, a uniform boundedness of the solution relative to the initial conditions, but the bound must be sufficiently small. On the other hand, the asymptotic stability is more important than stability. Also, the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. Thus the notion of practical stability is more suitable in several situations than Lyapunov stability.

Lyapunov's second method, also known as the direct method, is a widely recognized and commonly used technique for studying the stability of nonlinear systems. This method employs construction of a Lyapunov function. Converse Lyapunov results are an important tool in the analysis of stability of nonlinear time-varying systems and have been well studied (see [4], [10] – [12]).

In this paper we consider nonlinear time-varying differential systems of the form

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{1.1}$$

where $t \geq t_0 \geq 0$, $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in t and is locally Lipschitz in x.

The purpose of this paper is to obtain a converse Lyapunov theorem for uniform global practical exponential stability. In our proof of this result we use some arguments from [4, 10].

The paper is organized as follows. In section 2 we give some definitions and results about the practical global uniform stability and practical global uniform exponential stability. In section 3 we propose new sufficient conditions with the extended Lyapunov functions for the practical exponential stability of nonlinear time-varying system (1.1). A simple example is also discussed. In section 4 we present our main result - the converse of the Lyapunov theorem, proved in section 3. In addition, we give an illustrative example to demonstrate the applicability of the obtained principal result.

2. Auxiliary facts and results

In this section, we introduce some basic definitions and preliminary facts which we need in the sequel. We first recall the following definitions from [10].

Definition 2.1 (Uniform boundedness). *A solution of the system* (1.1) *is said to be globally uniformly bounded if for every $\alpha > 0$ there exists $c = c(\alpha)$ such that for all $t_0 \geq 0$*

$$\|x_0\| \leq \alpha \Rightarrow \|x(t)\| \leq c(\alpha), \quad \forall\, t \geq t_0.$$

Let $r \geq 0$ and $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$.

Definition 2.2. *(Uniform stability of B_r).*

(i) *B_r is uniformly stable if for all $\epsilon > r$ there exists $\delta = \delta(\epsilon) > 0$ such that for all $t_0 \geq 0$*

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall\, t \geq t_0;$$

(ii) B_r *is globally uniformly stable if it is uniformly stable and the solutions of system* (1.1) *are globally uniformly bounded.*

Definition 2.3 (Uniform attractivity of B_r). B_r *is globally uniformly attractive if for all* $\epsilon > r$ *and* $c > 0$ *there exists* $T(\epsilon, c) > 0$ *such that for all* $t_0 \geq 0$

$$\|x(t)\| < \epsilon \quad \forall\, t \geq t_0 + T(\epsilon, c), \quad \|x_0\| < c.$$

Definition 2.4 (Comparison functions). *A function* $\alpha : [0, a[\to [0, +\infty[$ *is said to be of class* $\mathcal{K}$, *if it is continuous, strictly increasing and* $\alpha(0) = 0$, *and it is said to be of class* $\mathcal{K}_\infty$ *if, in addition,* $a = +\infty$ *and* $\alpha(r) \to +\infty$ *as* $r \to +\infty$. *A continuous function* $\sigma : [0, +\infty[\to [0, +\infty[$ *is said to be of class* $\mathcal{L}$ ($\sigma \in \mathcal{L}$), *if it is decreasing and tends to zero as its argument tends to infinity. A function* $\beta : [0, +\infty[\times[0, +\infty[\to [0, +\infty[$ *is said to be of class* $\mathcal{KL}$, *if* $\beta(., t) \in \mathcal{K}$ *for any* $t \geq 0$, *and* $\beta(s, .) \in \mathcal{L}$ *for any* $s \geq 0$.

Definition 2.5. *(Lyapunov function] Let* $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ *be of class* C^1.

(i) $V(t, x)$ *is positive definite, that is, there exists a continuous, non-decreasing scalar function* $\alpha(x)$ *such that* $\alpha(0) = 0$ *and*

$$0 < \alpha(\|x\|) < V(t, x), \quad \forall x \neq 0.$$

(ii) $\dot{V}(t, x)$ *is negative definite, that is,*

$$\dot{V}(t, x) \leq -\gamma(\|x\|) < 0,$$

where γ *is a continuous non-decreasing scalar function such that* $\gamma(0) = 0$.

(iii) $V(t, x) \leq \beta(\|x\|)$, *where* β *is a continuous non-decreasing function and* $\beta(0) = 0$, *that is, the Lyapunov function is upper bounded.*

(iv) V *is radially unbounded, that is,* $\alpha(\|x\|) \to \infty$ *as* $\|x\| \to \infty$.

A sufficient condition for Globally Uniformly Practically Attractivity Stable (GUPAS) is the existence of a class $\mathcal{KL}$ of functions β and a constant $r > 0$ such that for any initial state x_0 the trajectory $x(t)$ satisfies:

$$\|x(t)\| \leq \beta(\|x_0\|, t) + r, \quad \forall t \geq t_0.$$

If the class $\mathcal{KL}$ consists of functions of the form $\beta(r, t) = kr^{-\lambda t}$, with $\lambda, k > 0$, then we say that the system (1.1) is Globally Uniformly Practically Exponentially Stable (GUPES).

It is worth to note that, if in the above definitions we take $r = 0$, then one deals with the standard concepts of Globally Uniformly Attractivity Stable (GUAS) and

Globally Uniformly Exponentially Stable (GUES), respectively. Moreover, in the rest of this paper, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq \| x(t) \| - r$, so that if in the above definition $r = 0$, then we get the classical definition of the uniform asymptotic stability of the origin viewed as an equilibrium point. The next definition concerns the practical global uniform exponential stability.

Definition 2.6 (Global uniform practical exponential stability of a ball). *B_R is globally uniformly exponentially stable if there exist positive constants γ, k, η, ξ, c, such that for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$,*

$$\|x(t)\| \leq k\|x_0\| \exp(-\gamma(t - t_0)) + \eta \exp(-\xi t) + c. \tag{2.1}$$

The system (1.1) is globally practically uniformly exponentially stable if there exists $R > 0$ such that B_R is globally uniformly exponentially stable.

To explain better the notion of practical stability, we consider the following scalar equation

$$\dot{x} = -x + \frac{\sin t}{1 + tx^2} \tag{E}$$

with $t_0 \in \mathbb{R}_+$. Let $V(x) = \frac{1}{2}x^2$ be a Lyapunov function candidate for equation (E). The derivative of V along the trajectories is given by

$$\dot{V}(x) = -x^2(1 - \frac{\sin t}{1 + tx^2}) \leq -x^2 + \left|\frac{\sin t}{t}\right|.$$

It follows that

$$\dot{V}(x) \leq -2V(x) + \left|\frac{\sin t}{t}\right|.$$

Note that the solutions of equation (E) can not be given explicitly. In this situation, we can not deduce the stability of the origin, but using Lyapunov function we can obtain estimates on the trajectories. Indeed, since $\left|\frac{\sin t}{t}\right|$ is bounded by 1, we obtain the following inequality:

$$V(x(t)) \leq (V(x(0)) - \frac{1}{2})e^{-2t} + \frac{1}{2}.$$

Hence, for $x(0) < -1$ and $x(0) > 1$, we have

$$|x(t)| \leq \left(x(0)^2 - 1\right)^{\frac{1}{2}} e^{-t} + 1.$$

So, for $x(0) < -1$ and $x(0) > 1$, B_1 is uniformly exponentially stable.

Note that we cannot study stability of the origin as an equilibrium point. The last inequality implies that $x(t)$ will be ultimately bounded by a small bound and since $\left|\frac{\sin t}{t}\right|$ tends to zero as t tends to $+\infty$, the ultimate bound approaches to zero, and so $x(t)$ converges to the origin. This implies the attractivity of the origin.

3. Global uniform practical exponential stability of a class of non linear time-varying systems

Consider the class of systems that can be modeled by (1.1). The practical uniform exponential stability can be established by showing existence of a Lyapunov function that satisfies certain conditions. The next theorem contains sufficient conditions for the system (1.1) to be practically globally uniformly exponentially stable. It is worth to notice that the origin is not required to be an equilibrium point for the system (1.1).

Theorem 3.1. *Consider the system* (1.1). *Let* $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a continuous differentiable Lyapunov function, such that for all* $t \geq 0$ *and* $x \in \mathbb{R}^n$

$$c_1\|x\|^r \leq V(t,x) \leq c_2\|x\|^r + a \tag{3.1}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^r + Me^{-\delta t} + K_1, \tag{3.2}$$

where c_1, c_2, c_3, r, a, δ *and* K_1 *are positive constants with*

$$r > 1 \text{ and } \delta < \frac{c_3}{c_2}.$$

Then the system (1.1) *is globally uniformly practically exponentially stable.*

Proof. Let $t_0 \geq 0$ be any initial time, $x(t)$ be any solution of (1.1) with $x(t_0) = x_0$, and let V be the Lyapunov function candidate of the system (1.1).

The derivative of V along the trajectories of the system (1.1) is given by

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\, f(t,x).$$

By the condition (3.1), we have $-\|x\|^r \leq \dfrac{a - V(t,x)}{c_2}$. Therefore, taking into account (3.2), for all $t \geq t_0$ and $x \in \mathbb{R}^n$, we can write

$$\dot{V}(t,x) \leq -\frac{c_3}{c_2}\, V(t,x) + \frac{a\, c_3}{c_2} + M\, e^{-\delta t} + K_1.$$

Thus, there exists $\lambda > 0$ such that for all $t \geq t_0 \geq 0$

$$V(t,x) \leq \lambda\, V(t_0,x_0)\, e^{-\frac{c_3}{c_2}(t-t_0)} + \frac{Mc_2}{c_3 - c_2\delta}\, e^{-\delta t} + ac_3 + c_2K_1.$$

It follows from (3.1) that $V(t_0,x_0) \leq c_2\|x_0\|^r + a$ and $\|x(t)\| \leq \left[\frac{V(t,x(t))}{c_1}\right]^{\frac{1}{r}}$. Therefore, we can write for all $t \geq t_0 \geq 0$

$$\|x(t)\| \leq \frac{1}{c_1^{\frac{1}{r}}}\left[\lambda\left(c_2\|x_0\|^r + a\right)e^{-\frac{c_3}{c_2}(t-t_0)} + \frac{Mc_2}{c_3 - c_2\delta}e^{-\delta t} + ac_3 + c_2K_1\right]^{\frac{1}{r}}$$

$$\leq \frac{1}{c_1^{\frac{1}{r}}}\left[\left(\lambda c_2\|x_0\|^r e^{-\frac{c_3}{c_2}(t-t_0)} + \lambda a e^{-\frac{c_3}{c_2}(t-t_0)}\right)^{\frac{1}{r}} + \left(\frac{Mc_2}{c_3 - c_2\delta}e^{-\delta t} + ac_3 + c_2K_1\right)^{\frac{1}{r}}\right]$$

$$\leq \left(\frac{\lambda c_2}{c_1}\right)^{\frac{1}{r}}\|x_0\|e^{-\frac{c_3}{c_2 r}(t-t_0)} + \left(\frac{\lambda a e^{\frac{c_3}{c_2}t_0}}{c_1}\right)^{\frac{1}{r}} e^{-\frac{c_3}{c_2 r}t}$$

$$+\left(\frac{Mc_2}{c_1(c_3 - c_2\delta)}\right)^{\frac{1}{r}} e^{-\frac{\delta}{r}t} + \left(\frac{ac_3 + c_2K_1}{c_1}\right)^{\frac{1}{r}}.$$

Denoting $\beta = \min\{\frac{c_3}{c_2 r}, \frac{\delta}{r}\}$, we obtain for all $t \geq t_0 \geq 0$

$$\|x(t)\| \leq \left(\frac{\lambda c_2}{c_1}\right)^{\frac{1}{r}}\|x_0\|e^{-\frac{c_3}{c_2 r}(t-t_0)} + \left[\left(\frac{\lambda a e^{\frac{c_3}{c_2}t_0}}{c_1}\right)^{\frac{1}{r}} + \left(\frac{Mc_2}{c_1(c_3 - c_2\delta)}\right)^{\frac{1}{r}}\right]e^{-\beta t}$$

$$+ \left(\frac{ac_3 + c_2K_1}{c_1}\right)^{\frac{1}{r}}.$$

The last inequality shows that the system (1.1) is GUPES. □

Example 3.1. Consider the following nonlinear differential equation:

$$\dot{x} = -x + \frac{1}{1+x^2}\, e^{-t}, \quad t \geq 0. \tag{3.3}$$

Let us take a Lyapunov function $V(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$, given by $V(t,x) = x^3 + e^{-t}$. Note that for all $x \in \mathbb{R}$ we have $|x|^3 \leq V(t,x) \leq |x|^3 + 1$ implying that the condition (3.1) holds with $c_1 = c_2 = 1$, $r = 3$ and $a = 1$. On the other hand, we have

$$\dot{V}(t,x) = -\,e^{-t} + 3\,x^2\,\dot{x} \leq -\,e^{-t} + 3\,x^2\,(-|x| + \frac{1}{1+x^2}\,e^{-t})$$

$$\leq -\,e^{-t} - 3\,|x|^3 + \frac{3x^2}{1+x^2}\,e^{-t}.$$

Therefore, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ we get $\dot{V}(t,x) \leq -3\,|x|^3 + 2\,e^{-t}$ implying that the condition (3.2) is satisfied with $c_3 = 3$, $M = 2$, $\delta = 1$, $r = 3$ and $K_1 = 0$. Thus, the system (3.3) is GUPES.

4. A CONVERSE THEOREM

The above discussions and results arise the following question: if the system (1.1) is GUPES, do there exist a function V which satisfies the hypotheses of Theorem 3.1? In [6], a converse theorem was established in the case where the origin is not an equilibrium point of the system, but it is assumed that there exists a non-negative constant f_0 such that $\|f(t,0)\| \leq f_0$ for all $t \geq 0$. In Theorem 4.1 that follows, we show that under this assumption there exists a function V that satisfies conditions similar (but not the same) to those of Theorem 3.1.

We first prove the following lemma which will be used later.

Lemma 4.1. *Consider the nonlinear system* (1.1). *Let* $\phi(\tau;t,x)$ *be a solution of the system that starts at* (t,x), *and let* $\phi_x(\tau;t,x) = \frac{\partial}{\partial x}\phi(\tau;t,x)$ *and* $\left\|\frac{\partial f}{\partial x}(t,x)\right\| \leq L$, *where* L *is a positive constant. Then*

$$\|\phi_x(\tau;t,x)\| \leq e^{L(\tau-t)}.$$

Proof. Note that ϕ_x is the solution of

$$\frac{\partial}{\partial\tau}\phi_x(\tau;t,x) = \frac{\partial f}{\partial x}(\tau,\phi(\tau;t,x))\phi_x, \quad \phi_x(t;t,x) = I.$$

We have

$$\left|\tfrac{d}{d\tau}\phi_x^T(\tau;t,x)\phi_x(\tau;t,x)\right| = \left|\tfrac{d}{d\tau}(\phi_x^T)\phi_x + \phi_x^T(\tfrac{d}{d\tau}\phi_x)\right|$$

$$\leq \left|(\tfrac{\partial f}{\partial x}(t,\phi)\phi_x)^T\phi_x + \phi_x^T(\tfrac{\partial f}{\partial x}(t,\phi)\phi_x)\right|$$

$$\leq \left|\phi_x^T(\tfrac{\partial f}{\partial x}(t,\phi))^T\phi_x + \phi_x^T(\tfrac{\partial f}{\partial x}(t,\phi)\phi_x)\right| \leq 2\left|\tfrac{\partial f}{\partial x}(t,\phi)\right|\|\phi_x\|^2.$$

Therefore,

$$-2L\|\phi_x\|^2 \leq \frac{\partial}{\partial t}(\|\phi_x\|^2) \leq 2L\|\phi_x\|^2.$$

Integrating the above inequality from t to τ, we get

$$\int_t^\tau -2L \leq \int_t^\tau \frac{\frac{\partial}{\partial t}(\|\phi_x\|)^2}{\|\phi_x\|^2} \leq \int_t^\tau 2L.$$

Therefore, for all $\tau \geq t$ we obtain

$$-2L(\tau-t) \leq \left[\log(\|\phi_x(s,t,x)\|^2\right]_t^\tau \leq 2L(\tau-t)$$

$$-2L(\tau-t) \leq \log\|\phi_x(\tau,t,x)\|^2 - \log\|\phi_x(t,t,x)\|^2 \leq 2L(\tau-t).$$

Since for $\|\phi_x(t,t,x)\| = \|I\| = 1$,

$$-2L(\tau-t) \leq \log\|\phi_x(\tau,t,x)\|^2 \leq 2L(\tau-t),$$

we obtain $\|\phi_x(\tau,t,x)\| \leq e^{L(\tau-t)}$. Lemma 4.1 is proved.

Theorem 4.1. *Consider the nonlinear system* (1.1). *Let f be a continuously differentiable function, such that the Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$ is bounded on $\mathbb{R}^n$ uniformly in t, and there exists a nonnegative constant f_0 such that $\|f(t,0)\| \leq f_0$ for all $t \geq 0$. Assume that the trajectories of the system* (1.1) *satisfy the condition* (2.1) *for all $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ and for some positive constants k, γ, β and α. Then there exist a natural number $r \geq 2$ and a function $V : [0,+\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}$ that satisfy the following inequalities:*

$$c_1\|x\|^r \leq V(t,x) \leq c_2\|x\|^r + a$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3\|x\|^r + Me^{-\delta t} + K_1$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|^{r-1} + Ne^{-\theta t} + K_2$$

for some positive constants c_1, c_2, c_3, c_4, a, M, N, δ, θ, K_1 and K_2.

Proof. The proof follows closely the proof of the converse theorem when the origin is exponentially stable (see [6], [10]). Let $\phi(\tau;t,x)$ be a solution of the system that starts at (t,x), that is, $\phi(t;t,x) = x$, and let L denote the bound of $\left[\frac{\partial f}{\partial x}\right]$. We have

$$\left|\tfrac{d}{d\tau}\phi^T(\tau;t,x)\phi(\tau;t,x)\right| = \left|2\phi^T(\tau;t,x)f(\tau,\phi(\tau;t,x))\right|$$

$$\leq 2\|\phi(\tau;t,x)\|\,\|f(\tau,\phi(\tau;t,x)) - f(t,0) + f(t,0)\| \leq 2L\|\phi(\tau;t,x)\|^2 + 2f_0\|\phi(\tau;t,x)\|.$$

Thus,

$$\frac{d}{d\tau}\phi^T(\tau;t,x)\phi(\tau;t,x) \geq -2L\|\phi(\tau;t,x)\|^2 - 2f_0\|\phi(\tau;t,x)\|. \tag{4.1}$$

Letting $v(\tau) = -\|\phi(\tau;t,x)\|$ and using (4.1), we deduce (as in [10], Example 3.9, pp. $103-104$) that $D^+v(\tau) \leq -Lv(\tau) + f_0$. By the comparison lemma (see [10], pp. $102-103$), we conclude that

$$\|\phi(\tau;t,x)\| + \frac{f_0}{L} \geq \left(\|x\| + \frac{f_0}{L}\right)e^{-L(\tau-t)}.$$

Next, using the inequality $(a+b)^n \leq 2^n(a^n + b^n)$, for all $n \in \mathbb{N}^*$, $a, b \geq 0$, we obtain

$$\left[\left(\|x\| + \tfrac{f_0}{L}\right)e^{-L(\tau-t)}\right]^r = \left(\|x\| + \tfrac{f_0}{L}\right)^r e^{-rL(\tau-t)}$$

$$\leq \left(\|\phi(\tau;t,x)\| + \tfrac{f_0}{L}\right)^r \leq 2^r\|\phi(\tau;t,x)\|^r + 2^r\left(\tfrac{f_0}{L}\right)^r.$$

Therefore

$$\|\phi(\tau;t,x)\|^r + \left(\frac{f_0}{L}\right)^r \geq \frac{1}{2^r}\left(\|x\| + \frac{f_0}{L}\right)^r e^{-rL(\tau-t)}. \tag{4.2}$$

Setting

$$V(t,x) = \int_t^{t+T} \left(\phi^T(\tau;t,x)\phi(\tau;t,x)\right)^{\frac{r}{2}} + \left(\frac{f_0}{L}\right)^r d\tau,$$

where T is a positive constant to be chosen later, we can write for all $t \geq 0$

$$\begin{aligned} V(t,x) &= \int_t^{t+T} \|\phi(\tau;t,x)\|^r + \left(\frac{f_0}{L}\right)^r d\tau \\ &\leq \int_t^{t+T} \left(k\|x\|e^{-\gamma(\tau-t)} + \alpha e^{-\beta\tau} + c\right)^r + \left(\frac{f_0}{L}\right)^r d\tau \\ &\leq \int_t^{t+T} 2^r(\alpha e^{-\beta\tau} + c)^r + 2^r k^r \|x\|^r e^{-r\gamma(\tau-t)} + \left(\frac{f_0}{L}\right)^r d\tau \\ &\leq \int_t^{t+T} 2^{2r}\alpha^r e^{-r\beta\tau} + 2^{2r}c^r + 2^r k^r \|x\|^r e^{-r\gamma(\tau-t)} + \left(\frac{f_0}{L}\right)^r d\tau \\ &\leq \frac{2^r k^r}{\gamma r}(1-e^{-r\gamma T})\|x\|^r + \frac{2^{2r}\alpha^r}{r\beta}(1-e^{-r\beta T}) + \left(\left(\frac{f_0}{L}\right)^r + 2^{2r}c^r\right)T. \end{aligned}$$

On the other hand, using (4.2), we have

$$V(t,x) \geq \frac{1}{2^r}\int_t^{t+T} \left(\|x\| + \frac{f_0}{L}\right)^r e^{-rL(\tau-t)} d\tau \geq \frac{1}{2^r rL}\left(1 - e^{-rLT}\right)\|x\|^r.$$

Thus, $V(t,x)$ satisfies the first inequality of Theorem 4.1 with

$$c_1 = \frac{1}{2^r rL}(1-e^{-rLT}),\ c_2 = \frac{2^r k^r}{\gamma r}(1-e^{-r\gamma T})$$

$$a = \left(\left(\frac{f_0}{L}\right)^r + 2^{2r}c^r\right)T + \frac{2^{2r}\alpha^r}{r\beta}(1-e^{-r\beta T}).$$

To calculate the derivative of V along the trajectories of the system, we use the following notation:

$$\phi_t(\tau;t,x) = \frac{\partial}{\partial t}\phi(\tau;t,x),\quad \phi_x(\tau;t,x) = \frac{\partial}{\partial x}\phi(\tau;t,x).$$

Then, we have

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) &= \left(\phi^T(t+T;t,x)\phi(t+T;t,x)\right)^{\frac{r}{2}} - \left(\phi^T(t;t,x)\phi(t;t,x)\right)^{\frac{r}{2}} \\ &+r\int_t^{t+T} \left[\left(\phi^T(\tau;t,x)\phi(\tau;t,x)\right)^{\frac{r}{2}-1}\phi^T(\tau;t,x)\phi_t(\tau;t,x)\right] d\tau \\ &+r\int_t^{t+T} \left[\left(\phi^T(\tau;t,x)\phi(\tau;t,x)\right)^{\frac{r}{2}-1}\phi^T(\tau;t,x)\phi_x(\tau;t,x)\right] f(t,x) d\tau \\ &= \|\phi(t+T;t,x)\|^r - \|x\|^r + r\int_t^{t+T} \left(\phi^T(\tau;t,x)\phi(\tau;t,x)\right)^{\frac{r}{2}-1}\phi^T(\tau;t,x) \\ &\quad \left(\phi_t(\tau;t,x) + \phi_x(\tau;t,x)\right) f(t,x) d\tau. \end{aligned}$$

As in [10], we obtain $\phi_t(\tau;t,x)+\phi_x(\tau;t,x)f(t,x)\equiv 0 \ \ \forall\ \tau\geq t$. Therefore, we can write

$$\begin{aligned}\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x) &= \|\phi(t+T;t,x)\|^r-\|x\|^r\\ &\leq \left(k\|x\|e^{-\gamma T}+\alpha e^{-\beta(t+T)}+c\right)^r-\|x\|^r\\ &\leq 2^r(\alpha e^{-\beta(t+T)}+c)^r+2^rk^r\|x\|^re^{-r\gamma T}-\|x\|^r\\ &\leq -\left(1-2^rk^re^{-r\gamma T}\right)\|x\|^r+2^{2r}\alpha^re^{-r\beta T}e^{-r\beta t}+2^{2r}c^r.\end{aligned}$$

By choosing $T=\dfrac{\log 2^{1+\frac{1}{r}}k}{\gamma}$, we obtain

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\leq -\frac{1}{2}\|x\|^r+\frac{2^r\alpha^r}{(2^{1+\frac{1}{r}}k)^{\frac{r\beta}{\gamma}}}e^{-r\beta t}+2^{2r}c^r.$$

Thus, the second inequality of Theorem 4.1 is satisfied with

$$c_3=\frac{1}{2},\quad M=\frac{2^r\alpha^r}{(2^{1+\frac{1}{r}}k)^{\frac{r\beta}{\gamma}}},\quad \delta=r\beta,\ K_1=2^{2r}c^r.$$

Now, by Lemma 4.1, we have $\|\phi_x(\tau;t,x)\|\leq e^{L(\tau-t)}$, where L is the bound of $\left[\frac{\partial f}{\partial x}\right]$. Hence, we can write

$$\begin{aligned}\left\|\frac{\partial V}{\partial x}\right\| &= \left\|r\int_t^{t+T}\left(\phi^T(\tau;t,x)\phi(\tau;t,x)\right)^{\frac{r}{2}-1}\left(\phi^T(\tau;t,x)\phi_x(\tau;t,x)\right)\right\|d\tau\\ &\leq r\int_t^{t+T}\|\phi(\tau;t,x)\|^{r-2}\|\phi^T(\tau;t,x)\|\|\phi_x(\tau;t,x)\|d\tau\\ &\leq r\int_t^{t+T}\|\phi(\tau;t,x)\|^{r-1}\|\phi_x(\tau;t,x)\|d\tau\\ &\leq r\int_t^{t+T}\left(k\|x\|e^{-\gamma(\tau-t)}+\alpha e^{-\beta\tau}+c\right)^{r-1}e^{L(\tau-t)}d\tau\\ &\leq r\int_t^{t+T}\left(2^{r-1}k^{r-1}\|x\|^{r-1}e^{-(r-1)\gamma(\tau-t)}+2^{r-1}(\alpha e^{-\beta\tau}+c)^{r-1}\right)e^{L(\tau-t)}d\tau\\ &\leq (2k)^{r-1}r\|x\|^{r-1}e^{((r-1)\gamma-L)t}\int_t^{t+T}e^{(L-(r-1)\gamma)\tau}d\tau\\ &+ 2^{2(r-1)}\alpha^{r-1}re^{-Lt}\int_t^{t+T}e^{(L-(r-1)\beta)\tau}d\tau+2^{2(r-1)}c^{r-1}r\int_t^{t+T}e^{L(\tau-t)}d\tau\\ &\leq (2k)^{r-1}r\|x\|^{r-1}e^{((r-1)\gamma-L)t}\int_t^{t+T}e^{(L-(r-1)\gamma)\tau}d\tau\end{aligned}$$

$$+ \quad 2^{2(r-1)}\alpha^{r-1}re^{-Lt}\int_t^{t+T} e^{(L-(r-1)\beta)\tau}d\tau + \frac{2^{2(r-1)}c^{r-1}r}{L}(e^{LT}-1), \ \forall\ \tau > t.$$

We have four possible cases: $L \neq (r-1)\gamma$ and $L \neq (r-1)\beta$, or $L \neq (r-1)\gamma$ and $L = (r-1)\beta$, or $L = (r-1)\gamma$ and $L \neq (r-1)\beta$, or $L = (r-1)\gamma$ and $L = (r-1)\beta$. Now we examine these four cases separately.

• Case 1: If $L \neq (r-1)\gamma$ and $L \neq (r-1)\beta$, then we have

$$\begin{aligned}\left\|\frac{\partial V}{\partial x}\right\| &\leq \frac{(2k)^{r-1}r}{L-(r-1)\gamma}\big(e^{(L-(r-1)\gamma)T}-1\big)\|x\|^{r-1}\\ &+ \frac{2^{2(r-1)}\alpha^{r-1}r}{L-(r-1)\beta}\big(e^{(L-(r-1)\beta)T}-1\big)e^{-(r-1)\beta t} + \frac{2^{2(r-1)}c^{r-1}r}{L}(e^{LT}-1).\end{aligned}$$

Thus, the last inequality of Theorem 4.1 holds with

$$c_4 = \frac{(2k)^{r-1}r}{L-(r-1)\gamma}\big(e^{(L-(r-1)\gamma)T}-1\big), \quad N = \frac{2^{2(r-1)}\alpha^{r-1}r}{L-(r-1)\beta}\big(e^{(L-(r-1)\beta)T}-1\big),$$

$$\theta = (r-1)\beta \ \ \textit{and} \ \ K_2 = \frac{2^{2(r-1)}c^{r-1}r}{L}(e^{LT}-1).$$

• Case 2: If $L \neq (r-1)\gamma$ and $L = (r-1)\beta$, then we have

$$\begin{aligned}\left\|\frac{\partial V}{\partial x}\right\| &\leq \frac{(2k)^{r-1}r}{L-(r-1)\gamma}\big(e^{(L-(r-1)\gamma)T}-1\big)\|x\|^{r-1} + 2^{2(r-1)}\alpha^{r-1}rTe^{-Lt}\\ &+ \frac{2^{2(r-1)}c^{r-1}r}{L}(e^{LT}-1).\end{aligned}$$

Thus, the last inequality of Theorem 4.1 holds with

$$c_4 = \frac{(2k)^{r-1}r}{L-(r-1)\gamma}\big(e^{(L-(r-1)\gamma)T}-1\big), \quad N = 2^{2(r-1)}\alpha^{r-1}rT,$$

$$\theta = L \ \ \textit{and} \ \ K_2 = \frac{2^{2(r-1)}c^{r-1}r}{L}(e^{LT}-1).$$

• Case 3: If $L = (r-1)\gamma$ and $L \neq (r-1)\beta$, then we have

$$\begin{aligned}\left\|\frac{\partial V}{\partial x}\right\| &\leq (2k)^{r-1}rT\|x\|^{r-1} + \frac{2^{2(r-1)}\alpha^{r-1}r}{L-(r-1)\beta}\big(e^{(L-(r-1)\beta)T}-1\big)e^{-(r-1)\beta t}\\ &+ \frac{2^{2(r-1)}c^{r-1}r}{L}(e^{LT}-1).\end{aligned}$$

Thus, the last inequality of Theorem 4.1 holds with

$$c_4 = (2k)^{r-1}rT, \quad N = \frac{2^{2(r-1)}\alpha^{r-1}r}{L-(r-1)\beta}\big(e^{(L-(r-1)\beta)T}-1\big),$$

$$\theta = (r-1)\beta \ \ \textit{and} \ \ K_2 = \frac{2^{2(r-1)}c^{r-1}r}{L}(e^{LT}-1).$$

- Case 4: If $L = (r-1)\gamma$ and $L = (r-1)\beta$, then we have

$$\left\|\frac{\partial V}{\partial x}\right\| \leq (2k)^{r-1} rT\|x\|^{r-1} + 2^{2(r-1)}\alpha^{r-1} rTe^{-Lt} + \frac{2^{2(r-1)}c^{r-1}r}{L}(e^{LT}-1).$$

Thus, the last inequality of Theorem 4.1 holds with

$$c_4 = (2k)^{r-1} rT,\ N = 2^{2(r-1)}\alpha^{r-1} rT,\ \theta = L \textit{ and } K_2 = \frac{2^{2(r-1)}c^{r-1}r}{L}(e^{LT}-1).$$

This completes the proof of Theorem 4.1.

Now we give an illustrative example to demonstrate the applicability of Theorem 4.1.

Example 4.1. Consider the following nonlinear differential equation

$$\dot{x} = -x^{\frac{6}{5}} + \frac{e^{-3t}x|\sin x|}{1+x^6}, \quad t \geq 0,$$

and observe that it is practically globally uniformly exponentially stable. Therefore, for a Lyapunov function $V(t,x) : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$, given by $V(t,x) = x^6 + e^{-t}$ and $r = 6$ we have

$$\|x\|^6 \leq V(t,x) \leq \|x\|^6 + 1, \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -6|x|^6 + 6|\sin x|e^{-3t}$$

$\left\|\frac{\partial V}{\partial x}\right\| \leq 6|x|^5 + 3$ with $c_1 = c_2 = a = 1,\ c_3 = c_4 = 6,\ M = 6|\sin x|,\ \delta = 3,\ K_1 = 0,\ K_2 = 3$ and $N = \theta = 0$.

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