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AN EXAMPLE OF ONE-STEP MLE-PROCESS IN VOLATILITY ESTIMATION PROBLEM

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Abstract. We consider the problem of construction of asymptotically efficient estimator for Pearson diffusion with unknown parameter in the volatility coefficient. The estimator-process is constructed in two steps. First we propose a preliminary consistent estimator obtained by the observations on the learning interval and then this estimator is used in construction of one-step MLE-process (maximum likelihood estimator process). It is shown that the obtained estimator-process is asymptotically efficient.

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1. Introduction

We consider an example of parameter estimation problem for the particular model of observations of Pearson-type diffusion process

$$\mathrm{d}X_t = -X_t\mathrm{d}t + \sqrt{\vartheta + X_t^2}\,\mathrm{d}W_t, \quad X_0, \quad 0 \le t \le T. \tag{1.1}$$

Here $\vartheta \in \Theta = (\alpha, \beta)\,, \alpha > 0$ is unknown parameter.

Note that this is particular case of the family of stochastic processes known as *Pearson diffusions* [10], section 1.3.7.

It is easy to see that in the case of continuous time observations the problem of parameter estimation is degenerated (singular), i.e., the unknown parameter ϑ can be estimated without error. Indeed, by Itô formula we can write

$$X_t^2 = X_0^2 + 2\int_0^t X_s\,\mathrm{d}X_s + \int_0^t \left[\vartheta + X_s^2\right]\mathrm{d}s.$$

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Hence for all $t \in (0, T]$ we have the equality

$$\hat{\vartheta} = t^{-1} \left[X_t^2 - X_0^2 - 2\int_0^t X_s \, dX_s - \int_0^t X_s^2 \, ds \right] \tag{1.2}$$

and $\hat{\vartheta} = \vartheta$. This effect is due to the singularity of the measures induced by the observations in the space of their realizations.

Such problems of parameter estimation in the diffusion coefficient are usually studied in the case of discrete time observations $X^n = (X_{t_0}, X_{t_1}, \ldots, X_{t_n})$, where $0 = t_0 < t_1 < \ldots < t_n = T$. Then the problem is no more singular and became an interesting statistical estimation problem. There is a diversity of the choice of observing times t_i. This work is the continuation of the study started in [2]. We take in our work the simplest way of equidistant observations, i.e., $t_j = j\delta, \delta = \frac{T}{n}$ and we study the properties of the estimators in the asymptotics of *high frequency* as $n \to \infty$. Our goal is to construct an asymptotically efficient estimator of the parameter ϑ. Note that the family of measures induced by the observations $X^k = (X_{t_0}, X_{t_1}, \ldots, X_{t_k})$ with t_k satisfying $t_k \leq t < t_{k+1}$ and fixed t are *locally asymptotically mixed normal* (LAMN) and for all estimators ϑ_k^* we have the lower bound on the risk

$$\varliminf_{\nu \to 0} \varliminf_{n \to \infty} \sup_{|\vartheta - \vartheta_0| < \nu} \mathbf{E}_\vartheta \ell \left(\sqrt{k} \left(\vartheta_k^* - \vartheta \right) \right) \geq \mathbf{E}_{\vartheta_0} \ell \left(\zeta_t \left(\vartheta_0 \right) \right). \tag{1.3}$$

As the loss functions $\ell(\cdot)$ can be taken, for example, polynomial $\ell(u) = |u|^p, p > 0$. For the definition of the random function $\zeta_t(\vartheta_0)$ see (1.7) below. An estimator ϑ_k^* is called *asymptotically efficient* if for all $\vartheta_0 \in \Theta$ we have

$$\lim_{\nu \to 0} \lim_{n \to \infty} \sup_{|\vartheta - \vartheta_0| < \nu} \mathbf{E}_\vartheta \ell \left(\sqrt{k} \left(\vartheta_k^* - \vartheta \right) \right) = \mathbf{E}_{\vartheta_0} \ell \left(\zeta_t \left(\vartheta_0 \right) \right). \tag{1.4}$$

The proof of this bound can be found in [1] and [4].

We construct the estimator in two steps. First we propose a consistent estimator $\bar{\vartheta}_N$ of this parameter based on the first N observations $X^N = (X_{t_0}, X_{t_1}, \ldots, X_{t_N})$ on the time interval $[0, \tau]$. Here $\tau = t_N = N\frac{T}{n}$ Then using this estimator and one-step type device we propose an asymptotically efficient estimator.

The first consistent estimator we obtain from the equality (1.2)

$$\bar{\vartheta}_N = \frac{n}{TN} \left[X_{t_N}^2 - X_0^2 - 2\sum_{j=1}^{N} X_{t_{j-1}} \left[X_{t_j} - X_{t_{j-1}} \right] - \sum_{j=1}^{N} X_{t_{j-1}}^2 \delta \right].$$

We just replaced the integrals by the corresponding integral sums. The consistency of this estimator follows immediately from the limits

$$\sum_{j=1}^{N} X_{t_{j-1}}\left[X_{t_j}-X_{t_{j-1}}\right]\longrightarrow\int_0^{\tau}X_s\,\mathrm{d}X_s,\qquad\sum_{j=1}^{N}X_{t_{j-1}}^2\delta\longrightarrow\int_0^{\tau}X_s^2\,\mathrm{d}s$$

and the relation (1.2).

The next step is to see the behavior of the error of estimation. Consider $\xi_N=\sqrt{N}\left(\bar\vartheta_N-\vartheta\right)$. We have

$$\xi_N=\frac{\sqrt{N}}{\tau}\left[\int_0^{\tau}X_s\left[2\mathrm{d}X_s+X_s\mathrm{d}s\right]-\sum_{j=1}^{N}X_{t_{j-1}}\left[2\left(X_{t_j}-X_{t_{j-1}}\right)+X_{t_{j-1}}\delta\right]\right]$$

and

$$\begin{aligned}X_{t_{j-1}}\left[X_{t_j}-X_{t_{j-1}}\right]-\int_{t_{j-1}}^{t_j}X_s\,\mathrm{d}X_s&=\int_{t_{j-1}}^{t_j}\left[X_{t_{j-1}}-X_s\right]\mathrm{d}X_s\\&=-\int_{t_{j-1}}^{t_j}X_s\left[X_{t_{j-1}}-X_s\right]\mathrm{d}s+\int_{t_{j-1}}^{t_j}\left[X_{t_{j-1}}-X_s\right]\sqrt{\vartheta_0+X_s^2}\,\mathrm{d}W_s\\&=O\left(\delta^{3/2}\right)+\int_{t_{j-1}}^{t_j}\left(\int_{t_{j-1}}^{s}\sqrt{\vartheta_0+X_r^2}\,\mathrm{d}W_r\right)\sqrt{\vartheta_0+X_s^2}\,\mathrm{d}W_s\\&=O\left(\delta^{3/2}\right)+\left(\vartheta_0+X_{t_{j-1}}^2\right)\left[\frac{\left(W_{t_j}-W_{t_{j-1}}\right)^2-\delta}{2}\right]\\&=O\left(\delta^{3/2}\right)+\left(\vartheta_0+X_{t_{j-1}}^2\right)\sqrt{\frac{\delta}{2}}\,w_j,\end{aligned}$$

where we used the estimate $X_{t_j}-X_{t_{j-1}}=O\left(\delta^{1/2}\right)$ and denoted

$$w_j=\frac{\left(W_{t_j}-W_{t_{j-1}}\right)^2-\delta}{\sqrt{2\delta}},\quad\mathbf{E}w_j=0,\ \mathbf{E}w_j^2=\delta,\ \mathbf{E}w_jw_i=0,i\neq j.$$

We have as well

$$X_{t_{j-1}}^2\delta-\int_{t_{j-1}}^{t_j}X_s^2\,\mathrm{d}s=\int_{t_{j-1}}^{t_j}\left[X_{t_{j-1}}^2-X_s^2\right]\mathrm{d}s=O\left(\delta^{3/2}\right).$$

Therefore we obtain the stable convergence (see [10])

$$\xi_N=\frac{\sqrt{2\delta N}}{\tau}\sum_{j=1}^{N}\left(\vartheta_0+X_{t_{j-1}}^2\right)w_j+o(1)\Longrightarrow\xi_\tau=\sqrt{\frac{2}{\tau}}\int_0^{\tau}\left(\vartheta_0+X_s^2\right)\mathrm{d}w(s).$$

More detailed analysis shows that we have the convergence of moments too: for any $p>0$

$$n^{\frac{p}{2}}\mathbf{E}_{\vartheta_0}\left|\bar\vartheta_N-\vartheta_0\right|^p\longrightarrow\mathbf{E}_{\vartheta_0}\left|\xi_\tau\right|^p.$$

The pseudo log-likelihood ratio function is

$$L\left(\vartheta, X^N\right) = -\frac{1}{2}\sum_{j=1}^{N} \ln\left(2\pi\left(\vartheta + X_{t_{j-1}}^2\right)\right) - \sum_{j=1}^{N} \frac{\left[X_{t_j} - X_{t_{j-1}} + X_{t_{j-1}}\delta\right]^2}{2\left(\vartheta + X_{t_{j-1}}^2\right)\delta}.$$

It is easy to see that the equation $\dot{L}\left(\vartheta, X^N\right) = 0$ has no solution, which can be written in explicit form. Hence the corresponding pseudo MLE can not be written in explicit form too. Remind that this estimator is asymptotically efficient [1], [3].

Our goal is to use the well-known one-step MLE device [8], [9] in the construction of one-step MLE-process. This estimator-process is asymptotically equivalent to the pseudo MLE but can be calculated in explicit form. This type of estimator-processes were proposed in [6] in the problem of approximation of the solution of backward stochastic differential equation for several models of observations (see the review of recent results in [5]).

Let us fix $t \in (\tau, T]$ and take such k that $t_k \leq t < t_{k+1}$. Hence $k \to \infty$ and $t_k \to t$ as $n \to \infty$. We consider the estimation of ϑ by the observations $X^k = (X_{t_0}, X_{t_1}, \ldots, X_{t_k})$. Recall that by the first X^N observations we already obtained the estimator $\bar{\vartheta}_N$. Denote the pseudo Fisher information as

$$\mathbb{I}_{t_k,n}(\vartheta_0) = \frac{1}{2}\sum_{j=1}^{k} \frac{\delta}{\left(\vartheta + X_{t_{j-1}}^2\right)^2} \longrightarrow \mathbb{I}_t(\vartheta_0) = \frac{1}{2}\int_0^t \frac{\mathrm{d}s}{(\vartheta + X_s^2)^2}.$$

The one-step MLE-process introduced first in [6] is

$$\vartheta_{t_k,n}^\star = \bar{\vartheta}_N + \sqrt{\delta}\sum_{j=1}^{k} \frac{\left[X_{t_j} - X_{t_{j-1}} + X_{t_{j-1}}\delta\right]^2 - \left(\bar{\vartheta}_N + X_{t_{j-1}}^2\right)\delta}{2\mathbb{I}_{t_k,n}\left(\bar{\vartheta}_N\right)\left(\bar{\vartheta}_N + X_{t_{j-1}}^2\right)^2\sqrt{\delta}}, \quad \tau \leq t_k \leq T.$$

Let us denote

$$\Delta_{t_k,n}\left(\vartheta, X^k\right) = \sum_{j=1}^{k} \frac{\left[X_{t_j} - X_{t_{j-1}} + X_{t_{j-1}}\delta\right]^2 - \left(\vartheta + X_{t_{j-1}}^2\right)\delta}{2\left(\vartheta + X_{t_{j-1}}^2\right)^2\sqrt{\delta}}, \quad \tau \leq t_k \leq T.$$

The main result of this work is the following theorem. theorem.

Theorem 1.1. *The one-step MLE-process $\vartheta_{t_k,n}^\star$ is consistent: for any $\nu > 0$*

$$\mathbf{P}_{\vartheta_0}\left(\max_{N \leq k \leq n}\left|\vartheta_{t_k,n}^\star - \vartheta_0\right| > \nu\right) \to 0 \tag{1.5}$$

and for all $t \in (\tau, T]$ the convergence

$$\delta^{-1/2}\left(\vartheta_{t_k,n}^\star - \vartheta_0\right) \Longrightarrow \zeta_t(\vartheta_0) \tag{1.6}$$

holds. Moreover, this estimator is asymptotically efficient in the sense (1.4).

Proof. The consistency of the estimator is proved following the same steps as it was done in [6] and we follow the main steps of the proof of the similar result in [6], where can be found the details. We have the presentation

$$\begin{aligned}\delta^{-1/2}\left(\vartheta^{\star}_{t_k,n}-\vartheta_0\right)&=\delta^{-1/2}\left(\bar\vartheta_N-\vartheta_0\right)+\frac{\Delta_{t_k,n}\left(\bar\vartheta_N,X^k\right)}{\mathbb{I}_{t_k,n}\left(\bar\vartheta_N\right)}\\&=\delta^{-1/2}\left(\bar\vartheta_N-\vartheta_0\right)+\frac{\Delta_{t_k,n}\left(\bar\vartheta_N,X^k\right)-\Delta_{t_k,n}\left(\vartheta_0,X^k\right)}{\mathbb{I}_{t_k,n}\left(\vartheta_0\right)}\\&+\Delta_{t_k,n}\left(\bar\vartheta_N,X^k\right)\left(\frac{1}{\mathbb{I}_{t_k,n}\left(\bar\vartheta_N\right)}-\frac{1}{\mathbb{I}_{t_k,n}\left(\vartheta_0\right)}\right)+\frac{\Delta_{t_k,n}\left(\vartheta_0,X^k\right)}{\mathbb{I}_{t_k,n}\left(\vartheta_0\right)}.\end{aligned}$$

We have the stable convergence

$$\frac{\Delta_{t_k,n}\left(\vartheta_0,X^k\right)}{\mathbb{I}_{t_k,n}\left(\vartheta_0\right)}\Longrightarrow\zeta_t\left(\vartheta_0\right)=\mathbb{I}_t\left(\vartheta_0\right)^{-1}\int_0^t\frac{\mathrm{d}w\left(s\right)}{\sqrt{2}\left(\vartheta_0+X_s^2\right)}. \tag{1.7}$$

From the continuity of Fisher information $\mathbb{I}_{t_k,n}\left(\vartheta\right)$ and consistency of $\bar\vartheta_N$ we obtain

$$\frac{1}{\mathbb{I}_{t_k,n}\left(\bar\vartheta_N\right)}-\frac{1}{\mathbb{I}_{t_k,n}\left(\vartheta_0\right)}\longrightarrow 0.$$

Further

$$\begin{aligned}&\frac{\Delta_{t_k,n}\left(\bar\vartheta_N,X^k\right)-\Delta_{t_k,n}\left(\vartheta_0,X^k\right)}{\mathbb{I}_{t_k,n}\left(\vartheta_0\right)}=\frac{\dot\Delta_{t_k,n}\left(\tilde\vartheta_N,X^k\right)\left(\bar\vartheta_N-\vartheta_0\right)}{\mathbb{I}_{t_k,n}\left(\vartheta_0\right)}\\&=-\frac{\mathbb{I}_{t_k,n}\left(\tilde\vartheta_N\right)}{\mathbb{I}_{t_k,n}\left(\vartheta_0\right)}\delta^{-1/2}\left(\bar\vartheta_N-\vartheta_0\right)+o\left(\delta^{1/2}\right)=-\delta^{-1/2}\left(\bar\vartheta_N-\vartheta_0\right)+o\left(\delta^{1/2}\right).\end{aligned}$$

Hence

$$\delta^{-1/2}\left(\vartheta^{\star}_{t_k,n}-\vartheta_0\right)=\frac{\Delta_{t_k,n}\left(\vartheta_0,X^k\right)}{\mathbb{I}_{t_k,n}\left(\vartheta_0\right)}+o\left(1\right)\Longrightarrow\zeta_t\left(\vartheta_0\right).$$

It can be shown that the moments converge too: for any $p>0$

$$\mathbf{E}_{\vartheta_0}\left|\delta^{-1/2}\left(\vartheta^{\star}_{t_k,n}-\vartheta_0\right)\right|^p\longrightarrow\mathbf{E}_{\vartheta_0}\left|\zeta_t\left(\vartheta_0\right)\right|^p.$$

Moreover this convergence is uniform on ϑ. Therefore the one-step MLE-process is asymptotically efficient estimator for polynomial loss functions.

Remark. The asymptotically efficient estimator process is constructed for the values $t\in(\tau,T]$. Note that it is possible to have such process (asymptotically) for all $t\in(0,T]$. To do this we have to consider the preliminary estimator $\bar\vartheta_N$ on the interval $[0,\tau_n]$ with $\tau_n\to 0$ but *sufficiently slowly*. As it follows from the proof of the similar result in [6] we have to take $N=n^{\kappa}$ with $\kappa\in(\frac{1}{2},1)$. Then $\xi_{\tau_n}\Longrightarrow\xi_0=\sqrt{2}\left(\vartheta_0+X_0^2\right)\eta$, where $\eta\sim\mathcal{N}(0,1)$. Now for all $t\in(0,T]$ we have

$$\eta_{t,T}\left(\vartheta_0\right)=\delta^{-1/2}\left(\vartheta^{\star}_{t_k,n}-\vartheta_0\right)\Longrightarrow\zeta_t\left(\vartheta_0\right).$$

More detailed analysis shows the weak convergence of the random process $\eta_{t,T}(\vartheta_0)$, $\tau_* \leq t \leq T$ with any $\tau_* \in (0,T)$ in the space of continuous on $[\tau_*, T]$ functions to $\zeta_t(\vartheta_0)$, $\tau_* \leq t \leq T$ (see [6] for details).

Список литературы

[1] Dohnal, G., "On estimating the diffusion coefficient", J. Appl. Probab., **24**, no. 1, 105 – 114 (1987).

[2] Gasparyan, S.B. and Kutoyants, Yu.A., "On approximation of the BSDE with unknown volatility in forward equation" Armenian J. Mathem., **7**, no. 1, 69 – 79 (2015).

[3] Genon-Catalot, V. and Jacod, J., "On the estimation of diffusion coefficient for multi-dimensional diffusion", Ann. IHP, Sec. B, **29**, no.1, 119 – 151 (1993).

[4] Jeganathan, P., "Some asymptotic properties of risk functions when the limit of the experiment is mixed normal", Sankhya: The Indian Journal of Statistics **45**, Series A, Pt.1, 66 – 87 (1983).

[5] Kutoyants, Yu.A., "Approximation of the backward stochastic differential equation. Small noise, large samples and high frequency cases", Proceedings of the Steklov Institute of Mathematics., **287**, 133 – 154 (2014).

[6] Kutoyants, Yu.A., "On multi-step MLE-processes for ergodic diffusion", submitted (2015).

[7] Kutoyants, Yu.A. and Zhou, L., "On estimation of the solution of BSDE in the case of small volatility", Submitted (2014).

[8] Le Cam, L. On the asymptotic theory of estimation and testing hypotheses. *Proc. 3rd Berkeley Symposium I*, 355 – 368 (1956).

[9] Lehmann, E.L. and Romano, J.P. Testing Statistical Hypotheses. (3rd ed.) Springer, N.Y. (2005)

[10] Sørensen, M., "Estimating functions for diffusion-type processes". In Statistical Methods for Stochastic Differential Equations, Kessler, Lindner and Sørensen (Ed's), Chapman & Hall/CRC, Boca Raton, 1 – 107 (2009).

[11] Wong, E., "Representation of martingales, quadratic variation and applications", SIAM Journal on Control, **9**, no. 4, 621 – 633 (1971).

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